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*Research Article*

# A Study on Ricci Solitons in Kenmotsu Manifolds

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We study and obtain results on Ricci solitons in Kenmotsu manifolds satisfying $R(\xi, X) \cdot B = 0$, $B(\xi, X) \cdot S = 0$, $S(\xi, X) \cdot R = 0$, $R(\xi, X) \cdot \overline{P} = 0$, and $\overline{P}(\xi, X) \cdot S = 0$, where $B$ and $\overline{P}$ are C-Bochner and pseudo-projective curvature tensor.

## 1. Introduction

A Ricci soliton is a natural generalization of an Einstein metric and is defined on a Riemannian manifold $(M, g)$. A Ricci soliton is a triple $(g, V, \lambda)$ with $g$ a Riemannian metric, $V$ a vector field, and $\lambda$ a real scalar such that

$$\mathscr{L}_V g + 2S + 2\lambda g = 0, \tag{1}$$

where $S$ is a Ricci tensor of $M$ and $\mathscr{L}_V$ denotes the Lie derivative operator along the vector field $V$. The Ricci soliton is said to be shrinking, steady, and expanding accordingly as $\lambda$ is negative, zero, and positive, respectively [1]. In this paper, we prove conditions for Ricci solitons in Kenmotsu manifolds to be shrinking, steady, and expanding.

In 1972, Kenmotsu [2] studied a class of contact Riemannian manifolds satisfying some special conditions and this manifold is known as Kenmotsu manifolds. Kenmotsu proved that a locally Kenmotsu manifold is a warped product $I \times_f N$ of an interval $I$ and a Kaehler manifold $N$ with warping function $f(t) = se^t$, where $s$ is a nonzero constant. Kenmotsu proved that if in a Kenmotsu manifold the condition $R(X, Y) \cdot R = 0$ holds, then the manifold is of negative curvature $-1$, where $R$ is the curvature tensor of type $(1, 3)$ and $R(X, Y)$ denotes the derivation of the tensor algebra at each point of the tangent space.

The authors in [3–7] have studied Ricci solitons in contact and Lorentzian manifolds. The authors in [8] have obtained some results on Ricci solitons satisfying $R(\xi, X) \cdot \widetilde{C} = 0$, $P(\xi, X) \cdot \widetilde{C} = 0$, $H(\xi, X) \cdot S = 0$ and $\widetilde{C}(\xi, X) \cdot S = 0$ and now we extend the work to $R(\xi, X) \cdot B = 0$, $B(\xi, X) \cdot S = 0$, $S(\xi, X) \cdot R = 0$, $R(\xi, X) \cdot \overline{P} = 0$ and $\overline{P}(\xi, X) \cdot S = 0$.

## 2. Preliminaries

An $n$-dimensional differential manifold $M$ is said to be an almost contact metric manifold [9] if it admits an almost contact metric structure $(\phi, \xi, \eta, g)$ consisting of a tensor field $\phi$ of type $(1, 1)$, a vector field $\xi$, a 1-form $\eta$, and a Riemannian metric $g$ compatible with $(\phi, \xi, \eta, g)$ satisfying

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1,$$
$$\eta \circ \phi = 0, \qquad \phi\xi = 0,$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \qquad g(X, \xi) = \eta(X), \tag{2}$$

for all vector fields $X, Y$ on $M$.

An almost contact metric manifold $M(\phi, \xi, \eta, g)$ is said to be Kenmotsu manifold [2] if

$$(\nabla_X \phi) Y = g(\phi X, Y)\xi - \eta(Y)\phi X. \tag{3}$$

From (3), we have

$$\nabla_X \xi = X - \eta(X)\xi, \tag{4}$$

where $\nabla$ denotes the Riemannian connection of $g$.

In an $n$-dimensional Kenmotsu manifold, we have

$$\eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \tag{5}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{6}$$

$$R(\xi, X)Y = \eta(Y)X - g(X,Y)\xi, \tag{7}$$

$$R(\xi, X)\xi = X - \eta(X)\xi, \tag{8}$$

where $R$ is the Riemannian curvature tensor.

