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A try to prove Riemann's Hypothesis

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Abstract

Riemann's conjecture (1859) states that:
The real part of every non trivial zero of Riemann's zeta function is 1/2.

The main contribution of this paper is to use the classical approach to this conjecture whose the key idea is to provide a self-adjoint operator whose real eigenvalues are the imaginary part of the non trivial zeros of Riemann's zeta function and whose existence, according to Hilbert and Pólya, proves Riemann's hypothesis.

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1 Introduction

In his book [1] of 1748, Leonhard Euler (1707-1783) proved what is now named *the Euler product formula*. This product is the result of the infinite sum:

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\in\mathbb{P}}(1-1/p^s)^{-1} \quad \text{for any integer variable } s > 1$$

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where $\mathbb{P}$ is the infinite set of primes.

In his article [2] of 1859, Riemann (1826-1866) extended the Euler definition to the complex variable s of the zeta function:

$$\zeta(s) = \prod_{p\in\mathbb{P}}(1 - 1/p^s)^{-1} \quad \textit{for any complex variable } s \neq 1$$

It is known that the trivial zeros of the function are the infinite set:

$$\{s_1\} = -2m \quad \text{for all integers } m > 0$$

Riemann's hypothesis can be seen as stating that:

Probably, the infinite set of the non trivial zeros $\{s_2\}$ of $\zeta(s)$ can be written:

$$\{s_2\} = \tfrac{1}{2} + it_n \quad \text{where } t_n \text{ is real.}$$

This conjecture is the first point of the eighth unresolved problem (among 23) that Hilbert listed in 1900 [3] as well as the second unresolved problem listed in 2000 by The Clay Mathematics Institute [4].

2 Preliminary notes

2.1 Hilbert-Pólya statement

Circa 1914, Hilbert et Pólya [5], independently from each other, have orally stated that Riemann's hypothesis would be proved if it could be shown that the imaginary parts t_n of the non trivial zeros of the symmetrical xi function $\xi(s)$ derived from $\zeta(s)$, corresponded to the real eigenvalues of an unbounded self-adjoint operator (here named $\hat{H}_\xi$) for which we could write:

$$\hat{H}_\xi \psi_k = E_k \psi_k \tag{1}$$

which is an equation of quantum physics where E is for energy.

So, the first and unique purely mathematical clue that we have is that this operator should be a square matrix of infinite dimension with real eigenvalues. This means that it could be written:

$$\hat{H}_\xi = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & t_{n-1} & 0 & 0 & 0 & \cdots \\ \cdots & 0 & t_n & 0 & 0 & \cdots \\ \cdots & 0 & 0 & t_{n+1} & 0 & \cdots \\ \cdots & 0 & 0 & 0 & t_{n+2} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \quad \text{all } t_i \text{ being real}$$

3 Proof of Riemann's Hypothesis

The proof will be established in two steps:

The first one establishes only a conditional proof.

The second one establishes the unconditional proof.

3.1 Conditional proof of Riemann Hypothesis

Proof. By definition, a complex number s is written:

$$s = x + iy \quad \text{where } x \text{ and } y \text{ are real and } i = \sqrt{-1}$$

By changing the conventional basis of coordinates (x, y) of the complex plane into the new one $(x' = \frac{1}{2} - x, y' = y)$, these complex numbers can be written:

$$s' = x' + iy' \text{ in the new basis}$$
or:
$$s' = (\tfrac{1}{2} - x) + iy \text{ using the change in coordinates.}$$

Condition. We suppose that the $\hat{H}_\xi$ operator exists and that it contains the infinitely many real eigenvalues t_n coming from the non trivial zeros s_2 of $\zeta(s)$.

