A Study of the Optimal Asset Allocation as Raising Taxes on the Rich

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Abstract

We study the effects of introducing taxation in classical continuous-time optimization problems with utility from consumption and optimal asset allocation as taxation on the rich. This paper applies the framework of original Merton's model to a new market model that consists of a risk asset as well as a riskless asset. Under the assumption that the risk asset's price is modeled as a geometric Brownian motion with an unpredictable jump to zero, the optimal problem is reformulated and analytically solved. The aim of this article is to analyze the portfolio strategies that are adopted a dynamic model of consumption, as the impact on optimal portfolio rules concerns the contribution-hedge strategy. We thus emphasize that the current practice of taxing the rich only is appropriate when trying to reduce the distortions of the taxation system on the portfolio behavior of the investor, and that taxation applied on contributions would be more adapted.

Key words: dynamic asset allocation, Merton portfolio problem, European put option.
JEL classification: D14, G11, G23.

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1. Introduction

Wealth inequality is the most obvious measure of the gap between the rich and the poor in society. In 2007, the top 1 percent of US households held about 35 percent of the economy's wealth, thereby owning more assets than the bottom 90 percent together (Wolff, 2010). To be clear, Oxfam's claim[1] that by 2016 the richest 1% could own as much or the same as the bottom 99% is not wildly implausible. These inequalities have spurred a vivid debate on whether wealth should be redistributed, and the possibility of achieving a more equitable wealth distribution has historically been a common rationale for taxing capital income.

This article responds the Piketty's argument at *Capital in the Twenty First Century*. His main proposal is a comprehensive international agreement to establish a progressive tax on individual wealth, defined to include every kind of asset. In recent years, there has been a renewed academic interest in the normative aspects of capital and wealth taxation (Diamond and Saez, 2011; Cagetti et al., 2009). Usual topics in these debates are long-term spending cuts of subsidies and social programs, or increments in consumption taxation. However, tax policy increasingly envisages taxing the "rich." Recently, Obama administration officials will seek to raise taxes on wealthy to finance cuts for middle class. The plan would also increase the top capital-gains tax rate, to 28 percent from 23.8 percent, for couples with incomes above $500,000 annually. People with higher income and wealth are suggested to bear a greater share of the tax burden. The pioneering work by Goolsbee (2000), concluded that the response of executive salaries was almost entirely a short-run shift in the timing of compensation rather than a permanent change and came almost entirely from a large increase in the exercise of stock options by the highest-income executives in anticipation of the rate increases. He estimated that the short-run elasticity of taxable income (ETI) with respect to the net of-tax share exceeded one, but concluded that the elasticity after one year was at most 0.4 and probably closer to zero.

Gruber and Saez (2002) expanded the previous literature in a number of important ways using a panel of tax returns that spanned several major shifts in tax rate regimes during the 1979-1990 period. The variation in tax rates from the long time period covered by their panel allowed them to more carefully examine and model mean

1 The wealthiest 1% will soon own more than the rest of the world's population, according to a study by anti-poverty charity Oxfam. The charity's research shows that the share of the world's wealth owned by the richest 1% increased from 44% in 2009 to 48% last year. On current trends, Oxfam says it expects the wealthiest 1% to own more than 50% of the world's wealth by 2016.Source from BBC news 18 January, 2015. http://www.bbc.co.uk/news/business-30875633

reversion and consider heterogeneity with respect to income and other taxpayer characteristics.

In contrast to Gruber and Saez, who focused on middle-income taxpayers during a period when tax rates were reduced, Carroll (1998) focused on a period when tax rates increased using a sample that included many high-income taxpayers. It is perhaps surprising then that he found an elasticity of taxable income with respect to the after-tax share of 0.4, about the same as the full-sample estimate of Gruber and Saez (2002) and lower than their estimate for high-income taxpayers. In this debate, the behavioral responses of the affluent to taxes are of particular interest. This focus is motivated by the notion that high income taxpayers may be more responsive to taxes both because they face higher marginal tax rates and may have more opportunities to respond to changes in tax policy. As a result, raising in tax rates at the top of the income distribution can have large implications for tax revenues and economic activity. Moreover, because the recent debates over future tax policy in the U.S. have focused predominantly on the taxation of the high end of the income distribution, these behavioral responses of the rich have received increased attention.

This paper offers the first analysis of the implications for dynamic asset allocation of taxation on the rich. Wealthy person also responds by increasingly favoring the higher-return risky asset. This stochastic process is expected to increase in real terms over time and might be correlated with the investment performance of the 'risky asset'.

Who Are the Rich?

Who is rich and who is not? The answer to that question depends on the measure of affluence chosen, and what dividing line one chooses. Some candidates for a measure of affluence are annual income, annual consumption, wealth, lifetime income and lifetime consumption; depending on the issue at hand, different measures may be more or less appropriate. Although conceptually attractive, a lack of data that tracks people over a lifetime precludes empirical examination of the latter two measures, although longitudinal data sets that follow people over a decade or more are now available. We assume the wealth process satisfies the following geometric Brownian motion (GBM), but with specification:

$$dW(t) = \begin{Bmatrix} \alpha_\omega W(t)dt + \sigma_\omega W(t)d{Z_t}^W & W(t) \geq W(r) \\ 0 & W(t), < W(r) \end{Bmatrix} \quad (1)$$

Where $W(r)$ is the wealth threshold to be rich people, α_ω return to assets, σ volatility of risky assets, and $d{Z_t}^W$ Wiener increment,$W(t)$ wealth taken into the wealthy taxpayers.

2. Economy Model

The rich investors take into consideration the risk and financial management decision for wealth utility maximization. Therefore adopt the option hedging strategies to reduce risk on the asset allocation of wealth, $W(t)$ wealth taken into rich man, $T-t$ is time to maturity, α instantaneous expected return to risky assets, r return to safe assets, $\mathcal{K}$ strike price. Solving the agent's utility maximization problem, we have define protected wealth

$$\mathbb{w}(t) \equiv \frac{1}{r}\left[1-e^{-r(T-t)}\right]-\mathcal{K}e^{-r(T-t)} \tag{2}$$

surplus wealth

$$\bar{\mathbb{w}}(t) \equiv W(t)-\mathbb{w}(t) \tag{3}$$

It values and replicates a put option on an `optimally invested' synthetic security $\mathbb{w}_\lambda(t)$;where the terminology and the subscript follow Cox and Huang (1989). Initial surplus optimally-invested wealth, $\mathbb{w}_\lambda(0)$ is just small enough to ensure that sufficient wealth remains to guarantee a nonnegative bequest. Remaining initial surplus wealth, $\bar{\mathbb{w}}(0)-\mathbb{w}_\lambda(0)$ is invested in a European put option on optimally-invested wealth. The put's value subsequently is given by

$$\mathcal{P}(\mathbb{w}_\lambda(t),t) \equiv E_t^{Q}\max[0,\mathcal{K}-\mathbb{w}_\lambda(T)], \tag{4}$$

where the superscript $\mathbb{Q}$ on the right-hand side denotes the value of an expectation taken under the risk-neutral measure. The option is self-funding through time, so that surplus wealth is conserved in the sense.