Let $(g, V, \lambda)$ be a Ricci soliton in an $n$-dimensional Kenmotsu manifold $M$. From (4) we have

$$(\mathscr{L}_\xi g)(X,Y) = 2[g(X,Y) - \eta(X)\eta(Y)]. \tag{9}$$

From (1) and (9) we get

$$S(X,Y) = -(\lambda+1)g(X,Y) + \eta(X)\eta(Y). \tag{10}$$

The above equation yields that

$$QX = -(\lambda+1)X + \eta(X)\xi, \tag{11}$$

$$S(X,\xi) = -\lambda\eta(X), \tag{12}$$

$$r = -\lambda n - (n-1), \tag{13}$$

where $S$ is the Ricci tensor, $Q$ is the Ricci operator, and $r$ is the scalar curvature on $M$.

*2.1. Example for 3-Dimensional Kenmotsu Manifolds.* We consider 3-dimensional manifold $M = \{(x,y,z) \in R^3; z \neq 0\}$, where $(x,y,z)$ are the standard coordinates in $R^3$. Let $\{E_1, E_2, E_3\}$ be linearly independent given by

$$E_1 = z\frac{\partial}{\partial x}, \qquad E_2 = z\frac{\partial}{\partial y}, \qquad E_3 = -z\frac{\partial}{\partial z}. \tag{14}$$

Let $g$ be the Riemannian metric defined by $g(E_1,E_2) = g(E_2,E_3) = g(E_1,E_3) = 0$, $g(E_1,E_1) = g(E_2,E_2) = g(E_3,E_3) = 1$, where $g$ is given by

$$g = \frac{1}{z^2}(dx\otimes dx + dy\otimes dy + dz\otimes dz). \tag{15}$$

The $(\phi,\xi,\eta)$ structure is given by

$$\eta = -\frac{1}{z}dz, \qquad \xi = E_3 = -z\frac{\partial}{\partial z},$$
$$\phi E_1 = -E_2, \qquad \phi E_2 = E_1, \qquad \phi E_3 = 0. \tag{16}$$

The linearity property of $\phi$ and $g$ yields that $\eta(E_3) = 1$, $\phi^2 U = -U + \eta(U)E_3$, $g(\phi U, \phi W) = g(U,W) - \eta(U)\eta(W)$, for any vector fields $U, W$ on $M$. By definition of Lie bracket, we have

$$[E_1,E_2] = 0, \qquad [E_1,E_3] = E_1, \qquad [E_2,E_3] = E_2. \tag{17}$$

Let $\nabla$ be the Levi-Civita connection; with respect to above metric $g$ is given by Koszula formula

$$\begin{aligned}2g(\nabla_X Y, Z) = {}& X(g(Y,Z)) + Y(g(Z,X)) \\ &- Z(g(X,Y)) - g(X,[Y,Z]) \\ &- g(Y,[X,Z]) + g(Z,[X,Y]),\end{aligned} \tag{18}$$

and by virtue of it we have

$$\begin{aligned}&\nabla_{E_1}E_3 = E_1, \qquad \nabla_{E_2}E_3 = E_2, \qquad \nabla_{E_3}E_3 = 0,\\ &\nabla_{E_1}E_2 = 0, \qquad \nabla_{E_2}E_2 = -E_3, \qquad \nabla_{E_3}E_2 = 0,\\ &\nabla_{E_1}E_1 = -E_3, \qquad \nabla_{E_2}E_1 = 0, \qquad \nabla_{E_3}E_1 = 0.\end{aligned} \tag{19}$$

Clearly (19) shows that $(\phi,\xi,\eta,g)$ satisfies (2), (3), and (4). Thus $M$ is a Kenmotsu manifold.

It is known that

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \tag{20}$$

With the help of (19) and (20), it can be easily verified that

$$\begin{aligned}&R(E_1,E_2)E_2 = -E_1, \qquad R(E_1,E_3)E_3 = -E_1,\\ &R(E_1,E_1)E_1 = 0, \qquad R(E_2,E_1)E_1 = -E_2,\\ &R(E_2,E_3)E_3 = -E_2, \qquad R(E_2,E_2)E_2 = 0,\\ &R(E_3,E_1)E_1 = -E_3, \qquad R(E_3,E_2)E_2 = -E_3,\\ &R(E_3,E_3)E_3 = 0.\end{aligned} \tag{21}$$

