A New Least-Squares Nonconforming Mixed Finite Element Analysis for Second Order Elliptic Problems

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Abstract In this paper a new least-squares nonconforming mixed finite element scheme for the second order elliptic problems is analyzed. The optimal order error estimates are obtained without requiring to LBB consistency condition under H^1 broken norm for primary solution and the flux solution. The superclose properties are presented by use of some typical characters of the elements and the mean value approach. In addition, the global superconvergence result is derived based on the interpolated postprocessing technique.
Keywords New least-squares scheme; nonconforming finite element; optimal error estimates; superclose and superconvergence

1 Introduction

Standard mixed finite element methods(MFEMs for short) for the Galerkin formulation have been developed in the last decades(see [3], [4], [18]). The main advantage of the above methods is that they can explicitly involve the derived flux as an independent variable, hence accurate nodal fluxes are obtained directly from the discretized mixed system, rather than by postprocessing in the traditional finite element schemes. But, to guaranttee stability the finite element spaces are required to satisfy the so-called LBB consistency condition, which makes some of the best known finite elements excluded.

Recently, more attention has been paid to the least-squares MFEMs since they are not subject to the above LBB consistency condition. For example, [1],[4],[15] and [16] considered the convergence analysis of conforming finite element methods. [10] studied the nonconforming Crouzeix-Raviart type linear triangular element[11] and the rotated $\mathcal{Q}_1$-element[17], and only obtained the convergence results.

On the other hand, the superconvergence of the MFEMs is one of the most active research subject

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for a long time. Various superconvergence results(see [9],[11]) have been established for the mixed finite element approximation on regular rectangular elements for second order elliptic problems. [5] and [6] studied the superclose properties with least-squares MFEMs by the kth order R-T elements[18] and BDFM elements[2](k≥ 1).

The main aim of this paper is to present a new least-squares nonconforming MFEM scheme for second order elliptic problems by a nonconforming element proposed in [13] combining with the lowest order R-T element. The optimal order error estimates are obtained without requiring to LBB consistency condition under H^1 broken norm for primary solution and the flux solution. At the same time, by means of some special tricks, such as the orthogonal property, mean value approach and the unconventional boundary estimation, the superclose and superconvergence results are yielded, which results in optimal order error estimate under L^2 norm.

This paper is organized in the following way. In section 2, we recall the least-squares MFEMs for second order elliptic problems. A new least-squares nonconforming MFEMs scheme is constructed, and the existence and uniqueness of solution for this scheme and convergence results are proved in Section 3. In last section, we state the superclose and superconvergence results of the present work.

Throughout this paper, C is a generic constant independent of discretization parameters which may take different values at different occurrences.

2 The Least-Squares Mixed Finite Element Scheme

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary $\Gamma = \partial\Omega$. Given a subdomain $e \subset \Omega$ with Lipschitz boundary ∂e, we introduce $L^2(e)$, and $(L^2(e))^2$, with inner product $(\cdot,\cdot)_{0,e}$ and norm $\|\cdot\|_{0,e}$, and introduce L^2-based Sobolev spaces $H^m(e)$ and $(H^m(e))^2$, with norm $\|\cdot\|_{m,e}$ and semi-norm $|\cdot|_{m,e}$($m \geq 1$ is an integer). In addition,we introduce $H_0^1(e) = \{v \in H^1(e); v|_{\partial e} = 0\}$ with norm $|\cdot|_{1,e}$, and $H(\mathrm{div}, e) = \{\boldsymbol{q} \in (L^2(e))^2; \mathrm{div}\boldsymbol{q} \in L^2(e)\}$ equipped with the norm

$$\|\boldsymbol{q}\|_{H(\mathrm{div},e)} = \|\boldsymbol{q}\|_{0,e} + \|\mathrm{div}\boldsymbol{q}\|_{0,e}.$$

In the case $e = \Omega$, we simplify the notation as follows: $|\cdot|_{m,e} \equiv |\cdot|_m, \|\cdot\|_{m,e} \equiv \|\cdot\|_m (m \geq 1), (\cdot,\cdot)_{0,e} \equiv (\cdot,\cdot), \|\cdot\|_{0,e} \equiv \|\cdot\|_0$.

Let $\Gamma = \Gamma_D \cup \Gamma_N$ with $\Gamma_D \cap \Gamma_N = \emptyset$, and let $\boldsymbol{n}$ be the unit outward normal vector to Γ. We additional introduce

$$H_{0,D}^1(\Omega) = \{v \in H^1(\Omega); v = 0, \text{ on } \Gamma_D\},$$
$$H_{0,N}(\mathrm{div};\Omega) = \{\boldsymbol{q} \in H(\mathrm{div};\Omega); \boldsymbol{q}\cdot\boldsymbol{n} = 0, \text{ on } \Gamma_N\}.$$

We consider the following second order elliptic problem:

$$\begin{cases} -\mathrm{div}(A\nabla u) + cu = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_D, \\ (-A\nabla u)\cdot\boldsymbol{n} = 0 & \text{on } \Gamma_N, \end{cases} \tag{2.1}$$

where $f \in L^2(\Omega)$, $A = (a_{ij}(x)) \in \mathbb{R}^{2\times 2}$ is sufficiently smooth, symmetric positive definite and the coefficients $(a_{ij}(x))$ are bounded, i.e., there exist positive constants α_1 and α_2 such that

$$\alpha_1\xi^T\xi \le \xi^T A\xi \le \alpha_2\xi^T\xi, \tag{2.2}$$

for all vectors $\xi \in \mathbb{R}^2$ and $x \in \overline{\Omega}$. Similarly, $c = c(x)$ is nonnegative and bounded with

$$0 \le c(x) \le C, \quad \forall x \in \overline{\Omega}. \tag{2.3}$$

Now we let $\boldsymbol{p} = -A\nabla u$, and rewrite (2.1) as

$$\begin{cases} \boldsymbol{p} + A\nabla u = 0 & \text{in } \Omega, \\ \text{div}\boldsymbol{p} + cu - f = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_D, \\ \boldsymbol{p}\cdot\boldsymbol{n} = 0 & \text{on } \Gamma_N. \end{cases} \tag{2.4}$$

