Traveling solitary wave solutions for the symmetric regularized long-wave equation

Mahmoud A.E. Abdelrahman and Mostafa M.A. Khater*
Mansoura University,Department of Mathematics, Faculty of Science,
35516 Mansoura, Egypt.
Email*: mostafa.khater2024@yahoo.com

November 13, 2015

Abstract In this paper, we employ the extended tanh function method to find the exact traveling wave solutions involving parameters of the symmetric regularized long- wave equation. When these parameters are taken to be special values, the solitary wave solutions are derived from the exact traveling wave solutions. These studies reveal that the symmetric regularized long-wave equation has a rich variety of solutions.

Keywords: The extended tanh function method; The symmetric regularized long-wave equation; Traveling wave solutions; Solitary wave solutions.

AMS subject classifications: 35A05, 35A20, 65K99, 65Z05, 76R50, 70K70

1 Introduction

Many models in mathematics and physics are described by nonlinear differential equations. Nowadays, research in physics devotes much attention to nonlinear partial differential evolution model equations, appearing in various fields of science, especially fluid mechanics, solid-state physics, plasma physics, and nonlinear optics. Large varieties of physical, chemical, and biological phenomena are governed by nonlinear partial differential equations. One of the most exciting advances of nonlinear science and theoretical physics has been the development of methods to look for exact solutions of nonlinear partial differential equations. Exact solutions to nonlinear partial differential equations play an important role in nonlinear science, especially in nonlinear physical science since they can provide much physical information and more insight into the physical aspects of the problem and thus lead to further applications. Nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction and convection are very important in nonlinear wave equations. In recent years, quite a few methods for obtaining explicit traveling and solitary wave solutions of nonlinear evolution equations have been proposed. A variety of powerful methods, tanh - sech method [1]-[3], The $\exp(-\varphi(\xi))$-expansion method [4]-[6],The extended $\exp(-\varphi(\xi))$-expansion method [7], sine - cosine method [8]-[9], modified simple equation method

[10, 12],F-expansion method [13]-[14], exp-function method [15, 16], trigonometric function series method [17], $(\frac{G'}{G})-$ expansion method [18]-[20], Jacobi elliptic function method [21]-[24], Extended tanh function method [25]-[27] and so on.
The objective of this article is to apply The extended tanh function method for finding the exact traveling wave solution of the symmetric regularized long- wave equation Which describe shallow water waves and plasma drift waves.
The rest of this paper is organized as follows: In Section 2, we give the description of extended tanh function method In Section 3, we use this method to find the exact solutions of the nonlinear evolution equations pointed out above. In Section 4, conclusions are given.

2 Description of method

Consider the following nonlinear evolution equation

$$F(u, u_t, u_x, u_{tt}, u_{xx}, ...) = 0, \tag{2.1}$$

where F is a polynomial in $u(x,t)$ and its partial derivatives in which the highest order derivatives and nonlinear terms are involved. In the following,we give the main steps of this method:
Step 1. We use the wave transformation

$$u(x,t) = u(\xi), \qquad \xi = x - ct, \tag{2.2}$$

where c is a constant, to reduce Eq.(2.1) to the following ODE:

$$P(u, u', u'', u''', ...) = 0, \tag{2.3}$$

where P is a polynomial in $u(\xi)$ and its total derivatives.
Step 2. Suppose that the solution of Eq.(2.3) has the form:

$$u(\xi) = a_0 + \sum_{i=1}^{m} \left(a_i\ \phi^i + b_i\ \phi^{-i}\right), \tag{2.4}$$

where a_i, b_i are constants to be determined, such that $a_m \neq 0$ or $b_m \neq 0$ and ϕ satisfies the Riccati equation

$$\phi' = b + \phi^2, \tag{2.5}$$

where b is a constant. Eq.(2.5) admits several types of solutions according to :
Case 1. If $b < 0$, then

$$\phi = -\sqrt{-b}\ tanh(\sqrt{-b}\ \xi), \quad or \quad \phi = -\sqrt{-b}\ coth(\sqrt{-b}\ \xi). \tag{2.6}$$

Case 2. If $b > 0$, then

$$\phi = \sqrt{b}\ tan(\sqrt{b}\ \xi), \quad or \quad \phi = -\sqrt{b}\ cot(\sqrt{b}\ \xi). \tag{2.7}$$

Case 3. If $b = 0$, then

$$\phi = -\frac{1}{\xi}. \tag{2.8}$$

Step 3. Determine the positive integer m in Eq.(2.4) by balancing the highest order derivatives and the nonlinear terms.
Step 4. Substitute Eq.(2.4) along Eq.(2.5) into Eq.(2.3) and collecting all the terms of the same power ϕ^i, $i = 0, \pm1, \pm2, \pm3, ...$ and equating them to zero, we obtain a system of algebraic equations, which can be solved by Maple or Mathematica to get the values of a_i and b_i.
Step 5. substituting these values and the solutions of Eq.(2.5) into Eq.(2.4) we obtain the exact solutions of Eq.(2.1).