From the above expression of the curvature tensor we obtain

$$\begin{aligned}S(E_1,E_1) = {}& g(R(E_1,E_2)E_2,E_1)\\ &+ g(R(E_1,E_3)E_3,E_1) = -2.\end{aligned} \tag{22}$$

Similarly we have

$$\begin{aligned}&S(E_2,E_2) = S(E_3,E_3) = -2,\\ &(\mathscr{L}_\xi g)(E_i,E_i) = 2[g(E_i,E_i) - \eta(E_i)\eta(E_i)].\end{aligned} \tag{23}$$

Now by $X = Y = E_i$, in (1), where $i = 1,2,3$ and by virtue of above equations we get the value of $\lambda$ which is strictly greater than 0. Thus this is an example of expanding Ricci solitons in Kenmotsu manifolds.

## 3. Ricci Soliton in a Kenmotsu Manifold Satisfying $R(\xi, X)\cdot B=0$

Bochner introduced a Kähler analogue of the Weyl conformal curvature tensor by purely formal considerations, which is now well known as the Bochner curvature tensor [10]. A geometric meaning of the Bochner curvature tensor is given by Blair in [11] by using the Boothby-Wang's fibration. In 1969, Matsumoto and Chūman [12] constructed the notion of C-Bochner curvature tensor in a Sasakian manifold and studied its several properties.

The C-Bochner curvature tensor [13] $B$ in $M$ is defined by

$$
\begin{aligned}
B(X,Y)Z = {} & R(X,Y)Z + \frac{1}{n+3} \\
& \times [g(X,Z)QY - S(Y,Z)X \\
& \quad - g(Y,Z)QX + S(X,Z)Y \\
& \quad + g(\phi X,Z)Q\phi Y - S(\phi Y,Z)\phi X \\
& \quad - g(\phi Y,Z)Q\phi X + S(\phi X,Z)\phi Y \\
& \quad + 2S(\phi X,Y)\phi Z + 2g(\phi X,Y)Q\phi Z \\
& \quad + \eta(Y)\eta(Z)QX - \eta(Y)S(X,Z)\xi \\
& \quad + \eta(X)S(Y,Z)\xi - \eta(X)\eta(Z)QY] \\
& - \frac{D+n-1}{n+3}[g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X \\
& \qquad + 2g(\phi X,Y)\phi Z] \\
& + \frac{D}{n+3}[\eta(Y)g(X,Z)\xi - \eta(Y)\eta(Z)X \\
& \qquad + \eta(X)\eta(Z)Y - \eta(X)g(Y,Z)\xi] \\
& - \frac{D-4}{n+3}[g(X,Z)Y - g(Y,Z)X],
\end{aligned} \tag{24}
$$

where $D = (r+n-1)/(n+1)$.

Taking $Z = \xi$ in (24) and using (6), (10), (11), we get

$$
B(X,Y)\xi = \left[1 - \frac{\lambda}{n+3} + \frac{4}{n+3}\right][\eta(X)Y - \eta(Y)X]. \tag{25}
$$

Similarly using (5), (10), (11), (12) in (24), we get

$$
\begin{aligned}
\eta(B(X,Y)Z) = {} & \left[1 - \frac{\lambda}{n+3} + \frac{4}{n+3}\right] \\
& \times [g(X,Z)\eta(Y) - g(Y,Z)\eta(X)].
\end{aligned} \tag{26}
$$

We assume that the condition $R(\xi,X)\cdot B = 0$, then we have

$$
\begin{aligned}
& R(\xi,X)B(Y,Z)W - B(R(\xi,X)Y,Z)W \\
& \quad - B(Y,R(\xi,X)Z)W - B(Y,Z)R(\xi,X)W = 0.
\end{aligned} \tag{27}
$$

Using (7) in (27), we get

$$
\begin{aligned}
& \eta(B(Y,Z)W)X - g(B(Y,Z)W,X)\xi + g(X,Y)B(\xi,Z)W \\
& \quad - \eta(Y)B(X,Z)W + g(X,Z)B(Y,\xi)W \\
& \quad - \eta(Z)B(Y,X)W + g(X,W)B(Y,Z)\xi \\
& \quad - \eta(W)B(Y,Z)X = 0.
\end{aligned} \tag{28}
$$