The least-squares minimization problem is to find $u \in U = H^1_{0,D}(\Omega)$ and $\boldsymbol{p} \in \boldsymbol{X} = H_{0,N}(\text{div};\Omega)$ such that

$$\boldsymbol{J}(u,\boldsymbol{p}) = \inf_{v\in U, \boldsymbol{q}\in\boldsymbol{X}} \boldsymbol{J}(v,\boldsymbol{q}),$$

where

$$\boldsymbol{J}(v,\boldsymbol{q}) = (\text{div}\boldsymbol{q} + cv - f, \text{div}\boldsymbol{q} + cv - f) + (\boldsymbol{q} + A\nabla v, \boldsymbol{q} + A\nabla v). \tag{2.5}$$

The corresponding variational problem is to find $u \in U = H^1_{0,D}(\Omega)$ and $\boldsymbol{p} \in \boldsymbol{X} = H_{0,N}(\text{div};\Omega)$ such that

$$a(u,\boldsymbol{p};v,\boldsymbol{q}) = (f, \text{div}\boldsymbol{q} + cv) \quad \forall v \in U, \quad \boldsymbol{q} \in \boldsymbol{X}, \tag{2.6}$$

where

$$a(u,\boldsymbol{p};v,\boldsymbol{q}) = (\text{div}\boldsymbol{p} + cu, \text{div}\boldsymbol{q} + cv) + (\boldsymbol{p} + A\nabla u, \boldsymbol{q} + A\nabla v). \tag{2.7}$$

The following lemma regarding the coerciveness of the bilinear form $a(\cdot;\cdot)$ was proved in [15].

Lemma 2.1. There exists a constant $C > 0$ such that for all $v \in U, \boldsymbol{q} \in \boldsymbol{X}$,

$$a(v,\boldsymbol{q};v,\boldsymbol{q}) \ge C(\|v\|_1^2 + \|\boldsymbol{q}\|^2_{H(\text{div};\Omega)}). \tag{2.8}$$

Thus, Lax-Milgram lemma guarantees that problem (2.6) has a unique solution $(u,\boldsymbol{p}) \in U \times \boldsymbol{X}$.

3 Construction of the Finite Element and Convergence Result

For convenience, let $\Omega \subset \mathbb{R}^2$ be a polygon on (x_1,x_2) plane with boundaries parallel to the axes, $\mathbf{T}_h$ be an axis-parallel rectangular subdivision of Ω. For a given $K \in \mathbf{T}_h$, let $K = [x_{1K} - h_{x_{1K}}, x_{1K} + h_{x_{1K}}] \times [x_{2K} - h_{x_{2K}}, x_{2K} + h_{x_{2K}}]$, the four vertices of K are $d_1 = (x_{1K} - h_{x_{1K}}, x_{2K} - h_{x_{2K}})$, $d_2 = (x_{1K} + h_{x_{1K}}, x_{2K} - h_{x_{2K}})$, $d_3 = (x_{1K} + h_{x_{1K}}, x_{2K} + h_{x_{2K}})$ and $d_4 = (x_{1K} - h_{x_{1K}}, x_{2K} + h_{x_{2K}})$, and the four edges are $l_i = \overline{d_i d_{i+1}} \pmod 4$. Let $\hat{K} = [-1,1] \times [-1,1]$ be the reference element on (ξ,η) plane, the four vertices of $\hat{K}$ are $\hat{d}_1 = (-1,-1)$, $\hat{d}_2 = (1,-1)$, $\hat{d}_3 = (1,1)$ and $\hat{d}_4 = (-1,1)$, and the four edges are $\hat{l}_i = \overline{\hat{d}_i\hat{d}_{i+1}} \pmod 4$, and and let $\boldsymbol{n}_i$ be the unit outward normal vector to l_i,$(i = 1,2,3,4)$.

For all $\hat{v} \in H^1(\hat{K})$, we define the finite element $(\hat{K}, \hat{P}, \hat{\Sigma})$ on $\hat{K}$ as follows[13]

$$\hat{\Sigma} = \{\hat{v}^1, \hat{v}^2, \hat{v}^3, \hat{v}^4, \hat{v}^5\}, \ \hat{P} = \text{span}\{1, \xi, \eta, \varphi(\xi), \varphi(\eta)\}, \tag{3.1}$$

where $\hat{v}^i = \dfrac{1}{|\hat{l}_i|}\displaystyle\int_{\hat{l}_i} \hat{v}d\hat{s}$, $i = 1,2,3,4$, $\hat{v}^5 = \dfrac{1}{|\hat{K}|}\displaystyle\int_{\hat{K}} \hat{v}d\xi d\eta$, $\varphi(t) = t^2$.

It can be easily checked that interpolation defined above is well-posed and the interpolation function $\hat{I}\hat{v}$ can be expressed by

$$\hat{I}\hat{v} = \hat{v}^5 + \frac{1}{2}(\hat{v}^2 - \hat{v}^4)\xi + \frac{1}{2}(\hat{v}^3 - \hat{v}^1)\eta + \frac{1}{2}(\hat{v}^2 + \hat{v}^4 - 2\hat{v}^5)\varphi(\xi) + \frac{1}{2}(\hat{v}^3 + \hat{v}^1 - 2\hat{v}^5)\varphi(\eta). \tag{3.2}$$

Then we define the associated finite element space U_h as

$$U_h = \Big\{v;\ \hat{v}|_{\hat{K}} = v|_K \circ F_K \in \hat{P}, \forall K \in \mathbf{T}_h, \int_F [v]ds = 0,\ F \subset \partial K\Big\}, \tag{3.3}$$

where $[v]$ stands for the jump of v across the edge F if F is an internal edge and it is equal to v itself if $F \subset \Gamma_D$.