$$\bar{\mathbb{w}}(t)=\mathbb{w}_\lambda(t)+\mathcal{P}(\mathbb{w}_\lambda(t),t) \tag{5}$$

Up to this point, the risk-neutral specialization of the process defined by Eq. (1) has instantaneous return r and (constant) instantaneous volatility $\bar{x}^*\sigma$ .Standard theory says that replicating the option specified by Eq. (4) with this asset and the safe asset requires going long by an amount

$$N(-d_2)\mathcal{K}e^{-r(T-t)} \tag{6}$$

in the safe asset, and short an amount

$$N(-d_1)\mathbb{w}_\lambda(t)e^{-\int_t^T \beta(\mu)d\mu} \tag{7}$$

in the synthetic risky asset, where

$$d_1=\frac{ln\left(\frac{\mathbb{w}_\lambda(t)}{\mathcal{K}}\right)+\left(r+\frac{(x^*\sigma)^2}{2}\right)(T-t)-\int_t^T \beta(\mu)d\mu}{\bar{x}\sigma(T-t)} \tag{8}$$

$$d_2=d_1-\bar{x}\sigma(T-t)$$

and $N(\,)$ denotes the Normal distribution. Replicating the put with the underlying risky asset at time t therefore requires going short an amount

$$\bar{x}^*(t)N(-d_1)\mathbb{w}_\lambda(t)\mathcal{K}e^{-\int_t^T \beta(\mu)d\mu} \tag{9}$$

in the underlying risky asset, where $\int_t^T \beta(\mu)d\mu$ is the same as investment(stock) dividend q.

2.1. Asset allocation

Followed by application of Eqs (2) and (3) to substitute out $\overline{\mathbb{w}}(t)$, the optimal dollar investment $A^*(\mathrm{t})$ in risky assets is

$$A^*(\mathrm{t}) = \bar{x}^*(t)\mathbb{w}_\lambda(t) - \bar{x}^*(t)N(-d_1)\mathbb{w}_\lambda(t)e^{-\int_t^T \beta(\mu)d\mu} \tag{10}$$

$$= \bar{x}^*(t)\left[\overline{\mathbb{w}}(t) - \mathcal{P}(\mathbb{w}_\lambda(t), t) - N(-d_1)\mathbb{w}_\lambda(t)e^{-\int_t^T \beta(\mu)d\mu}\right] \tag{11}$$

$$= \left(\frac{\alpha-r}{\gamma\sigma^2}\right)\left[\overline{\mathbb{w}}(t) - N(-d_2)\mathcal{K}e^{-r(T-t)}\right] \tag{12}$$

$$= \left(\frac{\alpha-r}{\gamma\sigma^2}\right)\left[W(t) - \frac{1}{r}(1-e^{-r(T-t)}) + \mathcal{K}e^{-r(T-t)}(1-N(-d_2))\ \right] \tag{13}$$

Divide Eq. (13) through by $W(t)$ to arrive at our main result:

Proposition 1

The optimal proportionate investment $x^*(t)$ in risky assets, in terms of the model's state variable and parameters, is given by

$$x^*(t) = \left(\frac{\alpha-r}{\gamma\sigma^2}\right)\left[1 - \frac{1}{rW(t)}(1-e^{-r(T-t)}) + \frac{\mathcal{K}}{W(t)}e^{-r(T-t)}(1-N(-d_2))\ \right] \tag{14}$$

Moreover, from equation (15) one can obtain optimal proportionate investment $x^*(\mathrm{t})$ in risky assets after imposition taxes τ as

$$\left(\frac{\alpha-r}{\gamma\sigma^2}\right)(1-\tau)\left[W(t) - \frac{1}{(1-\tau)r}(1-e^{-(1-\tau)r(T-t)}) + \mathcal{K}e^{-(1-\tau)r(T-t)}(1-N(-d_2))\ \right] \tag{15}$$

See e.g. Bruhn, K. (2013) for more detailed discussions of the equation (15) when there is under the chargeable wealthy people tax rules.

The right-hand side of Eq. (14) consists of three terms. The first, i.e., $\frac{\alpha-r}{\gamma\sigma^2}$ is familiar from Merton (1969). The second was introduced by Merton (1971). Its implications for dynamic asset allocation are discussed by Ingersoll (1987) and Karatzas and Shreve (1998), among others. Ingersoll offers the useful analogy of an `escrow' account, comprised of safe securities, set up at time zero, and then run down gradually, until time *T*. The third term is similar to Cox and Huang (1989) give several worked-out examples containing option-related components in their solutions. Carroll (2002) gives theory and evidence in support of the proposition that luxury bequests raise the average level of risky assets in portfolios, without considering dynamic asset allocation.

2.2 Two assets allocation model

We shall assume that the rich wealth funds can trade two assets continuously in an economy. The first asset is the money market account (the Bond) growing at a rate r . The second asset is a risky security (the stock). Following Merton, assume there is a single perishable consumption good as numeraire. The portfolio selection participants derive utility from intertemporal consumption ***C*** of this good and the terminal wealth at time T . We ignore labor income in this context. Throughout this paper, we are assume a probability space $(\Omega, \mathcal{F}, \mathcal{P})$and a filtration $\{\mathcal{F}_t\}$.Uncertainty in the models is generated by standard Brownian motion Z_t .The two equations governing the dynamics of the money market account (bond) and stock are now given as;

$$dB_t = rB_t dt \quad \text{or} \quad B_t = B_0 exp(rt) \tag{16}$$

and

$$dS_t = \alpha S_t dt + \sigma S_t dZ_t \tag{17}$$

or

$$S_t = S_0 exp\left\{\sigma Z_t + (\alpha - \frac{\sigma^2}{2})t_,\right\}, \forall t \in [0,1] \tag{18}$$