Hypothesis. We then suppose that these non trivial zeros lie anywhere in the complex plane with the two exceptions that they cannot lie on the real axis x or x' (reserved for trivial zeros s_1), which gives:

$$y \neq 0 \text{ and } y' \neq 0$$

nor on the conventional critical line $x = \frac{1}{2}$ that becomes the new imaginary axis y', which gives:

$$x \neq \tfrac{1}{2} \text{ and } x' \neq 0$$

Then, *each non trivial zero s_2 of $\zeta(s)$* could be written:

$$s_2 = x'_2 + iy'_2 \quad \text{with } x'_2 \neq 0 \text{ and } y'_2 \neq 0$$
or, using the change in coordinates:
$$s_2 = (\tfrac{1}{2} - x_2) + iy_2 \quad \text{with } x_2 \neq \tfrac{1}{2} \text{ and } y_2 \neq 0$$

But using the fact that $-x_2 = i^2 x_2$, they can be written:

$$s_2 = (\tfrac{1}{2} + i^2 x_2) + iy_2 = \tfrac{1}{2} + i(y_2 + ix_2) \quad \text{with } x_2 \neq \tfrac{1}{2} \text{ and } y_2 \neq 0$$
or:
$$s_2 = \tfrac{1}{2} + it_2 \quad \text{with } t_2 = y_2 + ix_2,\ x_2 \neq \tfrac{1}{2} \text{ and } y_2 \neq 0$$

and we get the result that the non trivial zeros can exist only when $x_2 = 0$ to have t_2 real, a result that is false because it is known that all non trivial zeros lie in the critical strip $0 < x = Re(s) < 1$ that excludes $x = 0$.

We thus get the contradiction to our hypothesis that the quantity $t_2 = y_2 + ix_2$ is unable, when the critical line $x_2 = \frac{1}{2}$ is excluded, to provide any real value $t_n \neq 0$ to the operator $\hat{H}_\xi$. Therefore, as it is known on one hand that infinitely many non trivial zeros *do exist* for $\zeta(s)$ and on the other hand that they cannot lie on the x axis nor *out of* the critical line $x = \frac{1}{2}$, it proves that they can only lie on this critical line. This, in turn, proves Riemann's hypothesis, conditionally to the existence of the $\hat{H}_\xi$ operator. □

3.2 Unconditional proof of Riemann's Hypothesis

Proof. Now, as Riemann's Hypothesis is conditionally proved, the $\hat{H}_\xi$ operator conditionally exists and contains the infinitely many real values t_n of the non trivial zeros $\{s_2\}$ of $\zeta(s)$.

To prove that the $\hat{H}_\xi$ operator *do exists*, we will consider the new and larger operator $\hat{H}_\zeta$ built with the zeros of both sets of zeros $\{s_1\}$ and $\{s_2\}$ of $\zeta(s)$ as eigenvalues, an operator that also contains the real values t_n but not as eigenvalues:

$$\hat{H}_\zeta = \begin{pmatrix} \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & -6 & 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & -4 & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & -2 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & \frac{1}{2}+it_1 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & 0 & \frac{1}{2}+it_2 & 0 & \dots \\ \dots & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}+it_3 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

As this new operator contains the real values t_n, it enables us, at any time but if it exists, to rebuild the operator $\hat{H}_\xi$ of Hilbert and Pólya. To simplify the writing, we set:

$$\begin{pmatrix} \dots & \dots & \dots & \dots \\ \dots & -6 & 0 & 0 \\ \dots & 0 & -4 & 0 \\ \dots & 0 & 0 & -2 \end{pmatrix} = (-2m)$$

and:

$$\begin{pmatrix} \frac{1}{2}+it_1 & 0 & 0 & \dots \\ 0 & \frac{1}{2}+it_2 & 0 & \dots \\ 0 & 0 & \frac{1}{2}+it_3 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix} = \left(\tfrac{1}{2}+it_n\right)$$

so that $\hat{H}_\zeta$ can be written:

$$\hat{H}_\zeta = \begin{pmatrix} (-2m) & (0) \\ (0) & \left(\frac{1}{2}+it_n\right) \end{pmatrix}$$