For a vector function $\hat{\boldsymbol{q}}$, we use Piola transformation: $\boldsymbol{q} = (\det(B_K))^{-1}B_K\hat{\boldsymbol{q}} \circ F_K^{-1}$, and let $\boldsymbol{X}_h$ be the Raviart-Thomas space of the lowest order[18], i.e.,

$$\begin{aligned} \boldsymbol{X}_h = \{\boldsymbol{q}_h \in \boldsymbol{X}; \boldsymbol{q}_h|_K = (\det(B_K))^{-1}B_K\hat{\boldsymbol{q}} \circ F_K^{-1}, \\ \hat{\boldsymbol{q}} \in Q_{1,0}(\hat{K}) \times Q_{0,1}(\hat{K}),\ \ \forall K \in \mathbf{T}_h,\ \boldsymbol{q}\cdot\boldsymbol{n} \text{ is continuous across } F,\ F \subset \partial K\}. \end{aligned} \tag{3.4}$$

Where $Q_{m,n}(\hat{K}) = P_m(\hat{K}) \times P_n(\hat{K})$. Here and later, $P_m(E)$ denotes the polynomial space in E with the degree no more than m. Obviously, $U_h \not\subset U,\ \boldsymbol{X}_h \subset \boldsymbol{X}$. So this is a nonconforming mixed finite element scheme.

It is easy to see that $\|\cdot\|_h = \Big(\sum_{K\in\mathbf{T}_h} |\cdot|_{1,K}^2\Big)^{\frac{1}{2}}$ is a norm over U_h.

For $v \in H^1(\Omega)$, let I_h be the associated interpolation operator on U_h satisfying $I_K = \hat{I} \circ F_K^{-1}$, $I_h\,|_K = I_K$, then we have

$$\begin{cases} \displaystyle\int_{l_i}(v - I_hv)ds = 0, \quad i = 1,2,3,4, \\ \displaystyle\int_K (v - I_hv)dx = 0, \end{cases} \tag{3.5}$$

here and later $dx = dx_1dx_2$.

The Raviart-Thomas interpolation on $\hat{K}$ is defined as

$$\begin{aligned} &\hat{\mathbf{\Pi}} : (H^1(\hat{K}))^2 \to Q_{1,0}(\hat{K}) \times Q_{0,1}(\hat{K}), \\ &\int_{\hat{l}_i}(\hat{\boldsymbol{q}} - \hat{\mathbf{\Pi}}\hat{\boldsymbol{q}})\cdot\hat{\boldsymbol{n}}_i d\hat{s} = 0, \quad i = 1,2,3,4. \end{aligned} \tag{3.6}$$

Then the associated finite element interpolation is

$$\begin{aligned} \mathbf{\Pi}_h : (H^1(\Omega))^2 \to \boldsymbol{X}_h, \\ \mathbf{\Pi}_h\boldsymbol{q}|_K = \hat{\mathbf{\Pi}}\hat{\boldsymbol{q}} \circ F_K^{-1}B_K^T(\det B_K)^{-1}, \end{aligned}$$

satisfying

$$\int_{l_i}(\boldsymbol{q} - \mathbf{\Pi}_h\boldsymbol{q})\cdot\boldsymbol{n}_i ds = 0, \quad i = 1,2,3,4. \tag{3.7}$$

Now we state prove the following important properties.

Lemma 3.1. Suppose $u \in H^2(\Omega),\ \boldsymbol{p} \in (H^2(\Omega))^2$, then, for anisotropic meshes, there hold

$$(\nabla_h(u - I_hu), \nabla_hv_h) = 0 \ \ \forall v_h \in U_h, \tag{3.8}$$

$$\|u - I_hu\|_0 \le Ch^2|u|_2, \quad \|u - I_hu\|_h \le Ch|u|_2, \tag{3.9}$$

$$(\mathrm{div}(\boldsymbol{p} - \mathbf{\Pi}_h\boldsymbol{p}), \mathrm{div}\boldsymbol{q}_h) = 0 \quad \forall \boldsymbol{q}_h \in \boldsymbol{X}_h, \tag{3.10}$$

$$\|\mathrm{div}(\boldsymbol{p} - \mathbf{\Pi}_h\boldsymbol{p})\|_0 \le Ch\|\boldsymbol{p}\|_2, \tag{3.11}$$

where ∇_h denotes the gradient operator element-by-element.

Proof. (3.8) and (3.9) have been proved in [13], (3.10) can be shown easily by use of the definition of $\mathbf{\Pi_h}$ and Green's formula, and (3.11) can be proved by noting that $\mathrm{div}\mathbf{\Pi_h}\boldsymbol{p} = \mathrm{P}_0\mathrm{div}\boldsymbol{p}$, where P_0 is the local L^2 projection, and L^2-norm estimate has nothing to do with the geometric condition of meshes. The proof is completed. □

Consider the corresponding finite element approximation to (2.6): to find $u_h \in U_h, \boldsymbol{q}_h \in \boldsymbol{X}_h$ such that

$$a_h(u_h, \boldsymbol{p}_h; v_h, \boldsymbol{q}_h) = (f, \mathrm{div}\boldsymbol{q}_h + cv_h) \quad \forall v_h \in U_h, \quad \boldsymbol{q}_h \in \boldsymbol{X}_h, \tag{3.12}$$

where

$$a_h(u_h, \boldsymbol{p}_h; v_h, \boldsymbol{q}_h) := (\mathrm{div}\boldsymbol{p}_h + cu_h, \mathrm{div}\boldsymbol{q}_h + cv_h) + (\boldsymbol{p}_h + A\nabla_h u_h, \boldsymbol{q}_h + A\nabla_h v_h), \tag{3.13}$$

The following orthogonality is available.