The parameter B_0 is the initial investment on the money market account which determines the speed of a mean-reversion to the stationary level. σ is the volatility of risky assets . The admissible trading strategies are (***D, I***) . The processes D and I are cumulative amount of sales and purchases of stock. The two processes satisfy $D(0)$= $\boldsymbol{I}(0)$=**0,** and both are non-decreasing, right continuous adapted. The evolution of the amount invested in the money market account and stock process can be expressed as:

$$\begin{cases} dB_t = rB_t dt - dI_t + dD_t \\ dS_t = \alpha S_t dt + \sigma S_t dZ_t + dI_t - dD_t \end{cases} \tag{19}$$

For tractability, quantitative derivation and insightful analytic solutions to optimal investment portfolio fund of the rich, we use CRRA utility function of the final wealth, that is, $U(W) = \frac{W^{1-\gamma}}{1-\gamma}, for\ 0 < \gamma < 1, \gamma$ is the constant relative risk aversion parameter (that is the relative risk premium). On behalf of the plan participants, the portfolio selection chooses optimal investment strategies and so as to maximize the final wealth at a deterministic time. Define the value function at time as;

$$J(C, B, S, t; T) = Max_{(D,I)} E\left[\frac{(B_T + S_T)^{1-\gamma}}{1-\gamma}\right], \tag{20}$$

where $W = B_T + S_T$ is the total investment from both the riskless and the risky assets.

Assumption1:

The participant makes intermediate consumption decision on the admissible consumption space $\mathbb{C}$, which satisfies

$$\int_0^t |C_s|\, ds < \infty, \ , \forall t \in [0, T] \tag{21}$$

Assumption 2*:*

The parameter values satisfy:

$$0 < \frac{\alpha - r}{\gamma\sigma^2} < 1. \qquad (22)$$

It guarantees that ***B*** and ***S*** would be chosen to be strictly positive. Consideration above assumptions, consumption is made through the money market account. The portfolio problem becomes:

$$J(C,B,S,t;T) = Max_{C_t,B_t,S_t:t>0}\, E\left[\int_0^T e^{-\theta t}\frac{C_t{}^{1-\gamma}}{1-\gamma}dt + e^{-\theta t}\frac{(B_T+S_T)^{1-\gamma}}{1-\gamma}\right] \qquad (23)$$

Subject to;

$$dB_t = rB_t dt - C_t dt - dI_t + dD_t$$
$$dS_t = \alpha S_t dt + \sigma S_t dZ_t + dI_t - dD_t$$

The constraints above are equivalent to:

$$dW_t = (rB_t + \alpha S_t - C_t)dt + \sigma S_t dZ_t. \qquad (24)$$

where the notation is: ***E*** expectations operator, T age at death (assumed known), θ rate of time preference and the time discount rate, γ is the coefficient of relative risk aversion and is assumed to greater than or equal to 1, W wealth, σ volatility of risky assets, and $d\boldsymbol{Z_t}$ Wiener increment, C consumption, The value function should also satisfy the terminal condition:

$$J(C,B,S,t;T) = \frac{(B_T+S_T)^{1-\gamma}}{1-\gamma} \qquad (25)$$

The first term of the value function, ***J*** represents discounted utility from consumption flows, while the second term captures the idea that terminal wealth gives utility to the participant as well for he can finance his consumption by using the benefit payment from time ***T*** upwards. Under this setting, we may establish that the result have located an optimum, the solution can be summarized as follows:

Proposition 2

The optimal surplus investment is a constant proportion of constrained optimally-invested surplus wealth, optimal amount invested in stock:

$$S^* = \frac{\alpha - r}{\gamma\sigma^2}W \text{ and } \quad x^*(t) = \frac{\alpha - r}{\gamma\sigma^2} \qquad (26)$$

Where

$$x^*(t) = S^*/\,W \ .$$

Appendix A provides the proof of equation (26).

Moreover, the agent's policy functions satisfy the following equation

$$C(t) = a(t)^{-\frac{1}{\gamma}}[W(t) + b(t)] \qquad (27)$$

$$S(t) = \frac{\alpha - r}{\gamma\sigma^2}[W(t) + b(t)]$$

The proof of equation (27) see Appendix B. Optimal investment policy involves investing a constant fraction of wealth in the stock, independent of the investor's horizon. As long as $\alpha > r$, the fund always holds the stock in its portfolio. Allowing for intermediate consumption does not change optimal investment policy. The ratio of the amount invested in stock and money market account is:

$$\pi^* = \frac{S^*}{B} = \frac{\frac{\alpha-r}{\gamma\sigma^2}W}{\left(1-\frac{\alpha-r}{\gamma\sigma^2}\right)W} = \frac{\alpha-r}{\gamma\sigma^2-\alpha+r}. \tag{28}$$

Now replacing **S** with the optimal value $S^* = \frac{\alpha-r}{\gamma\sigma^2}W$, in the HJB equation and

rearrange, we find the ordinary differential equation of a in time *t* as:

$$a(t)^{-\frac{1-\gamma}{\gamma}} \cdot \frac{\gamma}{1-\gamma} + \frac{a(t)'}{1-\gamma} + a(t)r + \frac{(\alpha-r)^2}{2\gamma\sigma^2}a(t) - \frac{\theta}{1-\gamma}a(t) = 0. \tag{29}$$

Formalizing it to:

$$\frac{da}{dt} = -\gamma a(t)^{-\frac{1-\gamma}{\gamma}} - \left[(1-\gamma)r + \frac{(1-\gamma)(\alpha-r)^2}{2\gamma\sigma^2} - \theta\right]a(t) \tag{30}$$

As a result, we will obtain[2]

$$a(t)^{\frac{1}{\gamma}} = \left(a(0)^{\frac{1}{\gamma}} - \frac{\gamma}{\eta}\right)e^{\frac{-\eta(T-t)}{\gamma}} + \frac{\gamma}{\eta} \quad \text{i.e. } a(t) = \left[\left(a(0)^{\frac{1}{\gamma}} - \frac{\gamma}{\eta}\right)e^{\frac{-\eta(T-t)}{\gamma}} + \frac{\gamma}{\eta}\right]^{\gamma} \tag{31}$$

Where $\eta = (1-\gamma)r + \frac{(1-\gamma)(\alpha-r)^2}{2\gamma\sigma^2} - \theta$

Thus, from extended eq. (24) we can have

$$dW(t) = [rW(t) + (\alpha-r)S(t) - C(t) + Y(t)]dt + \sigma S_t dZ_t \tag{32}$$

Substituting eq. (26)-(27) into wealth process obtained in above (32) gives us

$$dW(t) = \left[rW(t) + \frac{(\alpha-r)^2}{\gamma\sigma^2}(W(t)+b(t)) - a(t)^{-\frac{1}{\gamma}}[\![W(t+b(t)]\!] + Y(t)\right]dt +$$

$$\frac{(\alpha-r)}{\gamma\sigma}(W(t)+b(t))dZ_t \tag{32.1}$$

Define

$$b(t) = \int_t^T Y(\mu)e^{-r(\mu-t)}d\mu$$

Where $Y(t)$ is the labor income depending on age.We know that

$$db(t) = [-Y(t) + rb(t)]dt \tag{33}$$

Thus

$$d(W(t)+b(t)) = \left(r + \frac{(\alpha-r)^2}{\gamma\sigma^2} - a(t)^{-\frac{1}{\gamma}}\right)\left((W(t)+b(t)\right)dt + \frac{\alpha-r}{\gamma\sigma}\left((W(t)+b(t)\right)dZ(t)$$

(34)

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[2] The details see also Appendix C.