But the matrices $(-2m)$ and $\left(\frac{1}{2}+it_n\right)$ representing the sets of zeros $\{s_1\}$ and $\{s_2\}$ can symbolically be replaced by their parametric form:

$$\begin{gathered} -2m, \quad m \text{ being a positive integer parameter} \\ \tfrac{1}{2}+it_n, \quad t_n \text{ being a real parameter} \end{gathered}$$

The sets $\{s_1\}$ and $\{s_2\}$ can then be considered as the two infinite sets of roots of the polynomial of complex variable s:

$$\begin{gathered} P(s) = (s-s_1)(s-s_2) = s^2-(s_1+s_2)s+s_1s_2 \\ P(s,m,t) = s^2-(-2m+\tfrac{1}{2}+it_n)s-2m(\tfrac{1}{2}+it_n) \\ P(s,m,t) = s^2+(2m-(\tfrac{1}{2}+it_n))s-2m(\tfrac{1}{2}+it_n) \end{gathered}$$

which can be written either:

$$P(s,m,t) = \begin{pmatrix} s^2 & s & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & (2m-(\frac{1}{2}+it_n)) & 0 \\ 0 & 0 & -2m(\frac{1}{2}+it_n) \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{2}$$

or:

$$P(s,m,t) = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & (2m-(\frac{1}{2}+it_n)) & 0 \\ 0 & 0 & -2m(\frac{1}{2}+it_n) \end{pmatrix} \begin{pmatrix} s^2 \\ s \\ 1 \end{pmatrix} \tag{3}$$

Equation (2) gives, by multiplying the first two matrices:

$$P(s,m,t) = \begin{pmatrix} s^2 & (2m-(\frac{1}{2}+it_n))s & -2m(\frac{1}{2}+it_n) \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = E_k\psi_{E_k} \tag{4}$$

when we set:

$$E_k = \begin{pmatrix} s^2 & (2m - (\frac{1}{2} + it_n))s & -2m(\frac{1}{2} + it_n) \end{pmatrix}$$

and:

$$\psi_{E_k} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Now, as from (4) we also have:

$$P(s,m,t) = \begin{pmatrix} 1 & (2m - (\frac{1}{2} + it_n)) & -2m(\frac{1}{2} + it_n) \end{pmatrix} \begin{pmatrix} s^2 \\ s \\ 1 \end{pmatrix} = H_k \psi_{H_k} \tag{5}$$

when we set:

$$H_k = \begin{pmatrix} 1 & (2m - (\frac{1}{2} + it_n)) & -2m(\frac{1}{2} + it_n) \end{pmatrix}$$

and:

$$\psi_{H_k} = \begin{pmatrix} s^2 \\ s \\ 1 \end{pmatrix} = \hat{R} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \hat{R}\psi_{E_k}$$

where $\hat{R}$ is the 3-dimensional rotation matrix from the orthogonal basis ψ_{H_k} used to describe H_k to the orthogonal basis ψ_{E_k} used to describe E_k, and we have:

$$H_k \psi_{H_k} = H_k \hat{R} \psi_{E_k} = E_k \psi_{E_k}$$

Then, setting $\hat{H} = H_k \hat{R}$, we get:

$$\hat{H}\psi_{E_k} = E_k \psi_{E_k} \tag{6}$$

which is identical to equation (1). And the $\hat{H}_\xi$ operator looked for by Hilbert and Pólya can be built with the real values t_n of the existing operator:

$$\hat{H} = H_k \hat{R} = \begin{pmatrix} 1 & (2m - (\frac{1}{2} + it_n)) & -2m(\frac{1}{2} + it_n) \end{pmatrix} \hat{R}$$

As we can rebuild the self-adjoint operator $\hat{H}_\xi$ linked to $\zeta(s)$ via the function $P(s,m,t)$ and the existing operator $\hat{H}$, this self-adjoint operator $\hat{H}_\xi$ *do exists* and as we have proved earlier that Riemann hypothesis is true conditionally to the existence of the $\hat{H}_\xi$ operator, Riemann hypothesis is unconditionally proved. □