Lemma 3.2. Let $u \in U, \boldsymbol{p} \in \boldsymbol{X}$ and $u_h \in U_h, \boldsymbol{p}_h \in \boldsymbol{X}_h$ be the solutions of (2.6) and (3.12), respectively, then

$$a_h(u - u_h, \boldsymbol{p} - \boldsymbol{p}_h; v_h, \boldsymbol{q}_h) = 0 \quad \forall v_h \in U_h,\ \boldsymbol{q}_h \in \boldsymbol{X}_h. \tag{3.14}$$

Proof. From (2.6), (2.7), (3.12) and (3.13), for all $v_h \in U_h,\ \boldsymbol{q}_h \in \boldsymbol{X}_h$,

$$\begin{aligned} a_h(u, \boldsymbol{p}; v_h, \boldsymbol{q}_h) &= (\mathrm{div}\boldsymbol{p} + cu, \mathrm{div}\boldsymbol{q}_h + cv_h) + (\boldsymbol{p} + A\nabla u, \boldsymbol{q}_h + A\nabla_h v_h) \\ &= (f, \mathrm{div}\boldsymbol{q}_h + cv_h) \\ &= a_h(u_h, \boldsymbol{p}_h; v_h, \boldsymbol{q}_h). \end{aligned}$$

Which completes the proof. □

Lemma 3.3. For all $v_h \in U_h$ and $\boldsymbol{q}_h \in \boldsymbol{X}_h$, we have

$$\sum_{K\in\mathbf{T}_h} \int_{\partial K} \boldsymbol{q}_h \cdot \boldsymbol{n}_K v_h ds = 0, \tag{3.15}$$

$$(v_h, \mathrm{div}\boldsymbol{q}_h) = -(\nabla_h v_h, \boldsymbol{q}_h), \tag{3.16}$$

where $\boldsymbol{n}_K$ is the unit outward normal vector to the boundary ∂K of K.

Proof. From the constructions of spaces U_h and $\boldsymbol{X}_h$, we know that $\boldsymbol{q}_h|_{\partial K}$ is a constant and $\int_{\partial K}[v_h]ds = 0$. So, (3.15) holds. As to (3.16), it follows from the Green's directly.

The proof is completed. □

Remark 3.1 One can check that the above lemma is also valid for CNQ_1^{rot} element[12].

Furthermore, [7] has proved the following discrete Poincaré inequality for U_h: for all $v_h \in U_h$, there holds

$$\|v_h\|_0 \le C\|v_h\|_h. \tag{3.17}$$

Theorem 3.1. The bilinear form $a_h(\cdot;\cdot)$ is coercive, i.e., there exists a constant $C > 0$, such that for all $v_h \in U_h, \boldsymbol{q}_h \in \boldsymbol{X}_h$,

$$a_h(v_h, \boldsymbol{q}_h; v_h, \boldsymbol{q}_h) \ge C\left(\|v_h\|_h^2 + \|\boldsymbol{q}_h\|_{H(\mathrm{div};\Omega)}^2\right). \tag{3.18}$$

Proof. From (3.13), we have

$$\begin{aligned} a_h(v_h, \boldsymbol{q}_h; v_h, \boldsymbol{q}_h) &= (\mathrm{div}\boldsymbol{q}_h + cv_h, \mathrm{div}\boldsymbol{q}_h + cv_h) + (\boldsymbol{q}_h + A\nabla_h v_h, \boldsymbol{q}_h + A\nabla_h v_h) \\ &= (\mathrm{div}\boldsymbol{q}_h, \mathrm{div}\boldsymbol{q}_h) + 2(\mathrm{div}\boldsymbol{q}_h, cv_h) + c^2(v_h, v_h) \\ &\quad + (\boldsymbol{q}_h, \boldsymbol{q}_h) + 2(\boldsymbol{q}_h, A\nabla_h v_h) + (A\nabla_h v_h, A\nabla_h v_h) \\ &\quad + 2\beta(\boldsymbol{q}_h, \nabla_h v_h) - 2\beta(\boldsymbol{q}_h, \nabla_h v_h), \end{aligned}$$

where $\beta > 0$ is a constant.

Employing (3.16) yields

$$\begin{aligned} a_h(v_h, \boldsymbol{q}_h; v_h, \boldsymbol{q}_h) &= (\mathrm{div}\boldsymbol{q}_h, \mathrm{div}\boldsymbol{q}_h) - 2(\beta - c)(v_h, \mathrm{div}\boldsymbol{q}_h) + (\beta - c)^2(v_h, v_h) \\ &\quad + (\boldsymbol{q}_h, \boldsymbol{q}_h) + 2(\boldsymbol{q}_h, (A - \beta E)\nabla_h v_h) + ((A - \beta E)\nabla_h v_h, (A - \beta E)\nabla_h v_h) \\ &\quad - \beta(\beta - 2c)(v_h, v_h) + 2\beta(A\nabla_h v_h, \nabla_h v_h) - \beta^2(\nabla_h v_h, \nabla_h v_h) \\ &= (\mathrm{div}\boldsymbol{q}_h - (\beta - c)v_h, \mathrm{div}\boldsymbol{q}_h - (\beta - c)v_h) \\ &\quad + (\boldsymbol{q}_h + (A - \beta E)\nabla_h v_h, \boldsymbol{q}_h + (A - \beta E)\nabla_h v_h) \\ &\quad - \beta(\beta - 2c)(v_h, v_h) + 2\beta(A\nabla_h v_h, \nabla_h v_h) - \beta^2(\nabla_h v_h, \nabla_h v_h) \\ &\geq -\beta(\beta - 2c)(v_h, v_h) + 2\beta(A\nabla_h v_h, \nabla_h v_h) - \beta^2(\nabla_h v_h, \nabla_h v_h), \end{aligned} \tag{3.19}$$

where E is the identity matrix.