And wealth accumulation within lifetime, Let $X(t)$ be the total wealth, i.e. the sum of physical wealth and human wealth.

$X(t) = W(t) + b(t)$

Thus

$$dX(t) = \left(r + \frac{(\alpha-r)^2}{\gamma\sigma^2} - a(t)^{-\frac{1}{\gamma}}\right) X(t)dt + \frac{\alpha-r}{\gamma\sigma} X(t)dZ(t) \tag{35}$$

We can apply Euler equation and yield

$$X(t) = \left(\frac{a(t)}{a(0)}\right)^{\frac{1}{\gamma}} exp\left(\frac{r-\theta}{\gamma} + \frac{(\alpha-r)^2}{2\gamma\sigma^2}\right) t + \frac{\alpha-r}{\gamma\sigma} Z(t)X(0) \tag{36}$$

Using the boundary condition

$$a(T) = \chi(1-\zeta)^{1-\gamma} \tag{37}$$

Note that pre-tax the end-of-life wealth is

$$W(T) = X(T)$$

$$= \left(\frac{a(T)}{a(0)}\right)^{\frac{1}{\gamma}} exp\left[\left(\frac{r-\theta}{\gamma} + \frac{(\alpha-r)^2}{2\gamma\sigma^2}\right) T + \frac{\alpha-r}{\gamma\sigma} Z(T)\right] X(0)$$

$$= (\chi(1-\zeta)^{1-\gamma})^{\frac{1}{\gamma}}(a(0))^{-\frac{1}{\gamma}} exp\left[\left(\frac{r-\theta}{\gamma} + \frac{(\alpha-r)^2}{2\gamma\sigma^2}\right) T + \frac{\alpha-r}{\gamma\sigma} Z(T)\right] X(0) \tag{38}$$

And get the following after-tax result

$$W(T) = X(T) =$$

$$(\chi(1-\zeta)^{1-\gamma})^{\frac{1}{\gamma}} a(0)^{-\frac{1}{\gamma}} exp\left(\frac{(1-\tau)r-\theta}{\gamma} + \frac{(\alpha-r)^2}{2\gamma\sigma^2}\right) T + \frac{\alpha-r}{\gamma\sigma} Z(T)X(0) \tag{38.1}$$

The details see Appendix D.

3. Intergenerational connection with certain life

Now let $T, 2T, 3T \ \cdots \ nT$, $\cdots$be the rich at time of generation 1, 2, 3,…n, $\cdots$. Let $X_1 = X(T), X_2 = X(2T),\ X_3 = X(3T) \ \ \cdots,\ X_n = X(nT),\cdots$

Thus

$$X(n+1) = X(n+1)T \tag{39}$$

$$= (1-\xi)W\big((n+1)T\big) + z(0)$$

Combining with eq.38 yields

$$= \left(\frac{\chi(1-\xi)}{a(0)}\right)^{\frac{1}{\gamma}} exp\left[\left(\frac{(r-\theta)}{\gamma} + \frac{(\alpha-r)^2}{2\gamma\sigma^2}\right) T \ \ + \frac{\alpha-r}{\gamma\sigma} Z(T)\right] X(nT) + z(0)$$

$$= \left(\frac{\chi(1-\xi)}{a(0)}\right)^{\frac{1}{\gamma}} exp\left[\left(\frac{(r-\theta)}{\gamma} + \frac{(\alpha-r)^2}{2\gamma\sigma^2}\right) T \ \ + \frac{\alpha-r}{\gamma\sigma} Z(T)\right] X_n + z(0) \tag{40}$$

Let $\rho_{n+1} = \left(\frac{\chi(1-\xi)}{a(0)}\right)^{\frac{1}{\gamma}} exp\left[\left(\frac{(r-\theta)}{\gamma} + \frac{(\alpha-r)^2}{2\gamma\sigma^2}\right) T \ \ + \frac{(\alpha-r)}{\gamma\sigma} Z(T)\right]$

Note that ρ_{n+1} is lognormally distribution

Thus $X_{n+1} = \rho_{n+1}X_n + z(0)$ (41)

Thus the result of Sornette (2006) could be applied here.

3.1. Bequest distribution with Pareto tail

The next step invokes that the bequest distribution has a Pareto upper tail. Then by Reed (2006), we claim that the wealth distribution has an asymptotic Pareto upper tail. By Sornette (2006), the bequest follows a distribution with a Pareto upper tail, if there exists a ν such that $E\rho_{n+1}^{\nu} = 1$.Note that ρ_{n+1} is log-normally distributed. Thus

$$E\rho_{n+1}^{\nu} = \left(\frac{\chi(1-\xi)}{a(0)}\right)^{\frac{\nu}{\gamma}} exp\left[\nu\left(\frac{r-\theta}{\gamma} + \frac{(\alpha-r)^2}{2\gamma\sigma^2}\right)T \quad + \frac{1}{2}\nu^2 \frac{(\alpha-r)^2}{\gamma\sigma^2} T\right] = 1 \qquad (42)$$

Rearrange them and yield the pretax result

$$\nu\left(\frac{(r-\theta)}{\gamma} + \frac{(\alpha-r)^2}{2\gamma\sigma^2}\right) + \frac{1}{2}\nu^2 \frac{(\alpha-r)^2}{\gamma\sigma^2} = \frac{1}{T}\frac{\nu}{\gamma}\log\left(\frac{a(0)}{\chi(1-\xi)}\right)$$

$$\nu = \gamma\left(\frac{\frac{1}{T}\log\left(\frac{a(0)}{\chi(1-\xi)}\right) - r + \theta}{\frac{(\alpha-r)^2}{2\sigma^2}} - 1\right) \qquad (43)$$

By Sornette (2006), the starting wealth displays an asymptotic Pareto upper tail under tax system i.e.

$$P(x(0) > x) \sim x^{-\nu} \qquad (44)$$

where

$$\nu = \gamma\left(\frac{\frac{1}{T}\log\left(\frac{a(0)}{\chi(1-\xi)}\right) + \theta - (1-\tau)r}{\frac{(\alpha-r)^2}{2\sigma^2}} - 1\right)$$

and

$$G = \frac{1}{(2\nu-1)}$$ [3]

Above equation (44) states that a sufficient condition for the convergence of the wealth distribution to the Pareto distribution is that the wealth differentiation is driven only by luck. It can be shown that this condition is not only sufficient, but also necessary, to ensure the Pareto distribution; see Levy (2003).