3 The SRLW equation

Here, we will apply the extended tanh function method described in Sec.2 to find the exact traveling wave solutions and then the solitary wave solutions The SRLW equation [28].
Consider the SRLW equation be in the form

$$v_{tt} - v_{xx} + \left(\frac{v^2}{2}\right)_{xt} - v_{xxtt} = 0, \tag{3.1}$$

by using the transformation $v(\xi) = v(x,t)$, since $\xi = x + k\,t$. Where k is arbitrary constant to be determined later, we get

$$\left(k^2 - 1\right) v'' - k\left(\frac{v^2}{2}\right)'' - k^2 v'''' = 0. \tag{3.2}$$

By integration Eq.(3.2) twice with negligence of integral constant, we get

$$\left(k^2 - 1\right) v - \frac{k}{2} v^2 - k^2 v'' = 0. \tag{3.3}$$

Balancing v'' and $v^2 \Rightarrow m = 2$, so that, we assume the solution of Eq.(3.3) be in the form

$$v(\xi) = a_0 + a_1\Phi + a_2\Phi^2 + \frac{b_1}{\Phi} + \frac{b_2}{\Phi^2}. \tag{3.4}$$

Substituting Eq.(3.4) and it's derivatives into Eq.(3.3) and collecting the coefficients of ϕ^i, $i = 0, \pm1, \pm2, \pm3, ...$ and set it to zero we obtain the system of equation

$$-\frac{1}{2}\,k{a_2}^2 - 6\,k^2 a_2 = 0, \tag{3.5}$$

$$-\,k a_1 a_2 - 2\,k^2 a_1 = 0, \tag{3.6}$$

$$\left(k^2 - 1\right) a_2 - \frac{1}{2}\,k\left({a_1}^2 + 2\,a_0 a_2\right) - 8\,k^2 a_2 b = 0, \tag{3.7}$$

$$\left(k^2 - 1\right) a_1 - \frac{1}{2}\,k\left(2\,a_0 a_1 + 2\,a_2 b_1\right) - 2\,k^2 a_1 b = 0, \tag{3.8}$$

$$\left(k^2 - 1\right) a_0 - \frac{1}{2}\,k\left({a_0}^2 + 2\,a_2 b_2 + 2\,a_1 b_1\right) - k^2\left(2\,b_2 + 2\,a_2 b^2\right) = 0, \tag{3.9}$$

$$\left(k^2 - 1\right) b_1 - \frac{1}{2}\,k\left(2\,a_0 b_1 + 2\,a_1 b_2\right) - 2\,k^2 b_1 b = 0, \tag{3.10}$$

$$\left(k^2-1\right)b_2-\frac{1}{2}k\left({b_1}^2+2\,a_0b_2\right)-8\,k^2b_2b=0, \tag{3.11}$$

$$-\,kb_1b_2-2\,k^2b_1b^2=0, \tag{3.12}$$

$$-\frac{1}{2}k{b_2}^2-6\,k^2b_2b^2=0. \tag{3.13}$$

Solving above system by using Maple 16, we get

case 1.

$$b=-\frac{1}{4}\frac{k^2-1}{k^2},a_0=3\,\frac{k^2-1}{k},a_1=0,a_2=-12\,k,b_1=0,b_2=0.$$

case 2.

$$b=\frac{1}{4}\frac{k^2-1}{k^2},a_0=\frac{1-k^2}{k},a_1=0,a_2=-12\,k,b_1=0,b_2=0.$$

case 3.

$$b=-\frac{1}{16}\frac{k^2-1}{k^2},a_0=\frac{3}{2}\frac{k^2-1}{k},a_1=0,a_2=-12\,k,b_1=0,b_2=-\frac{3}{64}\frac{k^4-2\,k^2+1}{k^3}.$$

case 4.