Using (2.2), we get

$$2\beta(A\nabla_h v_h, \nabla_h v_h) \geq 2\beta\alpha_1(\nabla_h v_h, \nabla_h v_h). \tag{3.20}$$

Since $c \geq 0$, we have from (3.17)

$$-\beta(\beta - 2c)(v_h, v_h) \geq -\beta^2 C^2(\nabla_h v_h, \nabla_h v_h). \tag{3.21}$$

Combining (3.19)-(3.21),

$$a_h(v_h, \boldsymbol{q}_h; v_h, \boldsymbol{q}_h) \geq \beta(2\alpha_1 - \beta(C^2 + 1))(\nabla_h v_h, \nabla_h v_h). \tag{3.22}$$

Setting $\beta = \alpha_1/(1 + C^2)$, we deduce that

$$\beta(2\alpha_1 - \beta(C^2 + 1)) > 0.$$

Hence

$$a_h(v_h, \boldsymbol{q}_h; v_h, \boldsymbol{q}_h) \geq C(\nabla_h v_h, \nabla_h v_h) = C\|v_h\|_h^2. \tag{3.23}$$

Using (3.23) and the triangle inequality, we can obtain the desired result. □

Theorem 3.1 guarantees that problem (3.12) has a unique solution $(u_h, \boldsymbol{p}_h) \in U_h \times \boldsymbol{X}_h$.

Theorem 3.2. Let $(u, \boldsymbol{p}) \in U \times \boldsymbol{X}$ and $(u_h, \boldsymbol{p}_h) \in U_h \times \boldsymbol{X}_h$ be the solutions of (2.6) and (3.12), respectively, $u \in H^2(\Omega)$, $\boldsymbol{p} \in (H^2(\Omega))^2$, then for anisotropic meshes we have

$$\|u - u_h\|_h + \|\boldsymbol{p} - \boldsymbol{p}_h\|_{H(\mathrm{div},\Omega)} \leq Ch(\|u\|_2 + \|\boldsymbol{p}\|_2). \tag{3.24}$$

Proof. From (3.14) and (3.18), we get

$$\begin{aligned}
&\|u_h - I_h u\|_h^2 + \|\boldsymbol{p}_h - \boldsymbol{\Pi_h p}\|_{H(\mathrm{div},\Omega)}^2 \\
&\le C a_h(u_h - I_h u, \boldsymbol{p}_h - \boldsymbol{\Pi_h p}; u_h - I_h u, \boldsymbol{p}_h - \boldsymbol{\Pi_h p}) \\
&= C a_h(u - I_h u, \boldsymbol{p} - \boldsymbol{\Pi_h p}; u_h - I_h u, \boldsymbol{p}_h - \boldsymbol{\Pi_h p}) \\
&= C((\mathrm{div}(\boldsymbol{p} - \boldsymbol{\Pi_h p}), \mathrm{div}(\boldsymbol{p}_h - \boldsymbol{\Pi_h p})) + (c(u - I_h u), \mathrm{div}(\boldsymbol{p}_h - \boldsymbol{\Pi_h p})) \\
&\quad +(\mathrm{div}(\boldsymbol{p} - \boldsymbol{\Pi_h p}), c(u_h - I_h u)) + (c(u - I_h u), c(u_h - I_h u)) \\
&\quad +(\boldsymbol{p} - \boldsymbol{\Pi_h p}, \boldsymbol{p}_h - \boldsymbol{\Pi_h p}) + (\boldsymbol{p} - \boldsymbol{\Pi_h p}, A\nabla_h(u_h - I_h u)) \\
&\quad +(A\nabla_h(u - I_h u), \boldsymbol{p}_h - \boldsymbol{\Pi_h p}) + (A\nabla_h(u - I_h u), A\nabla_h(u_h - I_h u))) \\
&\le C(\|u - I_h u\|_h + \|\boldsymbol{p} - \boldsymbol{\Pi_h p}\|_{H(\mathrm{div},\Omega)})(\|u_h - I_h u\|_h + \|\boldsymbol{p}_h - \boldsymbol{\Pi_h p}\|_{H(\mathrm{div},\Omega)}). \qquad (3.25)
\end{aligned}$$

Hence,

$$\|u_h - I_h u\|_h + \|\boldsymbol{p}_h - \boldsymbol{\Pi_h p}\|_{H(\mathrm{div},\Omega)} \le C(\|u - I_h u\|_h + \|\boldsymbol{p} - \boldsymbol{\Pi_h p}\|_{H(\mathrm{div},\Omega)}).$$

By using (3.9), (3.11) and the triangle inequality, we have

$$\begin{aligned}
&\|u - u_h\|_h + \|\boldsymbol{p} - \boldsymbol{p}_h\|_{H(\mathrm{div},\Omega)} \\
&\le C(\|u - I_h u\|_h + \|\boldsymbol{p} - \boldsymbol{\Pi_h p}\|_{H(\mathrm{div},\Omega)}) \\
&\le Ch(\|u\|_2 + \|\boldsymbol{p}\|_2).
\end{aligned}$$

Which is the desired result. □

Remark 3.2. The patch test (3.15) is indispensable to prove the above Theorem 3.2. As far as we know, for many famous elements used in MFEMs, such as R-T elements[18] and BDFM elements[2], only the lowest order R-T element satisfies (3.15). This means that it is not easy to construct a suitable scheme to solve this kind problem for nonconforming finite elements.