4. Numeric illustration

[3] The Gini coefficient is a measure of the deviation of the Lorenz curve from the equidistribution line which is a line connecting [0, 0] and [1, 1], which is shown in black (α = ∞) in the Lorenz plot on the right. Specifically, the Gini coefficient is twice the area between the Lorenz curve and the equidistribution line. The Gini coefficient for the Pareto distribution is then calculated (for) to be $G = 1 - 2\left(\int_0^1 L(F)dF\right) = \frac{1}{(2\nu-1)}$, where α ≥ 1 (see Gastwirth 1972).

Firstly, we consider the parameterization of the model. Consideration is given to the resulting the wealthy tax policy implications and how these suggest the optimal rule of thumb is an appropriate rule for investor's asset *purchases*. Numeric results are then discussed, and we consider with Hence, we set $\alpha - r/\gamma\sigma^2$ to 0.25.Its value is $\frac{\alpha-r}{\gamma\sigma^2} = 0.25$, Merton (1973) describe exactly satisfies the requirement two assets sufficient liquidity market conditions. In Fig.1 parameters setting: initial wealth=\$100,000(thousand), strike price =\$92,000(thousand), initial age=45, final age=89, tax rate $\tau = 0.1, 0.2, 0.25, 0.3, 0.35, 0.4$,rate of time preference=2% p.a., investment dividend rate=3%, giving q as 0.03 , expected return to risky assets=2% p.a., volatility of risky assets=20% p.a., The latter two values are from Lockwood (2012, Table 3).

Figure 2 illustrates the simulated results show that my model replicates the Gini coefficient of the wealth distribution for a particular initial value of wealth and a particular set of model parameters.Gini and Lorenz curve parameters setting: $\theta = 0.04, r = 0.02, \gamma=2.5, \alpha = 0.08, \sigma = 0.2, \zeta = 0.19$, $\chi = 15$, $t \in [45,89]$.

The rich may have inherited more, either in terms of financial resources or in terms of human capital, broadly defined. If inherited endowment is the principal source of inequality (so that, people do not differ in what they make of their endowments), from a one-generation perspective there is little potential economic cost from a tax system that redistributes the fruits of this endowment. A longer horizon is required, however, because the incentive of parents to leave an endowment would arguably be affected by such taxation, and so could affect the incentive of potential bequeathors to work and to save. The rich may have different skills than everyone else, rather than more of the same kind of skills. This characterization certainly rings true, as the higher the bequest motive χ , or the lower the estate tax ζ,the smaller is v . Thus the impacts of χ , and ζon v are in line with our intuition about the role of bequest on wealth inequality: the more persistent the bequest process[4], the higher is the inequality in wealth distribution.

[4] Atkinson (1970) and his followers prefer to suppose that income is a continuous variable. It implies that the population is implicitly infinite, but the sample can be finite. Discrete variables and finite population are at first easy notions to understand while continuous variables and infinite population are more difficult to accept. But as far as computations and derivations are concerned, continuous variables lead to integral calculus which is an easy topic once we know some elementary theorems. Considering a continuous random variable opens the way for considering special parametric densities such as the Pareto or the lognormal which have played an important role in studying income distribution. Discrete mathematics are quite complicated.

A smaller v implies a fatter tail of wealth distribution. Castaneda et al. (2003) study the steady-state implications of abolishing estate taxation. They find that abolishing estate taxation brings about very little change in wealth inequality. Cagetti and De Nardi (2009) study the effect of abolishing estate taxation on the stationary wealth distribution in different policy change experiments. They also find that in each experiment abolishing estate taxation has little effect on the wealth inequality

5. Implications for policy advice

In figure 1 we plot the red-dotted line strips out the effect of our synthetic put on optimally invested wealth before tax, thereby shedding light on the empirical importance of looking beyond the solution resulting from unconstrained dynamic programming. Herein figure 1 is similar to the share of risky assets of the portfolio line with Ding, J. et al. (2014).At the initial age of 45, and in the case of the solution that rules out negative bequests (i.e., the solution that incorporates a synthetic put option), the estimated share of risky assets is 29.27%, so our example suggests that at the outset of retirement it is not important in practice to account for luxury bequests when allocating assets. This difference is consistent with the fact that the required synthetic put has considerable time value at the outset of retirement. At the final age of 89, and in the case of the solution that rules out negative bequests, the expected share of risky assets is 35.11%. Bodie et al.(1992) show that labor income can make a big difference to asset allocation early in working life. On the other hand, the synthetic put makes scarcely any difference to asset allocation late in retirement, consistent with decay over time in its value. The key behavioral assumption invoked by Merton's (1969) model is that investors only use securities that they know about in constructing their optimal portfolios. In sensitivity analysis on taxes rich, along with the tax rate increasing, the proportion of risky assets is also associated with the decline.

[Figure 1 here]

Figure 2, the Lorenz curve[5] is a graphical representation of the cumulative income distribution. Thus the straight line represents perfect equality. And any departure from

[5] It shows for the bottom p_1% of households, what percentage p_2% of the total income they have. The percentage of households is plotted on the x−axis, the percentage of income on the y−axis. It was developed by Max O. Lorenz in 1905 for representing inequality in the wealth distribution. As a matter of fact, if $p_1 = p_2$, the Lorenz curve is a straight line which says for instance that 50% of the households have 50% of the total income.

this 45-degree line represents inequality; see e.g., dotted line in figure 2. The simulation graph shows that the Lorenz curve on the red dashed line after tax the rich get close to 45-degree line, which displays the tax levied on the wealthy to reduce inequities and the distribution of wealth allocated more evenly. There is one important feature of the solution that should be pointed out: The wealth tax on the rich is fully demonstrated phenomenon tackling wealth inequality (can be reduced inequality).It play a corrective function on externalities of the gap between rich and poor.

