Sumudu decomposition method for solving fractional-Multi-order equation

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Abstract

In This paper, we propose a numerical algorithm for solving fractional-Multi-order equation by using Sumudu decomposition method (SDM). This method is a combination of the Sumudu transform method and decomposition method. We have apply the concepts of fractional calculus to the well known population growth modle inchaotic dynamic. The fractional derivative is described in the Caputo sense. The numerical results shows that the approach is easy to implement and accurate when applied to various fractional differentional equations.

Keywords: Caputo derivative; Adomian polynomials; Multi-order equation; Sumudu transform method; decomposition method.

1 INTRODUCTION

Ordinary and partial fractional differential equations have been the focus of many studies due to their frequent appearance in various applications in .uid mechanics, viscoelasticity, biology, physics and engineering [3]. Recently, a large amount of literatures developed concerning the application of fractional differential equations in non-linear dynamics. Consequently,considerable attentions have been given to the solutions of fractional differential equations of physical interest. Most fractional differential equations do not have exact solutions, so approximate and numerical techniques (see [9], [10],[15-18]), must be used. Recently, several numerical and approximate methods to solve the fractional differential equations have been given such as variational iteration method [23], homotopy perturbation method [24], Adomian decomposition method, homotopy analysis method, homotopy perturbation Sumdu transform method [14,20] and collocation method (see [19], [25]). Inspired and motivated by the ongoing research in this area, we introduce a new method called sumudu decomposition method (SDM) for solving the nonlinear equations in the present paper. It is worth mentioning that the proposed method is an elegant combination of the sumudu transform method and decomposition method which was rst introduced by Adomian [1, 2]. The proposed scheme provides the solution of the problem in a closed form while the mesh point techniques, such as Sumudu decomposition method (see[8], [12], [13]): The proposed algorithm provides the solution in a rapid convergent series which may lead to the solution in a closed form. This article considers the effectiveness of the sumudu decomposition method (SDM) in solving nonlinear fractional Multi-order equations.

2 Basic Definitions of Fractional Calculus

In this section, we present the basic de.nitions and properties of the fractional calculus theory, which are used further in this paper.

Definition 1 The Riemann-Liouville fractional integral operator of order $\alpha > 0$; for $t > 0$ is defined as [22]

$$\begin{aligned} J^{\alpha} f(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} \, f\,(\xi)\, d\,\xi, \\ J^{0} f(t) &= f(t). \end{aligned} \tag{1}$$

The Riemann-liouville derivative has certain disadvantage when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator D^{α} proposed by M Caputo in his work on the theory of viscoelasticity [5]

Definition 2 The Caputo fractional derivative of $f(t)$ of order $\alpha > 0$ with $t > 0$ is defined as [6]

$$\begin{aligned} D^{\alpha} f(t) &= J^{m-\alpha} D^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\xi)^{m-\alpha-1} f^{(m)}(\xi) d \\ form-1 &< \alpha \le m, m \in N, t > 0. \end{aligned} \tag{2}$$

Definition 3 The Sumudu transform is defined over the set of functions [26]

$$A = \left\{ f(t) \,\middle|\, \begin{array}{c} \exists, T_1, T_2 > 0, |f(t)| < M \, e^{\frac{|t|}{\tau_j}} \\ if \; t \in (-1)^j \times [0,\infty) \end{array} \right\}, \tag{3}$$

by the following formula:

$$f^{'}(u) = S\,[f(t)] = \int_0^1 f(ut)\, e^{-t}\, dt, \qquad u \in (T_1, T_2). \tag{4}$$

Definition 4 The Sumudu transform of Caputo fractional derivative is defined as follows [7]

$$S\,[D^{\alpha} f(t)] = u^{\alpha} S\,[f(t)] - \sum_{k=0}^{m-1} u^{-\alpha+k}\, f^{(k)}(0), \qquad m-1 < \alpha \le m. \tag{5}$$

3 Analysis of the SDM and FMOE

Consider the following nonlinear-Multi-order equation

$$D^{\alpha}y(t) = 1 - y^{2}(t) \qquad t\ > 0 \tag{6}$$

the parameter with α refers the fractional order of time derivatve with $0 < \alpha \leq 1$ and subject to the initial condition

$$y(0) = y^{0}, \quad y^{0} > 0. \tag{7}$$

where$D^{\alpha}y(t)$ is the Caputo fractional derivative, $x(t)$ represents the population size, t represents the time.