$$b=\frac{1}{16}\frac{k^2-1}{k^2},a_0=\frac{1}{2}\frac{k^2-1}{k},a_1=0,a_2=-12\,k,b_1=0,b_2=-\frac{3}{64}\frac{k^4-2\,k^2+1}{k^3}.$$

case 5.

$$b=-\frac{1}{4}\frac{k^2-1}{k^2},a_0=\frac{3\left(k^2-1\right)}{k},a_1=0,a_2=0,b_1=0,b_2=-\frac{3}{4}\frac{\left(k^2-1\right)^2}{k^3}.$$

case 6.

$$b=\frac{1}{4}\frac{k^2-1}{k^2},a_0=-\frac{k^2-1}{k},a_1=0,a_2=0,b_1=0,b_2=-\frac{3}{4}\frac{\left(k^2-1\right)^2}{k^3}.$$

So that, we will study each case and get the exact traveling wave solution and also the solitary wave solutions for Eq. (3.3).

For Case 1.

The exact traveling wave solution be in the form:

$$v(\xi)=3\,\frac{k^2-1}{k}-12\,k\,\phi^2, \tag{3.14}$$

the solitary wave solution be in the form:

case i. If $b<0$, we get

$$v(\xi)=3\,\frac{k^2-1}{k}-12\,k\,\left(-\sqrt{-b}\;tanh(\sqrt{-b}\;\xi)\right)^2,$$

or

$$v(\xi)=3\,\frac{k^2-1}{k}-12\,k\,\left(-\sqrt{-b}\;coth(\sqrt{-b}\;\xi)\right)^2.$$

case ii. If $b>0$, we get

$$v(\xi)=3\,\frac{k^2-1}{k}-12\,k\,\left(\sqrt{b}\;tan(\sqrt{b}\;\xi)\right)^2,$$

or

$$v(\xi) = 3\,\frac{k^2-1}{k} - 12\,k\,\left(\sqrt{b}\;cot(\sqrt{b}\;\xi)\right)^2.$$

case iii. If $b=0$, we get

$$v(\xi) = 3\,\frac{k^2-1}{k} - 12\,k\,\left(\frac{1}{\xi}\right)^2.$$

For Case 2.
The exact traveling wave solution be in the form:

$$v(\xi) = \frac{k^2-1}{k} - 12k\phi^2, \tag{3.15}$$

the solitary wave solution be in the form:
case i. If $b<0$, we get

$$v(\xi) = \frac{1-k^2}{k} - 12k\left(-\sqrt{-b}\;tanh(\sqrt{-b}\;\xi)\right)^2,$$

or

$$v(\xi) = \frac{1-k^2}{k} - 12k\left(-\sqrt{-b}\;coth(\sqrt{-b}\;\xi)\right)^2,$$

case ii. If $b>0$, we get

$$v(\xi) = \frac{1-k^2}{k} - 12k\left(\sqrt{b}\;tan(\sqrt{b}\;\xi)\right)^2,$$

or

$$v(\xi) = \frac{1-k^2}{k} - 12k\left(\sqrt{b}\;cot(\sqrt{b}\;\xi)\right)^2,$$

case iii. If $b=0$, we get

$$v(\xi) = \frac{1-k^2}{k} - 12k\left(\frac{1}{\xi}\right)^2.$$

For Case 3.
The exact traveling wave solution be in the form:

$$v(\xi) = \frac{3}{2}\frac{k^2-1}{k} - 12k\phi - \frac{3}{64}\frac{k^4-2k^2+1}{k^3}\frac{1}{\phi^2}, \tag{3.16}$$

the solitary wave solution be in the form:
case i. If $b<0$, we get

$$v(\xi) = \frac{3}{2}\frac{k^2-1}{k} + 12k\sqrt{-b}\;tanh(\sqrt{-b}\;\xi) - \frac{3}{64}\frac{k^4-2k^2+1}{k^3}\frac{1}{\left(-\sqrt{-b}\;tanh(\sqrt{-b}\;\xi)\right)^2},$$

or

$$v(\xi) = \frac{3}{2}\frac{k^2-1}{k} + 12k\sqrt{-b}\;coth(\sqrt{-b}\;\xi) - \frac{3}{64}\frac{k^4-2k^2+1}{k^3}\frac{1}{\left(-\sqrt{-b}\;coth(\sqrt{-b}\;\xi)\right)^2},$$