4 Superclose and Superconvergence Results

In order to give our the main results, we need the following two lemmas.

Lemma 4.1.[13] Suppose $\boldsymbol{p} \in (H^2(\Omega))^2$, then there holds

$$\sum_{K\in\mathbf{T}_h} \int_{\partial K} \boldsymbol{p} v_h \cdot \boldsymbol{n} ds \le Ch^2 |\boldsymbol{p}|_2 \|v_h\|_h. \qquad (4.1)$$

Lemma 4.2. Assume $\boldsymbol{p}_h \in (H^2(\Omega))^2$, then for all $v_h \in U_h, \boldsymbol{q}_h \in \boldsymbol{X}_h$, we have

$$|(\boldsymbol{p} - \boldsymbol{\Pi_h p}, \nabla_h v_h)| \le Ch^2 \|\boldsymbol{p}\|_2 \|v_h\|_h, \qquad (4.2)$$

$$|(\boldsymbol{p} - \boldsymbol{\Pi_h p}, \boldsymbol{q}_h)| \le Ch^2 \|\boldsymbol{p}\|_2 \|\boldsymbol{q}_h\|_{0,K}. \qquad (4.3)$$

$$|(\mathrm{div}(\boldsymbol{p} - \boldsymbol{\Pi_h p}), v_h)| \le Ch^2 \|\boldsymbol{p}\|_2 \|v_h\|_h, \qquad (4.4)$$

Proof. Firstly, we will prove (4.2) through the Bramble-Hilbert lemma[14].
Consider the functional

$$B(\hat{p}_1, \hat{v}_h) = \int_{\hat{K}} (\hat{p}_1 - \hat{\mathbf{\Pi}}\hat{p}_1)\hat{v}_{h\xi}.$$

It is easy to know

$$|B(\hat{p}_1, \hat{v}_h)| \le C\|\hat{p}_1\|_{2,\hat{K}} |\hat{v}_{h\xi}|_{0,\hat{K}}, \forall v_h \in U_h.$$

A direct computation shows that

$$B(\hat{p_1}, \hat{v}_h) = 0, \quad \forall \hat{p}_1 \in P_1(\hat{K}),\ v_h \in U_h.$$

According to Bramble-Hilbert lemma, we have

$$|B(\hat{p}_1, \hat{v}_h)| \le C|\hat{p}_1|_{2,\hat{K}}|\hat{v}_{h\xi}|_{0,\hat{K}}.$$

A scaling argument gives

$$|B(p_1, v_h)| \le Ch^2|p_1|_{2,K}|v_{hx_1}|_{0,K}, \forall v_h \in U_h. \tag{4.5}$$

Similarly, we have

$$|B(p_2, v_h)| \le Ch^2|p_2|_{2,K}|v_{hx_2}|_{0,K}, \forall v_h \in U_h. \tag{4.6}$$

Combining (4.5), (4.6) and summing all elements K, we get (4.2).
Similarly, we can derive (4.3).

Next, we prove (4.4). Applying Green's formula, we have

$$(\mathrm{div}(\boldsymbol{p} - \boldsymbol{\Pi_h p}), v_h) = -(\boldsymbol{p} - \boldsymbol{\Pi_h p}, \nabla_h v_h) + \sum_{K\in \mathbf{T}_h} \int_{\partial K} (\boldsymbol{p} - \boldsymbol{\Pi_h p}) v_h \cdot \boldsymbol{n} ds. \tag{4.7}$$

Using (4.1), we get

$$\sum_{K\in \mathbf{T}_h} \int_{\partial K} (\boldsymbol{p} - \boldsymbol{\Pi_h p}) v_h \cdot \boldsymbol{n} ds \le Ch^2 |\boldsymbol{p} - \boldsymbol{\Pi_h p}|_2 \|v_h\|_h = Ch^2|\boldsymbol{p}|_2\|v_h\|_h. \tag{4.8}$$

Thus (4.4) follows from the combination of (4.2), (4.7) and (4.8). The proof is completed. □

Now, we are ready to state the following superclose result.

Theorem 4.1. Under the assumptions of Theorem 3.2, and further assume $A, c \in W^{1,\infty}(\Omega)$, then there holds the following superclose property

$$\|u_h - I_h u\|_h + \|\boldsymbol{p}_h - \boldsymbol{\Pi_h p}\|_{H(\mathrm{div},\Omega)} \le Ch^2(\|u\|_2 + \|\boldsymbol{p}\|_2), \tag{4.9}$$

Proof. Let $\xi = u_h - I_h u, \theta = \boldsymbol{p}_h - \boldsymbol{\Pi_h p}$. it is easy to see from (3.14) and (3.18)

$$\begin{aligned}
&\|\xi\|_h^2 + \|\theta\|^2_{H(\mathrm{div},\Omega)} \le Ca_h(\xi,\theta;\xi,\theta) = Ca_h(u - I_h u, \boldsymbol{p} - \boldsymbol{\Pi_h p}; \xi, \theta)\\
&= C((\mathrm{div}(\boldsymbol{p} - \boldsymbol{\Pi_h p}), \mathrm{div}\theta) + (c(u - I_h u), \mathrm{div}\theta)\\
&\quad +(\mathrm{div}(\boldsymbol{p} - \boldsymbol{\Pi_h p}), c\xi) + (c(u - I_h u), c\xi)\\
&\quad +(\boldsymbol{p} - \boldsymbol{\Pi_h p}, \theta) + (\boldsymbol{p} - \boldsymbol{\Pi_h p}, A\nabla_h \xi)\\
&\quad +(A\nabla_h(u - I_h u), \theta) + (A\nabla_h(u - I_h u), A\nabla_h \xi)\\
&= \sum_{i=1}^{8} I_i.
\end{aligned} \tag{4.10}$$

Now, we start to estimate terms in (4.10) one by one.

