Decomposition of complete 3-uniform Hypergraphs into cycles of length n/2

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Abstract

A k-uniform hypergraph H is a pair (V, ε), where $V = \{v_1, v_2, \ldots, v_n\}$ is a set of n vertices and ε is a family of k-subset of V called hyperedges. A cycle of length l of H is a sequence of the form $(v_1, e_1, v_2, e_2, \ldots, v_l, e_l, v_1)$, where $v_1, v_2, \ldots, v_l$ are distinct vertices, and $e_1, e_2, \ldots, e_l$ are k-edges of H and $v_i, v_{i+1} \in e_i, 1 \le i \le l$, where addition on the subscripts is modulo n, $e_i \ne e_j$ for $i \ne j$. In this paper we show the decomposition of complete 3-uniform hypergraph K^3_{2m} into cycles of length m for m be prime.

Keywords: Hypergraph, Cycle, Hamilton Cycle.

1 Introduction

A k-uniform hypergraph H is a pair (V, ε), where $V = \{v_1 v_2, \ldots, v_n\}$ is a set of n vertices and ε is a family of k-subset of V called hyperedges. If ε consists of all k-subsets of V, then H is a complete k-uniform hypergraph on n vertices and is denoted by K^k_n. At the same time we may refer a vertex v_i to v_{i+n}. A cycle of length l of H is a sequence of the form

$$(v_1, e_1, v_2, e_2, \ldots, v_l, e_l, v_1),$$

where $v_1, v_2, \ldots, v_l$ are distinct vertices, and $e_1, e_2, \ldots,\ e_l$ are k-edges of H, satisfying

(i) $v_i,\ v_{i+1} \in e_i,\ 1 \le i \le l$, where addition on the subscripts is modular n, and

(ii) $e_i \ne e_j$ for $i \ne j$. This cycle is known as a Berge cycle[1]. A decompostion of H into cycles of length l is a partition of the hyperedges of H into cycles of length l.

A set of cycles of length l of complete 3-uniform hypergraph K^3_n, say $C_1, C_2, \ldots, C_m$, is called decompostion into cycles of length l if $\bigcup_{i=1}^{m} \varepsilon(C_i) = \varepsilon(K^3_n)$ and $\varepsilon(C_i) \cap \varepsilon(C_j) = \emptyset$ for $i \ne j$. In this paper, we use combinatorial method to distinguish and give the decomposition of complete hypergragh K^3_{2m} into cycles of length m with m be prime.

2 Results

For $1 \le r \le \dfrac{n}{2} - 1$, $n = 2m$, m is prime, let $D = D_{2r} \cup D_1 \cup D_{2m}$, where

$D_{2r} = \{(k_1, k_2):\ (k_1, k_2)$ is ordered odd 2-partition of $2r\}$,

$D_1 = \{(k_1, k_2):\ (k_1, k_2)$ is ordered even 2-partition of $2r$ and $1 \le k_1 < k_2 < \frac{n-k_1}{2}\}$,

$D_{2m} = \{(k_1, k_2):\ (k_1, k_2)$ is unordered even and $2k_1 + k_2 = 2m\}$.

Obviously,

$$| D_{2r} |= r, | D_1 |= 2 + 2(r - 7), | D_{2m} |= \frac{m-1}{2}.$$

$$| D_1 |= \sum_{i=1}^{m} | D_{2i} |= \sum_{i=1}^{m-1} i = \frac{m(m-1)}{2}, | D_2 |= \frac{(m-5)(m-1)}{6}, | D_{2m} |= \frac{(m-1)}{2}$$

Let $m > 3$, m be prime, for any $(k, l) \in D$ we consider two distinct sequence $C(i; k, l)(i = 0, 1)$ of triangles of $V = \{0,\ 1,\ \ldots,\ 2m - 1\}$ as follows: Given a $(k, l) \in D$ and integer j, define

$$C(i; k, l) = \{e_{ij} :\ \ i = 0,\ 1;\ j = 0,\ 1,\ 2,\ \ldots,\ m - 1\} \pmod{2m}, \tag{1}$$

$$e_{ij} = \{i + j(k + l), i + j(k + l) + k, i + (j + 1)(k + l)\} \pmod{2m}. \tag{2}$$

Lemma 1 Let $m > 3$ be a prime, then for any $(k, l) \in D$, $e_{ij}(k, l) = e_{ij'}(k, l)$ if and only if $j \equiv j' \pmod{m}$ for $i = 0, 1$.