case ii. If $b>0$, we get

$$v(\xi) = \frac{3}{2}\frac{k^2-1}{k} - 12k\sqrt{b}\;tan(\sqrt{b}\;\xi) - \frac{3}{64}\frac{k^4-2k^2+1}{k^3}\frac{1}{\left(\sqrt{b}\;tan(\sqrt{b}\;\xi)\right)^2},$$

or

$$v(\xi) = \frac{3}{2}\frac{k^2-1}{k} - 12k\sqrt{b}\ coth(\sqrt{b}\ \xi) - \frac{3}{64}\frac{k^4-2k^2+1}{k^3}\frac{1}{\left(-\sqrt{-b}\ cot(\sqrt{b}\ \xi)\right)^2},$$

case iii. If $b=0$, we get

$$v(\xi) = \frac{3}{2}\frac{k^2-1}{k} - 12k\left(\frac{1}{\xi}\right) - \frac{3}{64}\frac{k^4-2k^2+1}{k^3}\frac{1}{\left(\frac{1}{\xi}\right)^2},$$

For Case 4.
The exact traveling wave solution be in the form:

$$v(\xi) = \frac{1}{2}\frac{k^2-1}{k} - 12k\phi - \frac{3}{64}\frac{k^4-2k^2+1}{k^3}\frac{1}{\phi^2}, \tag{3.17}$$

the solitary wave solution be in the form:
case i. If $b<0$, we get

$$v(\xi) = \frac{1}{2}\frac{k^2-1}{k} + 12k\sqrt{-b}\ tanh(\sqrt{-b}\ \xi) - \frac{3}{64}\frac{k^4-2k^2+1}{k^3}\frac{1}{(-\sqrt{-b}\ tanh(\sqrt{-b}\ \xi))^2},$$

or

$$v(\xi) = \frac{1}{2}\frac{k^2-1}{k} + 12k\sqrt{-b}\ coth(\sqrt{-b}\ \xi) - \frac{3}{64}\frac{k^4-2k^2+1}{k^3}\frac{1}{(-\sqrt{-b}\ coth(\sqrt{-b}\ \xi))^2},$$

case ii. If $b>0$, we get

$$v(\xi) = \frac{1}{2}\frac{k^2-1}{k} - 12k\sqrt{b}\ tan(\sqrt{b}\ \xi) - \frac{3}{64}\frac{k^4-2k^2+1}{k^3}\frac{1}{\left(\sqrt{b}\ tan(\sqrt{b}\ \xi)\right)^2},$$

or

$$v(\xi) = \frac{1}{2}\frac{k^2-1}{k} - 12k\sqrt{b}\ coth(\sqrt{b}\ \xi) - \frac{3}{64}\frac{k^4-2k^2+1}{k^3}\frac{1}{\left(-\sqrt{-b}\ cot(\sqrt{b}\ \xi)\right)^2},$$

case iii. If $b=0$, we get

$$v(\xi) = \frac{1}{2}\frac{k^2-1}{k} - 12k\left(\frac{1}{\xi}\right) - \frac{3}{64}\frac{k^4-2k^2+1}{k^3}\frac{1}{\left(\frac{1}{\xi}\right)^2},$$

For Case 5.
The exact traveling wave solution be in the form:

$$v(\xi) = \frac{3\left(k^2-1\right)}{k} - \frac{3}{4}\frac{\left(k^2-1\right)^2}{k^3}\frac{1}{\phi^2}, \tag{3.18}$$

the solitary wave solution be in the form:
case i. If $b<0$, we get

$$v(\xi) = \frac{3\left(k^2-1\right)}{k} - \frac{3}{4}\frac{\left(k^2-1\right)^2}{k^3}\frac{1}{(-\sqrt{-b}\ tanh(\sqrt{-b}\ \xi))^2},$$

or

$$v(\xi)=\frac{3\left(k^2-1\right)}{k}-\frac{3}{4}\frac{\left(k^2-1\right)^2}{k^3}\frac{1}{\left(-\sqrt{-b}\;coth(\sqrt{-b}\;\xi)\right)^2}.$$

case ii. If $b>0$, we get

$$v(\xi)=\frac{3\left(k^2-1\right)}{k}-\frac{3}{4}\frac{\left(k^2-1\right)^2}{k^3}\frac{1}{\left(\sqrt{b}\;tan(\sqrt{b}\;\xi)\right)^2},$$

or

$$v(\xi)=\frac{3\left(k^2-1\right)}{k}-\frac{3}{4}\frac{\left(k^2-1\right)^2}{k^3}\frac{1}{\left(\sqrt{b}\;cot(\sqrt{b}\;\xi)\right)^2}.$$

case iii. If $b=0$, we get v$(\xi)=\frac{3\left(k^2-1\right)}{k}-\frac{3}{4}\frac{\left(k^2-1\right)^2}{k^3}\frac{1}{\left(\frac{1}{\xi}\right)^2}$.