[Figure 2 here]

6. Conclusion

We study the dynamics of the distribution of wealth in an economy with infinitely lived agents, intergenerational transmission of wealth, and redistributive taxing rich policy. We show that wealth accumulation with idiosyncratic investment risk and uncertain lifetimes can generate a Pareto wealth distribution. From a policy perspective, by levying a wealth transfer tax and redistributing revenue among the young generation, the government can further reduce the concentration of wealth. The higher the tax τ on the rich, the lower is the variance of wealth, while average wealth holdings are not affected. As a consequence, the coefficient of variation is reduced by the tax. Hence, the government can follow a wealthy taxation policy in order to reduce wealth inequality. We find this inequality-reducing effect of taxation (which would also be found in unintended-bequest setups) due to our assumption of a joy-of-giving motive which removes Becker–Tomes type "family wealth" considerations. While these results hold for the coefficient of variation as a measure of inequality, simulation suggests that they also hold for other, "more popular" measures like the Gini coefficient. Taxing bequests reduces not only the coefficient of variation but also the Gini coefficient. Future work could check whether the taxation result also survives under these more general specifications.

Appendix A.

To solve the optimal consumption and investment problem, the technique of stochastic dynamic optimization is used. We start with the Bellman equation:

$$J(C,B,S,t;T) = Max_{C,S}\left\{\frac{C^{1-\gamma}}{1-\gamma} + \frac{1}{1+\rho}E[J(C',B',S',t+\Delta t;T)]\right\} . \qquad \text{(A1)}$$

The actual utility over the time interval of length Δt is $\frac{C^{1-\gamma}}{1-\gamma}\Delta t$, and the discounting over such time interval is expressed by $\frac{1}{1+\rho\Delta t}$. Therefore the Bellman equation becomes:

$$J(C,B,S,t;T) = Max_{C,S}\left\{\frac{C^{1-\gamma}}{1-\gamma}\Delta t + \frac{1}{1+\rho\Delta t}E[J(C',B',S',t+\Delta t;T)]\right\} . \qquad \text{(A2)}$$

Multiplying both LHS and RHS by a factor of $1+\rho\Delta t$ and rearranging the terms, we get:

$$\rho J\Delta t = \underset{C,S}{Max} \left\{\frac{C^{1-\gamma}}{1-\gamma}\Delta t(1+\rho\Delta t)+E[\Delta J]\right\} \tag{A3}$$

Dividing by Δt and let it go to 0, the Bellman equation becomes:

$$\rho J = Max_S\left\{\frac{C^{1-\gamma}}{1-\gamma}+\frac{1}{dt}E[dJ]\right\} \tag{A4}$$

Ito's lemma states:

$$dJ = \left[\frac{dJ}{dt}+(rB+\alpha S-C)\frac{dJ}{dW}+\frac{1}{2}\sigma^2 S^2\frac{d^2J}{dW^2}\right]dt+\rho S\frac{dJ}{dW}dZ$$

Applying it to the Bellman equation, we get the corresponding Hamilton-Jacobi-Bellman (HJB) equation:

$$\frac{C^{1-\gamma}}{1-\gamma}+J_t+J_W(rB+\alpha S-C)+\frac{1}{2}J_{WW}\sigma^2S^2-\rho J=0 \tag{A5}$$

We derive optimal consumption policy from the HJB equation. First order condition with respect to consumption on the HJB equation yields:

$$J_W=\frac{\partial}{\partial C}\frac{C^{1-\gamma}}{1-\gamma}=C^{-\gamma} \tag{A6}$$

The optimal consumption is the given as:

$$C^*=(J_W)^{-\frac{1}{\gamma}}. \tag{A7}$$

Substituting the optimal consumption into the HJB equation yields:

$$\frac{C^{1-\gamma}}{1-\gamma}+J_t+J_W(rB+\alpha S-C^*)+\frac{1}{2}J_{WW}\sigma^2S^2-\rho J=0. \tag{A8}$$

To eliminate *B* from the equation, use the condition $W=B+S$

$$\frac{C^{1-\gamma}}{1-\gamma}+J_t+J_W(rW+(\alpha-r)S-C^*)+\frac{1}{2}J_{WW}\sigma^2S^2-\rho J=0. \tag{A9}$$

We conjecture that the value function J must be linear to $\frac{W^{1-\gamma}}{1-\gamma}$, and takes the form;

$$J(C,B,S,t;T)=a(t;T)\frac{W^{1-\gamma}}{1-\gamma} \tag{A10}$$

for a horizon dependent function
$a(t;T)>0, \forall t\in[0,T]$

Replacing $C^*\ by\ (J_W)^{-\frac{1}{\gamma}}=a^{-\frac{1}{\gamma}}W;\ J_t\ by\ a'\frac{W^{1-\gamma}}{1-\gamma}$;and $J\ by\ a'\frac{W^{1-\gamma}}{1-\gamma}$ in the HJB equation, it follows that:

$$\frac{a^{-\frac{1-\gamma}{\gamma}}}{1-\gamma}W^{1-\gamma}+a'\frac{W^{1-\gamma}}{1-\gamma}+aW^{-\gamma}\left(rW+(\alpha-r)S-a^{\frac{-1}{\gamma}}W\right)-\frac{\gamma}{2}aW^{-1-\gamma}\sigma^2S^2-$$

$$\rho a \frac{W^{1-\gamma}}{1-\gamma} = 0 \tag{A11}$$

First order condition on **s** gives the optimal amount invested in stock:

$$S^* = \frac{\alpha - r}{\gamma\sigma^2} W \tag{A12}$$

This completes the proof of equation 26.

Appendix B. Proof of Proposition 2.