Proof. Put $r = k + l$, by definition it is easy to see that

$$e_{ij+m} = \{i + (j + m)r, i + (j + m)r + k, i + (j + m + 1)r\}$$
$$\equiv \{i + jr, i + jr + k, i + (j + 1)r\} \pmod{m}$$

namely for $i = 0, 1$,

$$e_{ij+m}(k, r) \equiv e_{ij}(k, r) \pmod{m}\ .$$

Suppose $e_{ij}(k, l) = e_{ij'}(k, l)$ with $1 \le j, j' \le m - 1$. Set $t = j' - j$, we consider two cases:

Case 1. When $(k, l) \in D_1 \cup D_{2m}$, we have that

$$\{i + jr, i + jr + k, i + (j + 1)r\} \equiv \{i + j'r, i + j'r + k, i + (j' + 1)r\} \pmod{2m},$$

which implies that $\{0, r, k\} \equiv \{t\,r, tr + k, (t + 1)r\} \pmod{2m}$. If $tr \ne 0 \pmod{2m}$, (equivalently, $tl + k \ne k \pmod{2m}$ and $(t + 1)r \ne r \pmod{2m}$), then there are two subcases:

(i) $tr \equiv k \pmod{2m}$ $tr + k \equiv r \pmod{2m}$ and $(t + 1)r \equiv 0 \pmod{2m}$;

(ii) $tr \equiv r \pmod{2m}$ $tr + k \equiv 0 \pmod{2m}$ and $(t + 1)r \equiv k \pmod{2m}$.

Both cases imply that $3k \equiv 0 \pmod{2m}$, a contradiction. It shows that $tr \equiv 0 \pmod{2m}$. Recall that k and l are even, i.e., tr is even, so $\dfrac{tr}{2} \equiv 0 \pmod{m}$, which implies that $j \equiv j'$ $\pmod{m}$.

Case 2. When $(k_1, k_2) \in D_{2r}$. We have

$$\{i + jr, i + jr + k, i + (j + 1)r\} \equiv \{i + j'r, i + j'r + k, i + (j' + 1)r\} \pmod{2m}$$

which implies that $\{0,k,r\}\equiv\{tr,tr+k,(t+1)r\} \pmod{2m}$. If $tr\not\equiv 0 \pmod{2m}$, (equivalently, $tr+k\not\equiv k \pmod{2m}$ and $(t+1)r\not\equiv l \pmod{2m}$, then there are two subcases:

(i) $tr\equiv k \pmod{2m}, tr+k\equiv r \pmod{2m}$ and $(t+1)r\equiv 0 \pmod{2m}$;

(ii) $tr\equiv r \pmod{2m}, tr+k\equiv 0 \pmod{2m}$ and $(t+1)r\equiv k \pmod{2m}$.

From (i) we have $3k\equiv 0 \pmod{2m}$, a contradiction. From (ii) we have $3r\equiv 0 \pmod{2m}$, a contradiction. It shows that $tr\equiv 0 \pmod{2m}$. Since $(k,\ell)\in D_{2r}$, i.e., tr is even, so $\frac{tr}{2}\equiv 0 \pmod{m}$, which implies that $j\equiv j^{'} \pmod{m}$.

Lemma 2. Let $m>3$ be a prime and K_{2m}^3 is a complete 3-uniform hypergraph on $V=\{0,1,\dots,2\,m-1\}$, Then every edge sequence $C(i;k,l)$ defined in (1) and (2) is a cycle of length m for $i=0,1$.