For Case 6.

$$v(\xi)=\frac{-\left(k^2-1\right)}{k}-\frac{3}{4}\frac{\left(k^2-1\right)^2}{k^3}\frac{1}{\phi^2}, \tag{3.19}$$

the solitary wave solution be in the form:
case i. If $b<0$, we get

$$v(\xi)=\frac{-\left(k^2-1\right)}{k}-\frac{3}{4}\frac{\left(k^2-1\right)^2}{k^3}\frac{1}{\left(-\sqrt{-b}\;tanh(\sqrt{-b}\;\xi)\right)^2},$$

or

$$v(\xi)=\frac{-\left(k^2-1\right)}{k}-\frac{3}{4}\frac{\left(k^2-1\right)^2}{k^3}\frac{1}{\left(-\sqrt{-b}\;coth(\sqrt{-b}\;\xi)\right)^2}.$$

case ii. If $b>0$, we get

$$v(\xi)=\frac{-\left(k^2-1\right)}{k}-\frac{3}{4}\frac{\left(k^2-1\right)^2}{k^3}\frac{1}{\left(\sqrt{b}\;tan(\sqrt{b}\;\xi)\right)^2},$$

or

$$v(\xi)=\frac{-\left(k^2-1\right)}{k}-\frac{3}{4}\frac{\left(k^2-1\right)^2}{k^3}\frac{1}{\left(\sqrt{b}\;cot(\sqrt{b}\;\xi)\right)^2}.$$

case iii. If $b=0$, we get v$(\xi)=\frac{-\left(k^2-1\right)}{k}-\frac{3}{4}\frac{\left(k^2-1\right)^2}{k^3}\frac{1}{\left(\frac{1}{\xi}\right)^2}$.

Remark:
`All the obtained results have been checked with Maple 16 by putting them back into the original equation and found correct.`

4 Conclusion

The extended tanh function method has been applied in this paper to find the exact traveling wave solutions and then the solitary wave solutions of the symmetric regularized long-wave equation . Let us compare between our results obtained in the present article with the well-known results obtained by other authors using different methods as follows: Our results of the symmetric regularized long-wave equation are new and different from those obtained in [28]. The obtained exact solutions can be used as benchmarks against the numerical simulations in theoretical physics and fluid mechanics.

References

[1] W. Malfliet, Solitary wave solutions of nonlinear wave equation, Am. J. Phys., 60 (1992) 650-654.

[2] W. Malfliet, W. Hereman, The tanh method: Exact solutions of nonlinear evolution and wave equations, Phys.Scr., 54 (1996) 563-568.

[3] A. M. Wazwaz, The tanh method for travelling wave solutions of nonlinear equations, Appl. Math. Comput., 154 (2004) 714-723.

[4] Nizhum Rahman, Md. Nur Alam, Harun-Or-Roshid, Selina Akter and M. Ali Akbar, Application of $exp(-\varphi(\xi))$expansion method to find the exact solutions of Shorma-Tasso-Olver Equation, African Journal of Mathematics and Computer Science Research Vol. 7(1), pp. 1-6, February, 2014.

[5] 2) Mahmoud A.E. Abdelrahman and Mostafa M.A. Khater, the $\exp(-\phi(\xi))$-expansion method and its Application for Solving Nonlinear Evolution Equations. International Journal of Science and Research (IJSR) ISSN (Online): 2319-7064, Volume 4 Issue 2, 2143-2146. February (2015).

[6] Mahmoud A.E. Abdelrahman, Emad H. M. Zahran Mostafa M.A. Khater, Exact traveling wave solutions for power law and Kerr law non linearity using the $\exp(-\phi(\xi))$-expansion method . (GJSFR Volume 14-F Issue 4 Version 1.0).

[7] 13) Mostafa M. A. Khater, Extended $\exp(-\phi(\xi))$-expansion method for Solving the Generalized Hirota-Satsuma Coupled KdV System,GJSFR-F Volume 15 Issue 7 Version 1.0 (2015).