The agent lives from 0 to *T*. For the agent who the wealth pass through threshold to be the rich at time *u*,the value of his idiosyncratic risky asset at time *t* .Agents have portfolio selection problem between a risky asset and a riskless asset. Consumer's problem

$$J(W,t) = \underset{C_t, B_t, S_t : t>0}{Max} E\left[\int_t^T e^{-\theta(\mu-t)} \frac{C(\mu)^{1-\gamma}}{1-\gamma} d\mu + e^{-\theta(\mu-t)} \frac{(1-\zeta)(B_T+S_T)^{1-\gamma}}{1-\gamma}\right] \tag{B1}$$

$$dW(\mu) = \left[rW(\mu) + (\alpha - r)S_\mu - C(\mu) + Y(\mu)\right]d\mu + \sigma S_\mu dZ_\mu \tag{B2}$$

Define

$$b(t) = \int_t^T Y(\mu) e^{-r(\mu-t)} d\,\mu$$

We know that

$$db(t) = [-Y(t) + rb(t)]dt$$

Thus

$$d(W(t) + b(t)) = \left(r + \frac{(\alpha - r)^2}{\gamma\sigma^2} - a(t)^{-\frac{1}{\gamma}}\right)\big((W(t) + b(t)\big)dt + \frac{\alpha - r}{\gamma\sigma}\big((W(t) + b(t)\big)dZ(t)$$

By Hamilton-Jacobi-Bellman approach obtain

$$\theta J(W,t) = \underset{C_t, B_t, S_t : t>0}{Max} \left\{ \frac{C(t)^{1-\gamma}}{1-\gamma} + J_w(W,t)[rW(t) + (\alpha - r)S(t) - C(t) + Y(t)] + \frac{1}{2} J_{ww}(W,t)\sigma^2 S(t)^2 + J_t(W,t) \right\} \tag{B3}$$

We have the F.O.C.

$$C(t)^{-\gamma} = J_w(W,t) \tag{B4}$$

$$J_w(W,t)(\alpha - r) = -J_{ww}(W,t)\sigma^2 S(t) \tag{B5}$$

Guess

$$J(W,t) = \frac{a(t)}{1-\gamma}[W(t) + b(t)]^{1-\gamma} \tag{B6}$$

Where

$$b(t) = \int_t^T Y(\mu) e^{-r(\mu-t)} d\mu$$

$$J_w(W,t) = a(t)[W(t) + b(t)]^{-\gamma} \tag{B7}$$

$$J_{ww}(W,t) = -\gamma a(t)[W(t)+b(t)]^{-\gamma-1} \tag{B8}$$

After arrangement, We have

$$C(t) = a(t)^{-\frac{1}{\gamma}}[W(t)+b(t)] \tag{B9}$$

$$S(t) = \frac{\alpha - r}{\gamma\sigma^2}[W(t)+b(t)] \tag{B10}$$

■

Appendix C.

One can also derive the formula for constant relative risk aversion utility (CRRA), the analytic form of $J(W,t;T)$ itself can be obtained as follows

$$J(W,t;T) = a(t;T)\,\frac{W(t)^{1-\gamma}}{1-\gamma}$$

Where $a(t;T)$ satisfies the following ordinary differential equation

$$\frac{1}{1-\gamma}\frac{a'}{a} + \frac{\gamma}{1-\gamma}\,e^{-\frac{\theta}{\gamma}t}a^{-\frac{1}{\gamma}} + r + \frac{(\alpha-r)^2}{2\gamma\sigma^2} = 0 \tag{C1}$$

Hereafter the prime symbol is used to denote the derivative with respect to time and solve the **Bernoulli's equation** form, analytical solution of equation (C1) with zero bequest at time *T* can be obtained as:

$$a(t;T) = e^{-\theta t}\left(\frac{e^{\eta(T-t)}-1}{\eta}\right)^{\gamma} \tag{C2}$$

Where

$$\eta = \frac{1-\gamma}{\gamma}\left[r + \frac{(\alpha-r)^2}{2\gamma\sigma^2}\right] - \frac{\theta}{\gamma} \tag{C3}$$

with the terminal condition:

$$a(T,T) = 1 \tag{C4}$$

We can derive a at each time t numerically by discretization $a_t = a_{t-1} + \Delta a_t$ and work backward from the terminal time. Optimal consumption contains a horizon dependent fraction of wealth, which is independent of wealth at hand:

$$c_t^* = a(t;T)^{\frac{-1}{\gamma}}\,W_t \tag{C5}$$

It can be easily shown that in the infinite horizon case, optimal consumption is a constant proportion of wealth:

$$c_t^* = \frac{1}{\gamma}\left[\theta - (1-\gamma)r - \frac{(1-\gamma)(\alpha-r)^2}{2\gamma\sigma^2}\right]W_t \tag{C6}$$

as the same given by Merton (1969). ■

Appendix D.

Furthermore, take into account taxes on rich condition, the investor makes contingent plans for a bequest χ , The consumption C is made through the money market account. The participant has a CRRA utility function over consumption and terminal wealth, and that maximize expected utility, Agent's problem

$$\max_{c(t),S(t)} \left\{ E_t \int_{\mathrm{t}}^{T} \frac{C(\mu)^{1-\gamma}}{1-\gamma} e^{-\Theta(\mu-t)} d\mu + \chi \frac{[(1-\zeta)W(T)]^{1-\gamma}}{1-\gamma} e^{-\Theta(T-t)} \right\} \quad \text{(D1)}$$

subject to a budget constraint

$$dW(\mu) = [(1-\tau)rw(\mu) + (\alpha - (1-\tau)r)S(\mu) - C(\mu) + Y(\mu)]d\mu + (1-\tau)\sigma W(\mu)dZ(t)$$

where τ is capital income tax rate. ζ is estate tax rate. The agent's human wealth

$$b(t) = \int_{\mathrm{t}}^{\mathrm{T}} Y(\mu) e^{-(1-\tau)r(\mu-t)} d\mu$$

Plugging these expressions into the HJB, we have the agent's policy functions after tax are

$$c(t) = a(t)^{-\frac{1}{\gamma}}[W(t) + b(t)] \quad \text{(D2)}$$

$$S(t) = \frac{(1-\tau)\alpha - (1-\tau)r}{\gamma\sigma^2(1-\tau)^2}[W(t) + b(t)] \quad \text{(D3)}$$

And

$$d[W(t) + b(t)] = \left[(1-\tau)r + \frac{(\alpha - r)^2}{\gamma\sigma^2} - a(t)^{-\frac{1}{\gamma}}\right][W(t) + b(t)]dt + \frac{\alpha - r}{\gamma\sigma}[W(t) + b(t)]dZ(t) \quad \text{(D4)}$$

From equation (E4), we know

$$dX(t) = \left[(1-\tau)r + \frac{(\alpha-r)^2}{\gamma\sigma^2} - a(t)^{-\frac{1}{\gamma}}\right] X(t)dt + \frac{\alpha - r}{\gamma\sigma} X(t)dZ(t) \quad \text{(D5)}$$