Proof. For any $(k,l)\in D$ by the definition of e_{ij}, and from Lemma 1, we can see that, for any two edges $e_j\ e_{j'}$ in $C(i;k,l)$, we know exactly that $e_j\neq e_{j'}$ if and only if $j\neq j'$ (mod m) and $j=0,1,\dots,m-1$, so $|C(i;k,l)|=m$. For any $(k,l)\in D$, by the definition of $C(i;k,l)$, then $e_{ij}\cap e_{i(j+1)}=\{i+(j+1)(k+l)\}$ for $j=0,1,\dots,m-1$ and each $e_{ij}\cap e_{i(j+1)}$ is different from other for different j. Suppose $i+(j+1)(k+l)\equiv i+(j^{'}+1)(k+l) \pmod{2m}$ and $0\le j,j'\le m-1$ for $i=0,1$, then $j(k+l)\equiv j^{'}(k+l) \pmod{2m}$, since $k+l$ is even, so $j\equiv j' \pmod{m}$, i.e., $\{e_j\cap e_{j+1}\ \ j=0,1,\dots,m-1\} \pmod{m}$ are distinct vertices of V. It satisfies the following two conditions:

(1) $j(k+l) \pmod{m}, (j+1)(k+l) \pmod{m}\in e_j$. $0\le j\le m-1$,

(2) $e_i\neq e_j$, for $i\neq j$.

Which proves that $C(i;k,l)$ is a cycle of length m for $i=0,1$.

Lemma 3. Let (k,l) and $(k^{'},l^{'})$ be two distinct pairs of D. Then $C(i;k,l)\cap C(i;k^{'},\ell^{'})=\varnothing$ for $i=0,1$.

Proof. By the definition of $C(i;k,l)$, put the reduced residues modulo $2m$ equidistantly and clockwise on a circle. Take three of them, say, a,b and c, then $\{a,b,c\}\in C(i;k,l)$ for some $(k,l)\in D$ if and only if the spaces among the three elements are in turn k, l and $2m-(k+l)$. Therefore, if $e_j(k,l)=e_j^{'}(k^{'},l^{'})$ then the cycle permutations $k,l,2m-(k+l)=k^{'},l^{'},2m-(k^{'}+l^{'})$. We now discuss the following cases

Case 1: Let $(k,l),(k^{'}l^{'})\in D_{2r}$. Note that there are only $2m-(k+l)$ and $2m-(k^{'}+l^{'})$ are even. We therefore obtain that $k=k^{'}$ and $2m-(k+l)=2m-(k^{'}+l^{'})$, which yields that $(k,l)=(k^{'}l^{'})$.

Case 2: Let $(k,l),(k^{'}l^{'})\in(D_m\cup D_1)$. Suppose $(k,l)\neq(k^{'}l^{'})$. put $k=l$ and $k^{'}=l$, by the define of D_m and D_1 there are $k<l<\frac{2m-k}{2}$ and $k^{'}<l^{'}<\frac{2m-k^{'}}{2}$, such that $k<l=k^{'}$ and $k^{'}<l^{'}=k$, a contradiction. put $k=2m-k^{'}-l^{'}$ and $k^{'}=2m-k-l$.

Note that there are only $2m-(k+l)$ and $2m-(k^{'}+l^{'})$ are even. We therefore obtain that $k=k^{'}$ and $2m-(k+l)=2m-(k^{'}+l^{'})$, which yields that $(k,\ell)=(k^{'}l^{'})$.

Set $r=k+l,k,r=k+l$, then $\{k_1,k_1+k_2,m-(2k_1+k_2\,)\}=\{k,r-l,2m-r\}$, namely $e_{ij}=\{i+j(k_1+k_2),i+j(k_1+k_2)+k_1,i+(j+1)(k_1+k_2)\} \pmod{2m}=\{i+j\,l,i+j\,r,i+(j+1)\,l\} \pmod{2m}$. Therefore, if $e_{ij}(k,r)=e_{ij'}(k',r')$, (or $e_j(k,r)=e_{j'}(k',r')$, or $e_j(l,r)=e_{j'}(l',r')$), then the cycle permutations $(k,r-k,n-r)$ and $(k',r'-k',n-r')$ (or $(k,r,n-r)$ and $(l',r'-l',n-r')$ or $(l,r-l,n-r)$ and $(l',r'-l',n-r')$) are identical. We only need consider this case, if $e_j(k,l\,)=e_{j'}(k',l'\,)$, then the cycles permute $(k,r-k,n-r)$

$=(k',r'-k',n-r')$, i.e. $C(k,l\,)=C(k',l'\,)$. For other two cases, the discussions are similar. We now complete the proof by three cases.