By taking an inner product with $\xi$, we have

$$
\begin{aligned}
& \eta(B(Y,Z)W)\eta(X) - g(B(Y,Z)W,X) \\
& \quad + g(X,Y)\eta(B(\xi,Z)W) - \eta(Y)\eta(B(X,Z)W) \\
& \quad + g(X,Z)\eta(B(Y,\xi)W) - \eta(Z)\eta(B(Y,X)W) \\
& \quad + g(X,W)\eta(B(Y,Z)\xi) - \eta(W)\eta(B(Y,Z)X) = 0.
\end{aligned} \tag{29}
$$

By using (25), (26) in (29), we have

$$
\begin{aligned}
& \left[1 - \frac{\lambda}{n+3} + \frac{4}{n+3}\right][g(Y,W)g(Z,X) - g(Z,W)g(Y,X)] \\
& \quad - g(B(Y,Z)W,X) = 0.
\end{aligned} \tag{30}
$$

In view of (24) in (30), then we have

$$
\begin{aligned}
& \left[1 - \frac{\lambda}{n+3} + \frac{4}{n+3}\right][g(Y,W)g(Z,X) - g(Z,W)g(Y,X)] \\
& - g(R(Y,Z)W,X) \\
& - \frac{1}{n+3}[g(Y,W)S(Z,X) - S(Z,W)g(Y,X) \\
& \qquad - g(Z,W)S(Y,X) + S(Y,W)g(Z,X) \\
& \qquad + g(\phi Y,W)S(\phi Z,X) - S(\phi Z,W)g(\phi Y,X) \\
& \qquad - g(\phi Z,W)S(\phi Y,X) + S(\phi Y,W)g(\phi Z,X) \\
& \qquad + 2S(\phi Y,Z)g(X,\phi W) + 2g(\phi Y,Z)S(X,\phi W) \\
& \qquad + \eta(W)\eta(Z)S(Y,X) - \eta(X)\eta(Z)S(Y,W) \\
& \qquad + \eta(Y)\eta(X)S(Z,W) - \eta(W)\eta(Y)S(Z,X)] \\
& - \frac{D}{n+3}[\eta(X)\eta(Z)g(Y,W) - \eta(W)\eta(Z)g(Y,X) \\
& \qquad + \eta(W)\eta(Y)g(Z,X) - \eta(Y)\eta(X)g(Z,W)] \\
& + \frac{D+n-1}{n+3}[g(\phi Y,W)g(\phi Z,X) \\
& \qquad - g(\phi Z,W)g(\phi Y,X) \\
& \qquad + 2g(\phi Y,Z)g(X,\phi W)] \\
& + \frac{D-4}{n+3}[g(Y,W)g(Z,X) - g(Z,W)g(Y,X)] = 0.
\end{aligned} \tag{31}
$$

Taking $X = Y = e_i$ in (31) and summing over $i = 1, 2, \ldots, n$. By virtue of (10), (11), (12), and on simplification, we get

$$\begin{aligned} S(Z,W) &= \left[\frac{-(n+4)\lambda - 2n - 3}{n+3} + \frac{-n^2 - 6n + 8}{n+3} - \frac{r}{n+3}\right] g(W,Z) \\ &\quad + \left[\frac{(\lambda+1)(n+4) - 2 + n}{n+3} + \frac{r + 4(n-1)}{n+3}\right] \eta(W)\eta(Z). \end{aligned} \tag{32}$$

Putting $Z = W = \xi$ in (32) and by virtue of (10) and (13), we have

$$\lambda = (n-1). \tag{33}$$

Therefore, $\lambda$ positive that is, the Ricci soliton in Kenmotsu manifold is expanding.