[8] A. M. Wazwaz, A sine-cosine method for handling nonlinear wave equations, Math. Comput. Modelling, 40 (2004) 499-508.

[9] C. Yan, A simple transformation for nonlinear waves, Phys. Lett. A 224 (1996) 77-84.

[10] 8) Mostafa M. A. Khater, the Modified Simple Equation Method and its Applications in Mathematical Physics and Biology, Global Journal of Science Frontier Research: F Mathematics and Decision Sciences, Volume 15 Issue 4 Version 1.0 (2015).

[11] M. L. Wang, Exct solutions for a compound KdV-Burgers equation, Phys. Lett. A 213 (1996) 279-287.

[12] Emad H. M. Zahran and Mostafa M.A. Khater, The modified simple equation method and its applications for solving some nonlinear evolutions equations in mathematical physics, (Jokull journal- Vol. 64. Issue 5 - May 2014).

[13] Y. J. Ren, H. Q. Zhang, A generalized F-expansion method to find abundant families of Jacobi elliptic function solutions of the (2+1)-dimensional Nizhnik-Novikov-Veselov equation, Chaos Solitons Fractals, 27 (2006) 959-979.

[14] J. L. Zhang, M. L. Wang, Y. M. Wang, Z. D. Fang, The improved F-expansion method and its applications, Phys.Lett.A 350 (2006) 103-109.

[15] J. H. He, X. H. Wu, Exp-function method for nonlinear wave equations, Chaos Solitons Fractals 30 (2006) 700-708.

[16] H. Aminikhad, H. Moosaei, M. Hajipour, Exact solutions for nonlinear partial differential equations via Exp-function method, Numer. Methods Partial Differ. Equations, 26 (2009) 1427-1433.

[17] Z. Y. Zhang, New exact traveling wave solutions for the nonlinear Klein-Gordon equation, Turk. J. Phys., 32 (2008) 235-240.

[18] M. L. Wang, J. L. Zhang, X. Z. Li, The $(\frac{G'}{G})$- expansion method and travelling wave solutions of nonlinear evolutions equations in mathematical physics, Phys. Lett. A 372 (2008) 417-423.

[19] S. Zhang, J. L. Tong, W.Wang, A generalized $(\frac{G'}{G})$- expansion method for the mKdv equation with variable coefficients, Phys. Lett. A 372 (2008) 2254-2257.

[20] E.H.M.Zahran and mostafa M.A. khater, Exact solutions to some nonlinear evolution equations by the $(\frac{G'}{G})$ -expansion method equations in mathematical physics, J$\ddot{o}$kull Journal, Vol. 64, No. 5; May 2014.

[21] C. Q. Dai , J. F. Zhang, Jacobian elliptic function method for nonlinear differential difference equations, Chaos Solutions Fractals, 27 (2006) 1042-1049.

[22] E. Fan , J .Zhang, Applications of the Jacobi elliptic function method to special-type nonlinear equations, Phys. Lett. A 305 (2002) 383-392.

[23] S. Liu, Z. Fu, S. Liu, Q.Zhao, Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations, Phys. Lett. A 289 (2001) 69-74.

[24] Emad H. M. Zahran and Mostafa M.A. Khater, Exact Traveling Wave Solutions for the System of Shallow Water Wave Equations and Modified Liouville Equation Using Extended Jacobian Elliptic Function Expansion Method, American Journal of Computational Mathematics (AJCM) Vol.4 No.5 (2014).

[25] S. A. EL-Wakil, M.A.Abdou, New exact travelling wave solutions using modified extented tanh-function method, Chaos Solitons Fractals, 31 (2007) 840-852.

[26] E. Fan, Extended tanh-function method and its applications to nonlinear equations, Phys. Lett. A 277 (2000) 212-218.

[27] Mahmoud A.E. Abdelrahman, Emad H. M. Zahran Mostafa M.A. Khater, Exact Traveling Wave Solutions for Modified Liouville Equation Arising in Mathematical Physics and Biology, (International Journal of Computer Applications (0975 8887) Volume 112 - No. 12, February 2015).

[28] Fakir Chand , Anand K Malik, Exact Traveling Wave Solutions of Some Nonlinear Equations Using $\left(\frac{G'}{G}\right)$-expansion Method methods, International Journal of Nonlinear Science Vol.14(2012) No.4,pp.416-424.