Case 1: There are two subcases if $k,\ k'$ are odd:

(1) r and r' are odd,

(2) r and r' are even.

From the subcase (1) we know $n-r$, $n-r'$ are even, and either (i) $k=k'$, $n-r=nr'$ and $r-k=r'-k'$ or (ii) $k=k'$, $n-r=r'-k'$ and $r-k=n-r'$. By (i) we have that $(k,r)=(k',r')$, a contradiction. Sameness, (ii) implies that $(k,l\,),(k',l'\,)\in D_2$ and $n=k+l+l'$, therefore have $k+l+l'<n$, a contradiction.

From the subcase (2), we have that $n-r$ and $n-r'$ are odd, so we obtain that (i) $k=k'$, $r-k=r'-k'$ and $n-r=n-r'$, or (ii) $k=k'$, $r-k=n-r'$ and $n-r=r'-k'$ or (iii) $k=r'-k'$, $r-k=n-r'$ and $n-r=k'$, or (iv) $k=r'-k'$, $r-k=k'$ and $n-r=n-r'$ or (v) $k=n-r',r-k=r'-k'$ and $n-r=k'$, or (vi) $k=n-r'$, $r-k=k'$ and $n-r=r'-k'$.

By (i) we immediately have $(k,\ l\,)=(k',r')$, a contradiction. By (ii), we have the following: if $(k,l),(k',l'\,)\in D_1$, then $(k,\ r)=(k',r')$, a contradiction. If $(k,r)((k',r'))\in D_2$, and $(k',r')((k,r))\in D_2$, without loss of generality assume $(k,r)\in D_1$ and $(k',r')\in D_2$, we obtain that $r=l$, a contradiction. If $(k,l\,),(k',l'\,)\in D_2$, then $l>\frac{n-k}{2}$, a contradiction.

For (iii), (iv), (v) and (vi), the discussions are similar. Recall k are odd, so we have $e_j(k,l\,)=e_{j'}(k',l'\,)$, which implies that, $(k,r-k,n-r)$=$(k',r'-k',n-r')$. i.e., $C(k,l\,)=C(k',l'\,)$.

Case 2. There are three subcases if k,k' are even:

(1) r and r' are odd;

(2) r and r' are even;

(3) r (r') is odd and r' (r) is even.

For (1) we have that $n-r$ and $n-r'$ are even, so (i) $r-k=r'-k'$, $k=k'$ and $n-r=n-r'$ or (ii) $r-k=r'-k'$, $k=n-r'$ and $n-r=k'$.

By (i) we have $(k,r)=(k',r')$, a contradiction. Sameness, (ii) implies that: if $(k,l),(k',l'\,)\in D_1$, then $(k,l\,)=(k',l'\,)$, a contradiction. If $(k,l\,)((k',l'\,))\in D_2$, then $2n=r+r'+k+k')$, a contradiction. If $(k,r)((k',r'))\in D_2$, and $(k',\ l'\,)((k,l\,))\in D_2$, without loss of generality we assume that $(k,l\,)\in D_1$ and $(k',l')\in D_2$, we obtain that $n=2k'+l'$, a contradiction.

For (2), we have $n-r$ and $n-r'$ are odd, so we obtain (i) $n-r=n-r',k=k'$ and $r-k=r'-k'$ or (ii) $n-r=n-r',k=r'-k'$ and $r-k=k'$.

By the subcase (i) we have that $(k,l\,)=(k',l'\,)$, a contradiction. Sameness, the subcases (ii) implies that $(k,l\,),(k',l'\,)\in D_2$ and $(k,l\,)=(k,k'),(k',l'\,)=(l,k\,)$, therefore implies $k<l,\ l<k$, a contradiction.