Hence we state the following theorem:

**Theorem 1.** *A Ricci soliton in a Kenmotsu manifold satisfying $R(\xi, X) \cdot B = 0$ is expanding.*

## 4. Ricci Soliton in a Kenmotsu Manifolds Satisfying $B(\xi, X) \cdot S$=0

The condition $B(\xi, X) \cdot S = 0$ implies that

$$S(B(\xi,X)Y,Z) + S(Y,B(\xi,X)Z) = 0. \tag{34}$$

By using (10) in (34), we have

$$\begin{aligned} &\eta(Z)\eta(B(\xi,X)Y) - (\lambda+1) g(B(\xi,X)Y,Z) \\ &\quad - (\lambda+1) g(Y,B(\xi,X)Z) + \eta(Y)\eta(B(\xi,X)Z) = 0, \end{aligned} \tag{35}$$

the above equation implies that

$$\begin{aligned} &[\eta(Z)\eta(B(\xi,X)Y) + \eta(Y)\eta(B(\xi,X)Z)] \\ &\quad = (\lambda+1)[g(B(\xi,X)Y,Z) + g(Y,B(\xi,X)Z)]. \end{aligned} \tag{36}$$

By using (24) and (26) in (36), we have

$$\begin{aligned} &2\eta(X)\eta(Y)\eta(Z)\left[1 - \frac{\lambda}{(n+3)} + \frac{4}{(n+3)}\right] \\ &\quad - \left[1 - \frac{\lambda}{(n+3)} + \frac{4}{(n+3)}\right] \\ &\quad \times [g(X,Z)\eta(Y) + g(X,Y)\eta(Z)] = 0. \end{aligned} \tag{37}$$

Put $X = Y = \xi$ in (37) then the equation is identically satisfied and we do not get the value for $\lambda$. So, we proceed as follows: Taking $X = Y = e_i$ in (37) and summing over $i = 1, 2, \ldots, n$ and by virtue of (13) and $\eta(Z) \neq 0$ conditions, we obtain

$$\lambda = n + 7. \tag{38}$$

Therefore, $\lambda$ is positive that is Ricci soliton in Kenmotsu manifolds satisfying $B(\xi, X) \cdot S = 0$ is expanding.

Hence we can state the following theorem.

**Theorem 2.** *A Ricci soliton in a Kenmotsu manifold satisfying $B(\xi, X) \cdot S = 0$ is expanding.*

## 5. Ricci Soliton in a Kenmotsu Manifold Satisfying $S(\xi, X) \cdot R = 0$

Using the following equations:

$$\begin{aligned} &S((X,\xi)\cdot R)(U,V)W \\ &\quad = ((X \wedge_S \xi)\cdot R)(U,V)W = (X \wedge_S \xi) R(U,V)W \\ &\qquad + R((X \wedge_S \xi)U,V)W + R(U,(X \wedge_S \xi)V)W \\ &\qquad + R(U,V)(X \wedge_S \xi)W, \end{aligned} \tag{39}$$

where the endomorphism $X \wedge_S Y$ is defined by

$$(X \wedge_S Y)Z = S(Y,Z)X - S(X,Z)Y, \tag{40}$$

we have

$$\begin{aligned} &S((X,\xi)\cdot R)(U,V)W \\ &\quad = S(\xi,R(U,V)W)X - S(X,R(U,V)W)\xi \\ &\qquad + S(\xi,U)R(X,V)W - S(X,U)R(\xi,V)W \\ &\qquad + S(\xi,V)R(U,X)W - S(X,V)R(U,\xi)W \\ &\qquad + S(\xi,W)R(U,V)X - S(X,W)R(U,V)\xi. \end{aligned} \tag{41}$$

By using the condition $S(\xi, X) \cdot R = 0$, and by virtue of (10), (12), we have

$$\begin{aligned} &-\lambda\eta(R(U,V)W)X \\ &\quad - [-(\lambda+1)g(X,R(U,V)W) + \eta(X)\eta(R(U,V)W)]\xi \\ &\quad - \lambda\eta(U)R(X,V)W \\ &\quad - [-(\lambda+1)g(X,U) + \eta(X)\eta(U)]R(\xi,V)W \\ &\quad - \lambda\eta(V)R(U,X)W \\ &\quad - [-(\lambda+1)g(X,V) + \eta(X)\eta(V)]R(U,\xi)W \\ &\quad - \lambda\eta(W)R(U,V)X \\ &\quad - [-(\lambda+1)g(X,W) + \eta(X)\eta(W)]R(U,V)\xi = 0. \end{aligned} \tag{42}$$