Noticing (3.10), we have $I_1 = 0$.

Using (2.3) and (3.9), I_2 and I_4 can be estimated respectively as

$$I_2 \le Ch^2\|u\|_2\|\mathrm{div}\theta\|_0, I_4 \le Ch^2\|u\|_2\|\xi\|_h. \tag{4.11}$$

For $\varphi \in W^{1,\infty}(\Omega)$, we define $\bar{\varphi}|_K = \frac{1}{|K|}\int_K \varphi dx$, then $|\varphi - \bar{\varphi}| \le Ch\|\varphi\|_{1,\infty,\Omega}$.

By (3.11),(3.17) and (4.4), I_3 can be estimated as

$$\begin{aligned} I_3 &= \int_\Omega c\mathrm{div}(\boldsymbol{p}-\boldsymbol{\Pi_h p})\xi dx \\ &= \int_\Omega (c-\bar{c})(\mathrm{div}(\boldsymbol{p}-\boldsymbol{\Pi_h p})\xi dx + \sum_{K\in\mathbf{T}_h}\bar{c}\int_K \mathrm{div}(\boldsymbol{p}-\boldsymbol{\Pi_h p})\xi dx \\ &\leq Ch^2\|\boldsymbol{p}\|_2\|\xi\|_h. \end{aligned} \tag{4.12}$$

Similarly, by (4.2) and (4.3), we get

$$I_5 \leq Ch^2\|\boldsymbol{p}\|_2\|\theta\|_0, I_6 \leq Ch^2\|\boldsymbol{p}\|_2\|\xi\|_h. \tag{4.13}$$

As to I_7 and I_8, by use of (3.8), we have

$$\begin{aligned} I_7 &= \sum_{K\in\mathbf{T}_h}\int_K A\nabla(u-I_hu)\theta dx \\ &= \sum_{K\in\mathbf{T}_h}\int_K (A-\bar{A})\nabla(u-I_hu)\theta dx + \sum_{K\in\mathbf{T}_h}\bar{A}\int_K \nabla(u-I_hu)\theta dx \\ &\leq Ch^2\|u\|_2\|\theta\|_{H(\mathrm{div};\Omega)}, \end{aligned} \tag{4.14}$$

$$\begin{aligned} I_8 &= \sum_{K\in\mathbf{T}_h}\int_K A^2\nabla(u-I_hu)\nabla\xi dx \\ &= \sum_{K\in\mathbf{T}_h}\int_K (A^2-\overline{A^2})\nabla(u-I_hu)\nabla\xi dx + \sum_{K\in\mathbf{T}_h}\overline{A^2}\int_K \nabla(u-I_hu)\nabla\xi dx \\ &\leq Ch^2\|u\|_2\|\xi\|_h, \end{aligned} \tag{4.15}$$

respectively.

Combining (4.10)-(4.15), we obtain the desired result. □

Remark 4.1. We point out that [11] only obtained the convergence of order $O(h)$ for the non-conforming linear triangular C-R element and the rotated $\mathcal{Q}_1$-element. However, we can prove that the above theorem is also valid for the latter element on square meshes and for CNQ_1^{rot} element on anisotropic rectangular meshes, respectively. Whether Theorem 4.1 holds for the above C-R element still remains open. On the other hand, we can derive the L^2-norm estimation with $O(h^2)$ order directly through Theorem 4.1, the triangle inequality and interpolation theory.

Consequently, we will use proper postprocessing interpolation operators to get global superconvergence for the primary solution u and the flux solution $\boldsymbol{p}$. For this purpose, we assume that $\mathbf{T}_h$ is obtained from $\mathbf{T}_{2h}$(where $\mathbf{T}_{2h}$ is an anisotropic rectangular partition of Ω) by dividing each element K of $\mathbf{T}_{2h}$ into four congruent rectangles K_1, K_2, K_3 and K_4(see Fig 5.1).

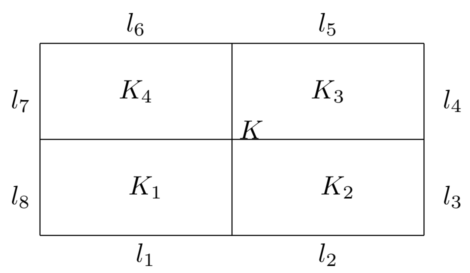

Fig. 5.1 The macro element K

Then, we can define the interpolation operators I_{2h} and $\boldsymbol{J}_{2h}$ as follows[13,14]

$$\begin{cases} I_{2h}u|_K \in P_2(K), \ \forall K \in \mathbf{T}_{2h}, \\ \int_{l_i \cup l_{i+1}} (I_{2h}u - u)ds = 0, \ i = 1,3,5,7, \\ \int_{K_i \cup K_{i+2}} (I_{2h}u - u)dx = 0, \ i = 1,2, \end{cases} \qquad \begin{cases} \boldsymbol{J}_{2h}u|_K \in Q_{11}(K) \times Q_{11}(K), \ \forall K \in \mathbf{T}_{2h}, \\ \int_{l_i} (J_{2h}p_1 - p_1)ds = 0, \ i = 3,4,7,8, \\ \int_{l_i} (J_{2h}p_2 - p_2)ds = 0, \ i = 1,2,5,6, \end{cases} \tag{4.16}$$

where $\boldsymbol{J}_{2h}\boldsymbol{p} = (J_{2h}p_1, J_{2h}p_2)$ and $Q_{11}(K) = P_1(K) \times P_1(K)$.