For (3), without loss of generality we assume that r is odd and r' is even, we have that $n-r$ is even and $n-r'$ is odd, so we obtain (i) $r-k=r'-k',k=k'$ and $n-r=n-r'$ or (ii) $r-k=r'-k',k=n-r'$ and $n-r=k'$.

The subcase (i) implies that $(k',l')\in D_2$ and $k'=2l'$, a contradiction. The subcase (ii), imply that $(k,l)\in D_1$, and $n=3k+l$, a contradiction. Recall k,k' are even, so we have $e_j(k,l)=e_{j'}(k',l')$, which implies that, $(k,r-k,n-r)$=$(k',r'-k',n-r')$, i.e. , $C(k,l)=C(k',l')$.

Case 3. If $k(k')$ is odd and $k'(k)$ is even. Without loss of generality we assume k is odd and k' is even. Then there are two subcases:

(1) r is odd and r' is even;

(2) r and r' are odd.

For (1), we have $n-r$ is odd and $n-r'$ is even, so we obtain that (i) $k=n-r', r-k=k'$ and $n-r=r'-k'$ or (ii) $k=n-r'r-k=r'-k'$ and $n-r=k'$.

By the subcase (i) we have that $(k,l),(k',l')\in D_2$, and $2n=2k+l+2k'+l'$, a contradiction. Sameness, the subcases (ii) implies that $(k,l),(k',l')\in D_2$ and $n=k+l+l'$, therefore implies $(k,l),(k',l')\in D_2$, and $r=k$, a contradiction.

For (2), we have that $n-r$ and $n-r'$ are even, so we obtain that (i) $k=r'-k', -k=n-r'$ and $n-r=k'$, or (ii) $k=r'-k', r-k=k'$ and $n-r=n-r'$.

By the subcase (i) we have that $(k,l)\in D_2$, and $2n=2k+l+2k'+l'$, a contradiction. Sameness, the subcases (ii) implies that $(k,l)\in D_2$ and $k=l', k'=l$, a contradiction. Recall $k(k')$ is odd and $k'(k)$ is even, so we must have $e_j(k,l)=e_{j'}(k',l')$, which implies that, $(k,r-k,n-r)$=$(k',r'-k',n-r')$, i.e., $C(k,l)=C(k',l')$. The proof is completed. □

Theorem 1 *Let $n=2m, m$ be prime, then the decomposition*

$$K_n^3=\bigcup_{(k,\ l)\in D} C(k,\ l)\ \bigcup_{(k,\ l)\in D_2} C(l,\ k)$$

is a cycle of length m decomposition.

Proof. Let $V=\{0,1,\ldots,n-1\}$. By Lemma 3, for any $(k,l)\in D$, $C(l,k)$ is a Hamilton cycle of K_n^3. Therefore, we shall prove that

$$\{\,C(k,l),C(l,k):(k,l)\in D\,\}$$

is a decomposition of K_n^3 into Hamiltonian cycles. By lemma 1, $|\,D_2\,|=\frac{n^2-6n+5}{12}, |\,D\,|=\frac{n^2-1}{12}$ and by lemma 4, let (k,l) and (k',l') be two distinct pairs of D, then the cycle defined in (5) or (6) satisfying $C(k,l)\cap C(k',l')=\emptyset$ or $C(k,\ l)\cap C(l',k')=\emptyset$ or $C(l,k)\cap C(l',k')=\emptyset$, and because $|\ C(k,l)\ |=n, |\,C(l,k)\,|=n$, so

$$\begin{aligned}&|\,C(k,l)\,|\cdot|\,D\,|+|\,C(l,k)\,|\cdot|\,D_2\,|\\&=n\cdot\frac{n^2-1}{12}+n\cdot\frac{n^2-6n+5}{12}\\&=\frac{n\,(n-1)\,(n-2)}{3!},\end{aligned}$$

which equals the size of $|\ (K_n^3)\ |$, that is

$$K_n^3=\bigcup_{(k,l)\in D} C(k,l)\ \bigcup_{(k,l)\in D_2} C(l,k).$$

The proof is completed. □

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