By taking an inner product with $\xi$ and by virtue of (5), (6), (7), and (8), we have

$$\begin{aligned} &-(\lambda+1)\eta(X)[g(U,W)\eta(V) - g(V,W)\eta(U)] \\ &\quad + \lambda[g(V,W)\eta(X)\eta(U) - g(U,W)\eta(X)\eta(V) \\ &\qquad - g(U,X)\eta(V)\eta(W) + g(V,X)\eta(U)\eta(W)] \\ &\quad + (\lambda+1)g(X,R(U,V)W) \\ &\quad + (\lambda+1)g(X,U)[\eta(W)\eta(V) - g(V,W)] \\ &\quad + (\lambda+1)g(X,V)[g(U,W) - \eta(W)\eta(U)] \\ &\quad + g(V,W)\eta(X)\eta(U) - g(U,W)\eta(X)\eta(V) = 0. \end{aligned} \tag{43}$$

Taking $X = U = e_i$ and summing over $i = 1, 2, \ldots, n$, we obtain

$$2(\lambda+1)[g(V,W)-\eta(W)\eta(V)] + (\lambda+1)S(V,W)-(\lambda+1)(n-1)g(V,W) + (n-1)\eta(V)\eta(W)=0. \quad (44)$$

Taking $V = W = \xi$ in (44) and by virtue of (12), (13), we obtain

$$-\lambda(\lambda+n)=0. \quad (45)$$

This implies either

$$\lambda = 0 \quad \text{or} \quad \lambda = -n. \quad (46)$$

Therefore for any $\lambda = 0$ or $\lambda = -n$ the Ricci soliton in Kenmotsu manifolds satisfying $S(\xi, X)\cdot R = 0$ is either steady or shrinking.

Hence we can state the following theorem.

**Theorem 3.** *A Ricci soliton in a Kenmotsu manifold satisfying $S(\xi, X)\cdot R = 0$ is either steady or shrinking.*

## 6. Ricci Soliton in a Kenmotsu Manifolds Satisfying $R(\xi, X)\cdot \overline{P} = 0$

The Pseudo-projective curvature tensor $\overline{P}$ is defined by

$$\overline{P}(X,Y)Z = aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y] - \frac{r}{n}\left(\frac{a}{n-1}+b\right)[g(Y,Z)X-g(X,Z)Y], \quad (47)$$

where $a, b \neq 0$ are constants. Taking $Z = \xi$ in (47) and using (6), (10), (11), we get

$$\overline{P}(X,Y)\xi = \left[a + b\lambda + \frac{r}{n}\left(\frac{a}{n-1}+b\right)\right][\eta(X)Y-\eta(Y)X]. \quad (48)$$

Similarly using (5), (10), (11), (12) in (47), we get

$$\eta\left(\overline{P}(X,Y)Z\right) = \left[a + b(\lambda+1) + \frac{r}{n}\left(\frac{a}{n-1}+b\right)\right] \times [g(X,Z)\eta(Y)-g(Y,Z)\eta(X)]. \quad (49)$$

We assume that the condition $R(\xi, X)\cdot \overline{P} = 0$, then we have

$$R(\xi,X)\overline{P}(U,V)W - \overline{P}(R(\xi,X)U,V)W - \overline{P}(U,R(\xi,X)V)W - \overline{P}(U,V)R(\xi,X)W = 0. \quad (50)$$

Using (7) in (50), we find

$$\begin{aligned}\eta\left(\overline{P}(U,V)W\right)X - g\left(X,\overline{P}(U,V)W\right)\xi &- \eta(U)\overline{P}(X,V)W + g(X,U)\overline{P}(\xi,V)W \\ &- \eta(V)\overline{P}(U,X)W + g(X,V)\overline{P}(U,\xi)W \\ &- \eta(W)\overline{P}(U,V)X + g(X,W)\overline{P}(U,V)\xi = 0.\end{aligned} \quad (51)$$

By taking an inner product with $\xi$ then we get

$$\begin{aligned}\eta\left(\overline{P}(U,V)W\right)\eta(X) - g\left(X,\overline{P}(U,V)W\right) &- \eta(U)\eta\left(\overline{P}(X,V)W\right) + g(X,U)\eta\left(\overline{P}(\xi,V)W\right) \\ &- \eta(V)\eta\left(\overline{P}(U,X)W\right) + g(X,V)\eta\left(\overline{P}(U,\xi)W\right) \\ &- \eta(W)\eta\left(\overline{P}(U,V)X\right) + g(X,W)\eta\left(\overline{P}(U,V)\xi\right) = 0.\end{aligned} \quad (52)$$