At the same time, the postprocessing operators I_{2h} and $\boldsymbol{J}_{2h}$ also satisfy:

$$\begin{cases} I_{2h}I_h u = I_{2h}u, \\ \|I_{2h}u - u\|_h \le Ch^2\|u\|_3, \\ \|I_{2h}v_h\|_h \le C\|v_h\|_h, \quad \forall v_h \in U_h, \end{cases} \qquad \begin{cases} \boldsymbol{J}_{2h}\boldsymbol{\Pi}_h\boldsymbol{p} = \boldsymbol{J}_{2h}\boldsymbol{p}, \\ \|\boldsymbol{J}_{2h}\boldsymbol{p} - \boldsymbol{p}\|_0 \le Ch^2\|\boldsymbol{p}\|_2, \\ \|\boldsymbol{J}_{2h}\boldsymbol{q}_h\|_0 \le C\|\boldsymbol{q}_h\|_0, \quad \forall \boldsymbol{q}_h \in \boldsymbol{X}_h, \end{cases} \tag{4.17}$$

So we can get the following superconvergence result.

Theorem 4.2. Let $(u, \boldsymbol{p})$ and $(u_h, \boldsymbol{p}_h)$ be the solutions of (2.6) and (3.12), respectively. Furthermore, assume $u \in H^3(\Omega) \cap H_0^1(\Omega), \boldsymbol{p} \in (H^2(\Omega))^2$, then there holds

$$\|u - I_{2h}u_h\|_h + \|\boldsymbol{p} - \boldsymbol{J}_{2h}\boldsymbol{p}_h\|_0 \le Ch^2(\|u\|_3 + \|\boldsymbol{p}\|_2). \tag{4.18}$$

Proof. Noticing (4.16), we have

$$\begin{aligned} &\|u - I_{2h}u_h\|_h + \|\boldsymbol{p} - \boldsymbol{J}_{2h}\boldsymbol{p}_h\|_0 \\ &\le \|u - I_{2h}I_h u\|_h + \|I_{2h}(u_h - I_h u)\|_h + \|\boldsymbol{p} - \boldsymbol{J}_{2h}\boldsymbol{\Pi}_h\boldsymbol{p}\|_0 + \|\boldsymbol{J}_{2h}(\boldsymbol{p}_h - \boldsymbol{\Pi}_h\boldsymbol{p})\|_0 \\ &\le \|u - I_{2h}u\|_h + C\|u_h - I_h u\|_h + \|\boldsymbol{p} - \boldsymbol{J}_{2h}\boldsymbol{p}\|_0 + C\|\boldsymbol{p}_h - \boldsymbol{\Pi}_h\boldsymbol{p}\|_0 \\ &\le Ch^2(\|u\|_3 + \|\boldsymbol{p}\|_2), \end{aligned} \tag{4.19}$$

which is the desired result. □

References

[1] P.B. Bochev and M.D. Gunzburger, Finite element methds of least-squares type, SIAM Review, 40(4)(1998), 789-834.

[2] F. Brezzi, J. Douglas, M. Fortin and L. Marini, Efficient rectangular mixed finite elements in two and three spaces variables, RAIRO Model. Math. Anal. Numer., 21(4)(1987), 581-604.

[3] F. Brezzi and M. Fortin, Mixed and Hybrid Finite Element Methods, Springer-Verlag, 1991.

[4] Z. Cai, R. Lazarov, T.A. Manteuffel and S.F. McCormick, First order system least-squares for second order partial diffential equation:Part I, SIAM J. Numer. Anal., 31(6)(1994), 1785-1799.

[5] Y.P. Chen and M.P. Zhang, Superconvergence of least-squares mixed finite element for symmetric elliptic problems, Appl. Numer. Math., 48(2)(2004), 195-204.

[6] Y.P. Chen and D.H. Yu, Superconvergence of least-squares mixed finite element for second-order elliptic problems, J. Comput. Math., 21(6)(2003), 825-832.

[7] P.G. Ciarlet, The Finite Element Method for Elliptic Problems, North-Holland, Amsterdam. 1978.

[8] M. Crouzeix and P.A. Raviart, Conforming and nonconforming finite element methods for solving the stationary Stokes equations, RAIRO Anal. Numer., 7(1)(1973), 33-74.

[9] J. Douglas and J.P. Wang, Superconvergence of mixed finite element methods on rectangular domain, Calcoco, 26(1989), 121-134.

[10] H.Y. Duan and G.P. Liang, Nonconforming elements in least-squares mixed finite element methods, Math. Comp., 73(245)(2003), 1-18.

[11] R. Duran, Superconvergence for rectangular mixed finite elements, Numer. Math., 58(3)(1990), 2-15.

[12] J. Hu and Z.C. Shi, Constrained quadrilateral nonconforming rotated Q1 element, J. Comput. Math., 23(6)(2005), 561C586.

[13] Q. Lin, L. Tobiska and A.H. Zhou, Superconvergence and extrapolation of non-conforming low order finite elements applied to Poisson equation, IMA J. Numer. Anal., 25(1)(2005), 160-181.

[14] Q. Lin and J.F. Lin, Finite Element Methods: Accuracy and Improvement, Beijing: Science Press, 2006.

[15] A.I. Pehivanov, G.F. Carey and R.D. Lazarov, Least-Squares mixed finite elements for second order elliptic problems. SIAM J. Numer. Anal., 31(5)(1994), 1368-1377

[16] A.I. Pehivanov and G.F. Carey. Error estimates for least-squares mixed finite elements. RAIRO Math. Model. Numer. Anal., 28(5)(1994), 499-516.

[17] R. Rannacher and S. Turek, Simple nonconforming quadrelateral Stokes element, Numerical Methods for PDEs, 8(1)(1992), 97-111.

[18] P.A. Raviart and J.M. Thomas, A mixed finite element method for second order elliptic problems, Math. Aspects of the Finite Element Method. Lecture Notes in Math. Springer, Berlin, 606(1977), 292-315.