Taking the Sumudu transform (denoted throughout this paper by S) on both sides of **Eq**.(6), we have

$$S\left[D^{\alpha}y(t)\right] = S\left[1 - y^{2}(t)\right], \tag{8}$$

Using the differentiation property of the Sumudu transform and the initial conditions in **Eq**.(8), we have

$$S\left[y(t)\right] = y^{0} + u^{\alpha}S\left[1 - y^{2}(t)\right], \tag{9}$$

Operating with the Sumudu inverse on both sides of **Eq**.(9) we get

$$y(t) = F(t) + S^{-1}\left[u^{\alpha}S\left[1 - y^{2}(t)\right]\right], \tag{10}$$

where $F(t)$ represent the prescribed initial conditions.

Now, pplying SDM. And assuming that the solution of **Eq**.(10) is in the form

$$y(t) = \sum_{m=0}^{\infty} y_m(t), \tag{11}$$

and the nonlinear term of **Eq**.(10) can be decomposed as

$$N\ y(t) = \sum_{m=0}^{\infty} A_m(t), \tag{12}$$

where A_m are He.s polynomials, which can be calculated with the formula [4,7,11] :

$$A_m = \frac{1}{m!}\frac{d^m}{dp^m}\left[N\ \left(\sum_{i=0}^{\infty} p^i\ y_i\ (t)\right)\right]_{p=0}, \quad m = 0, 1, 2, ... \tag{13}$$

$$
\begin{aligned}
A_0 &= f(u_0), \\
A_1 &= u_1 f^{'}(u_0), \\
A_2 &= u_2 f^{'}(u_0) + \frac{1}{2!} u_1^2 f^{''}(u_0), \\
A_3 &= u_3 f^{'}(u_0) + u_1 u_2 f^{''}(u_0) + \frac{1}{3!} u_1^3 f^{'''}(u_0),
\end{aligned}
$$

The first few components of Adomian polynomials, are given by

$$
\begin{aligned}
A_0 &= y_0^2, \\
A_1 &= 2y_0 y_1, \\
A_2 &= 2y_0 y_1 + y_1 y_1, \\
& . \\
& . \\
& .
\end{aligned}
$$

Substituting **Eq**.(11) and (12) in **Eq**.(10), we get

$$
\sum_{m=0}^{\infty} y_m(t) = F(t) + S^{-1}\left[u^{\alpha} S\left[1 - \left(\sum_{m=0}^{\infty} A_m(t)\right)\right]\right], \tag{14}
$$

On comparing both sides of **Eq**.(14); we get

$$
\begin{aligned}
y_0(t) &= F(t), \\
y_1(t) &= S^{-1}\left[u^{\alpha} S\left[1 - A_0\right]\right], \\
y_2(t) &= S^{-1}\left[u^{\alpha} S\left[1 - A_1\right]\right], \\
& . \\
& . \\
.y_{m+1}(t) &= S^{-1}\left[u^{\alpha} S\left[1 - A_m\right]\right],
\end{aligned} \tag{15}
$$

4 Approximate solution of the FMOE

We start with the initial approximate $y_0(t) = 0.9$; and by using the SDM **Eq**.(15); we can directly obtain the components of the solution. Consequently, the exact solution may be obtained by using (11)On comparing both sides of (15) we get

$$
\begin{aligned}
y_0(t) &= 0.9,\\
y_1(t) &= \frac{0.19t^{\alpha}}{\Gamma(\alpha+1)},\\
y_2(t) &= \frac{\Gamma(\alpha+1)t^{\alpha}-0.342t^{2\alpha}}{\Gamma(2\alpha+1)},\\
y_3(t) &= \frac{\Gamma(\alpha+1)\Gamma^2(\alpha+1)t^{\alpha}-(0,342t^{2\alpha}\Gamma^2(\alpha+1)+0,361t^{3\alpha}\Gamma(\alpha+1))}{\Gamma(2\alpha+1)\Gamma^2(\alpha+1)},\\
&\vdots
\end{aligned}
$$

If we $\alpha \to 1$ in **Eq.**(16)or solve **Eq.**(6) and (7) with $\alpha = 1$; we obtain

$$y(t) = 0.9 + 1.19t - 0.342t^2 - 0.1805t^3 + ...$$

The numerical results of the proposed problem (6) are given in Figures 1 and 2 with different values of α in the interval $[0;3]$ with $M = 3$ and $Y\ _0 = 0.9$ Where in Figure 1, we presented a comparison between the behavior of the exact solution and the approximate solution using the introduced technique at $\alpha = 1$ (Figure 1(a)),

5 Conclusions

This present analysis exhibits the applicability of the Sumudu decomposition method to solve fractional-Multi-order equation. The work emphasized our belief that the method is a reliable technique to handle linear and nonlinear fractional differential equations. It provides the solutions in terms of convergent series with easily computable components in a direct way without using linearization, restrictive assumptions. The numerical results obtained with the proposed techniques are in an excellent agreement with the exact solution. All numerical results are obtained using Maple 16.

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