By using (48), (49) in (52), we have

$$- g\left(X,\overline{P}(U,V)W\right) + \left[a + b(\lambda+1) + \frac{r}{n}\left[\frac{a}{n+1}+b\right]\right] \times [g(X,V)g(U,W) - g(X,U)g(V,W)] = 0. \quad (53)$$

In view of (47) in (53), we have

$$\begin{aligned}&- ag(X,R(U,V)W) \\ &- b[(\lambda+1)\{g(V,X)g(U,W) - g(V,W)g(U,X)\} + \eta(V)\eta(W)g(U,X) - g(V,X)\eta(U)\eta(W)] \\ &+ [a + b(\lambda+1)] \times [g(X,V)g(U,W) - g(X,U)g(V,W)] = 0.\end{aligned} \quad (54)$$

Taking $X = U = e_i$ in (54) and summing over $i = 1, 2, \ldots, n$, and on simplification, we get

$$aS(V,W) = -a(n-1)g(V,W) - b(n-1)\eta(V)\eta(W). \quad (55)$$

Putting $V = W = \xi$ in (55) and by virtue of (12), (13), we get the following equation:

$$\lambda = \frac{(n-1)(a+b)}{a}. \quad (56)$$

Since $(a+b)/a \neq 0$ implies that $\lambda > 0$, that is, the Ricci soliton in Kenmotsu manifold satisfying $R(\xi, X)\cdot\overline{P} = 0$ is expanding, hence we state the following theorem.

**Theorem 4.** *A Ricci soliton in a Kenmotsu manifold satisfying $R(\xi, X)\cdot \overline{P} = 0$ is expanding.*

## 7. Ricci Soliton in a Kenmotsu Manifolds Satisfying $\overline{P}(\xi, X)\cdot S = 0$

The condition $\overline{P}(\xi, X)\cdot S = 0$ implies that

$$S\left(\overline{P}(\xi,X)Y,Z\right) + S\left(Y,\overline{P}(\xi,X)Z\right) = 0. \quad (57)$$

By using (10) in (57), we have

$$\eta(Z)\eta\left(\overline{P}(\xi,X)Y\right)-(\lambda+1)g\left(\overline{P}(\xi,X)Y,Z\right) - (\lambda+1)g\left(Y,\overline{P}(\xi,X)Z\right)+\eta(Y)\eta\left(\overline{P}(\xi,X)Z\right)=0, \tag{58}$$

that is,

$$\left[\eta(Z)\eta\left(\overline{P}(\xi,X)Y\right)+\eta(Y)\eta\left(\overline{P}(\xi,X)Z\right)\right] = (\lambda+1)\left[g\left(\overline{P}(\xi,X)Y,Z\right)+g\left(Y,\overline{P}(\xi,X)Z\right)\right]. \tag{59}$$

By using (47) and (48) in (59), we have

$$\left[a+\frac{r}{n}\left[\frac{a}{n-1}+b\right]\right] \times\left[2\eta(X)\eta(Y)\eta(Z) - g(X,Z)\eta(Y)-g(X,Y)\eta(Z)\right]=0. \tag{60}$$

Put $X=Y=\xi$ in (60); then the equation is identically satisfied and we do not get the value for $\lambda$. So, we proceed as follows: taking $X=Y=e_i$, summing over $i=1,2,\ldots,n$, and by virtue of (13) and $\eta(Z)\neq 0$ conditions we obtain

$$\lambda=\frac{(n-1)^2(a-b)}{n[a+b(n-1)]}. \tag{61}$$

Therefore, if $a=b$ in (61) then $\lambda=0$; that is, Ricci soliton in Kenmotsu manifolds satisfying $\overline{P}(\xi,X)\cdot S=0$ is steady. If $a\neq b$ then either $\lambda>0$ for $a>b$ or $\lambda<0$ for $a<b$, that is, the Ricci soliton in Kenmotsu manifold satisfying $\overline{P}(\xi,X)\cdot S=0$ is expanding or shrinking.