The end-of-life wealth post tax is

$$W(T) = X(T)$$

$$= (\chi(1-\zeta)^{1-\gamma})^{\frac{1}{\gamma}} a(0)^{-\frac{1}{\gamma}} exp\left(\frac{(1-\tau)r - \theta}{\gamma} + \frac{(\alpha-r)^2}{2\gamma\sigma^2}\right) T + \frac{\alpha - r}{\gamma\sigma} Z(T)X(0) \quad \text{(D6)} \ \blacksquare$$

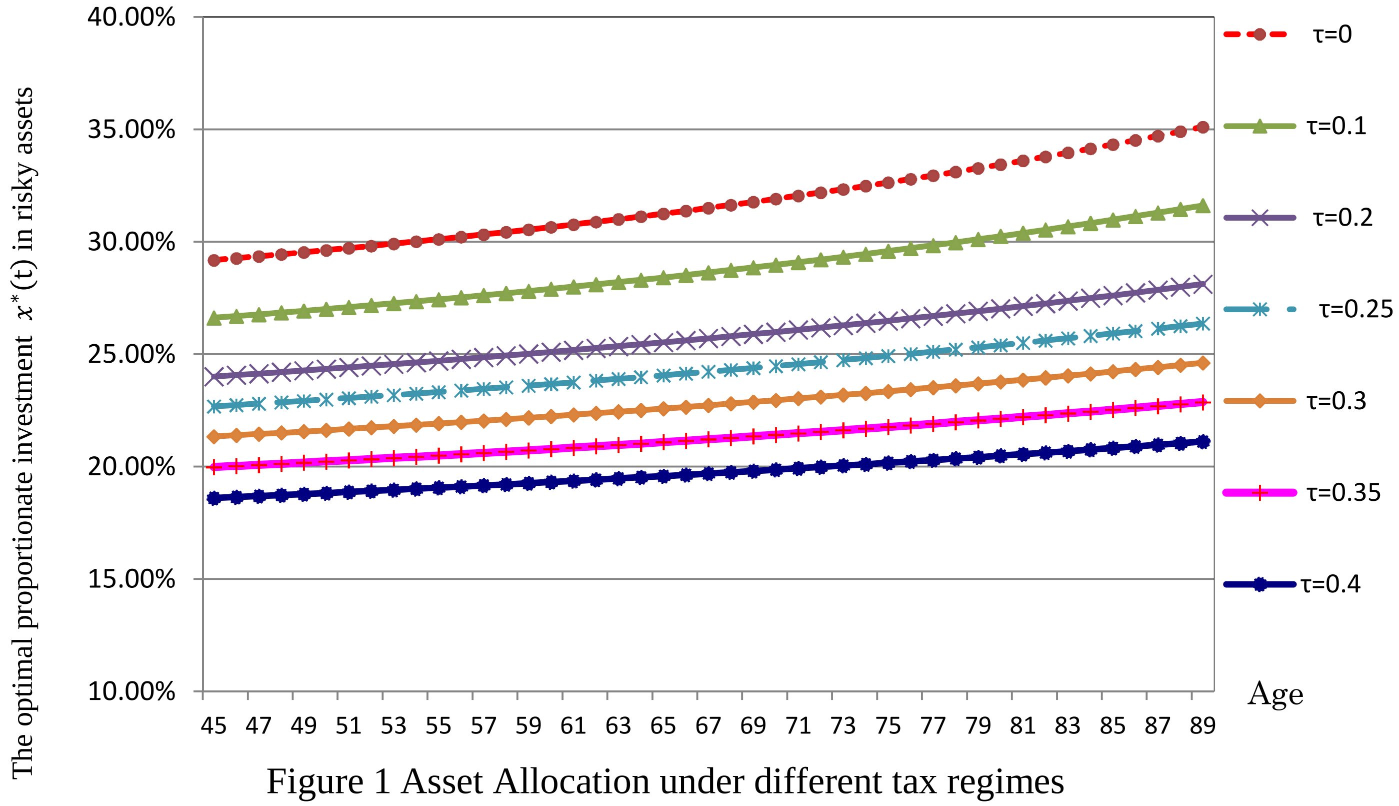

Figure 1 Asset Allocation under different tax regimes

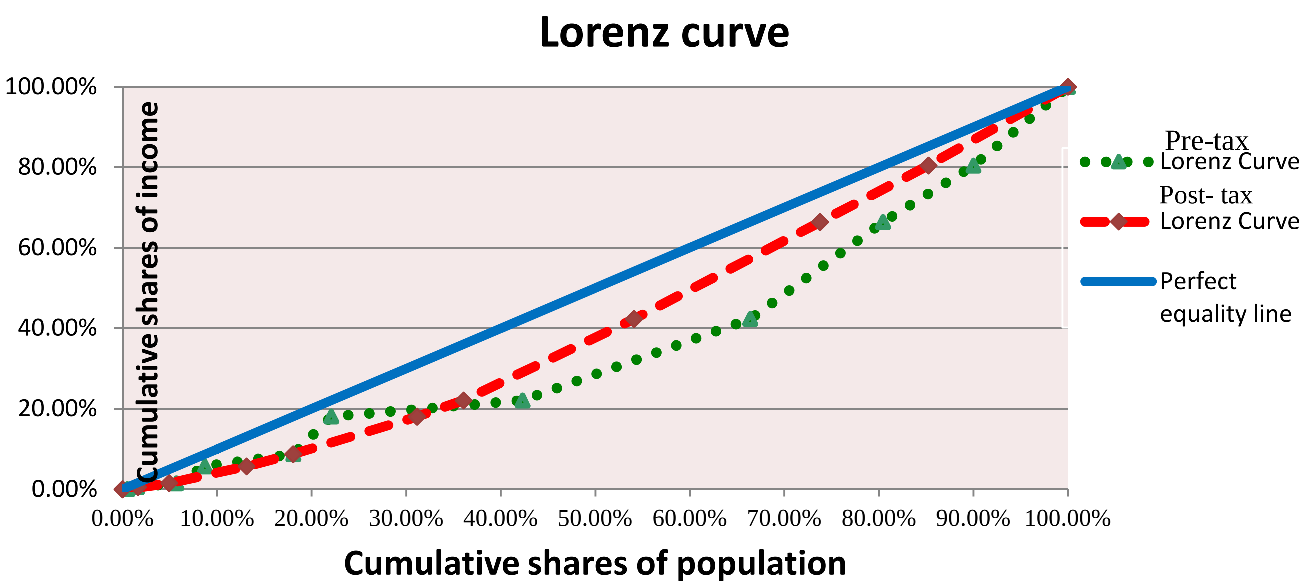

Figure 2: Compared with pre-tax and pro- tax Lorenz curve

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