Remark. As H_k can also be written:

$$H_k = \begin{pmatrix}1 & 1 & 1\end{pmatrix} \begin{pmatrix}1 & 0 & 0 \\ 0 & (2m-(\frac{1}{2}+it_n)) & 0 \\ 0 & 0 & -2m(\frac{1}{2}+it_n)\end{pmatrix} = \begin{pmatrix}1 & 1 & 1\end{pmatrix} \hat{A}$$

when we set:

$$\hat{A} = \begin{pmatrix}1 & 0 & 0 \\ 0 & (2m-(\frac{1}{2}+it_n)) & 0 \\ 0 & 0 & -2m(\frac{1}{2}+it_n)\end{pmatrix} \tag{7}$$

we get from (3) and (2) that:

$$\begin{pmatrix}1 & 1 & 1\end{pmatrix} \hat{A} \begin{pmatrix}s^2 \\ s \\ 1\end{pmatrix} = P(s,m,t) = \begin{pmatrix}s^2 & s & 1\end{pmatrix} \hat{A} \begin{pmatrix}1 \\ 1 \\ 1\end{pmatrix} \tag{8}$$

But, for any $s = x + iy$, we have:

$$\begin{aligned} P(s,m,t) &= s^2 + \left(2m - (\tfrac{1}{2}+it_n)\right) s - 2m\left(\tfrac{1}{2}+it_n\right) \\ &= (x+iy)^2 + \left(2m - (\tfrac{1}{2}+it_n)\right)(x+iy) - 2m\left(\tfrac{1}{2}+it_n\right) \\ &= \left(x^2 - y^2 + (2m-\tfrac{1}{2})x + yt_n\right) + i\left(-xt_n + y(2m-\tfrac{1}{2})\right) - m - 2mit_n \\ &= \left(x^2 - y^2 + (2m-\tfrac{1}{2})x + yt_n - m\right) + i\left(-xt_n + y(2m-\tfrac{1}{2}) - 2mt_n\right) \end{aligned}$$

and $P(s,m,t)$ will be real only when:

$$-xt_n + y(2m-\tfrac{1}{2}) - 2mt_n = 0$$

and so, for all of the infinitely many lines in the complex plane such that:

$$y = \frac{t_n}{2m-\frac{1}{2}}x + \frac{2mt_n}{2m-\frac{1}{2}}$$

where m and t_n are the discrete values defined earlier

Then, for all the points of all these lines, we have:

$$P(s,m,t)_{lines} = \left(x^2 - y^2 + (2m-\frac{1}{2})x + yt_n\right) = V(x,y), \text{ a real value} \tag{9}$$

and therefore for the mono-term matrix (V(x,y)) we have:

$$(V(x,y)) = (\overline{V(x,y)}) = (\overline{V(x,y)})^T \tag{10}$$

where $(\overline{V(x,y)})$ is the conjugate matrix of $(V(x,y))$ and $(\overline{V(x,y)})^T$ is the conjugate transpose of $(V(x,y))$. So, from (8), (9) and (10), we can write:

$$P(s,m,t)_{lines} = V(x,y) = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \hat{A} \begin{pmatrix} s^2 \\ s \\ 1 \end{pmatrix} = \left(\overline{\begin{pmatrix} s^2 & s & 1 \end{pmatrix} \hat{A} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} \right)^T$$

which proves that the operator $\hat{A}$ that also provides the real t_n to $\hat{H}_\xi$, verifies the equation of the observables in quantum physics, which is generally written:

$$< \psi_1 \mid \hat{A} \mid \psi_2 >= \left(< \psi_2 \mid \hat{A} \mid \psi_1 > \right)^T$$

where $\hat{A}$ is a self-adjoint operator associated to a physical quantity A, <x| and |x> are the bra and ket operators and ψ_1 and ψ_2 are the states of the physical quantity A before and after the measuring of A.

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