Hence we can state the following theorem.

**Theorem 5.** *A Ricci soliton in a Kenmotsu manifolds satisfying $\overline{P}(\xi,X)\cdot S=0$ is steady for $a=b$, expanding for $a>b$ and shrinking for $a<b$.*

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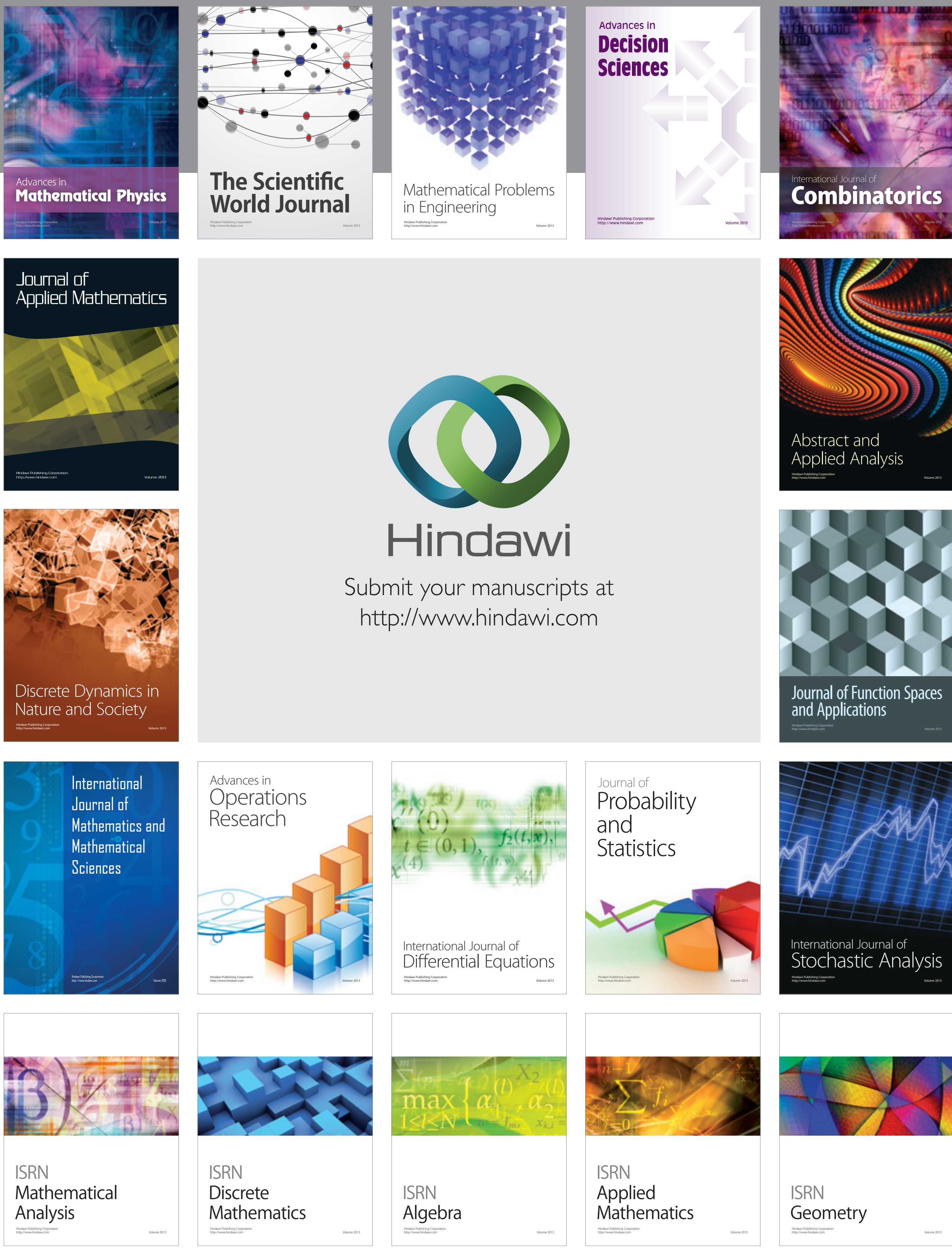
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