On a General Even Order Structure on a Differentiable Manifold

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Abstract

K. Yano defined and studied the structures defined by a tensorfield $f(\neq 0)$ of type $(1, 1)$ satisfying $f^3 + f = 0, f^4 \pm f^2 = 0([1],[3])$. In this paper, we have considered the structure of order $2n$ defined by $(1, 1)$ tensorfield $f$ where $n$ is a positive integer. Certain interesting results have been obtained. Local coordinate system is introduced in the manifold and it has been shown that there exist complementary distributions $L^*$ and $M^*$ and a positive definite Riemannian metric $G$ such that they are orthogonal.

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1 General even order structure

Let \( M \) be an \( m \)-dimensional differentiable manifold of differentiability class \( C^\infty \). Suppose there exists on \( M \), a tensor field \( f \neq 0 \) of type \((1, 1)\) satisfying

\[
f^{2n} + af^n + bI_{2n} = 0
\]

where \( n \) is a positive integer \((n > 1)\), \( a,b \) scalars not equal to zero and \( I_{2n} \) denotes the unit tensor field.

Then we say that the manifold \( M \) is equipped with general even order structure. We now prove the following theorem:

**Theorem 1.1** The general even order structure is not unique.

**Proof.** Let \( \mu \) be a non-singular real valued function and \( f' \) a tensorfield of type \((1, 1)\) on \( M \) such that

\[
\mu f' = f \mu
\]

(1.2)

Then, by (1.2)

\[
\mu f'^2 = f(\mu f') = f^2 \mu
\]

In a similar manner, we have

\[
\mu f'^3 = f^3 \mu, \ldots, \mu f'^{2n} = f^{2n} \mu
\]

Therefore

\[
\mu(f'^2)^n + a\mu(f')^n + b\mu I_{2n} = (f^{2n})\mu + a(f)^n \mu + bI_{2n} \mu
\]

\[
= (f^{2n} + af^n + bI_{2n})\mu
\]

by (1.1)

\[
= 0
\]

Since \( \mu \) is non-singular we have

\[
(f')^{2n} + a(f')^n + bI_{2n} = 0
\]

Thus \( f' \) gives to \( M \) another general even order structure. Therefore, such structure is not unique. \( \square \)
Theorem 1.2 The rank of the general even order structure is equal to dimension of the manifold.

Proof. Let $M$ be of dimension $m$. If $X$ be a vector field on $M$ such that

$$f(X) = 0 \Rightarrow f^2(X) = f^3(X) = \ldots = f^n(X) = 0.$$ 

Also $f^{2n}(X) = 0$. Hence from (1.1) it follows that $X = 0$.

Hence kernel of $f$ contains only zero vector field. So if $\nu(f)$ be nullity of $f$, $\nu(f) = 0$.

If $\rho(f)$ be rank of $f$, then from a well known theorem of Linear Algebra

$$\rho(f) + \nu(f) = \text{dimension of } M$$

As $\nu(f) = 0$, therefore

$$\rho(f) = m$$

Hence we have the theorem.

\[ \square \]

Theorem 1.3 Let $f$ and $f'$ be two general even order structures on a differentiable manifold $M$ such that the equation (1.2) holds. If $V$ is an eigenvector of $f'$ corresponding to some eigenvalue, $\mu V$ is the eigenvector of $f$ corresponding to same eigenvalue.

Proof. As given, $V$ is the eigenvector of $f'$ for the eigenvalue $\lambda$. Then

$$f'V = \lambda V$$

Therefore

$$\mu f^n(V) = \mu(\lambda V)$$

or by (1.2)

$$f(\mu V) = \lambda(\mu V)$$

So $\mu V$ is the eigenvector of $f$ for the same eigenvalue $\lambda$.

\[ \square \]

Theorem 1.4 The dimension $m$ of the manifold $M$ equipped with general even order structure satisfying the equation (1.1) for $a^2 < 4b$ is even.
**Proof.** Let \( V \) be eigenvector of \( f \) corresponding to eigenvalue \( \lambda \). So

\[
 f(V) = \lambda V, \quad f^2(V) = \lambda^2 V, \ldots, f^n(V) = \lambda^n V, \ldots
\]

Hence by virtue of the equation (1.1), it follows that

\[
 \lambda^{2n} + a\lambda^n + b = 0
\]

which has solution of the form

\[
 \lambda^n = \frac{-a \pm \sqrt{a^2 - 4b}}{2}
\]

If \( a^2 < 4b \), the values of \( \lambda^n \) are complex. Hence the eigenvalues of \( f \) are complex numbers. Since complex roots occur in pair, hence number of the eigenvalues must be even. Consequently dimension of \( M \) is even and \( m = 2n \) as \( f \) has \( 2n \) non-zero distinct eigenvalues.

\[
 \square
\]

2 **Necessary and sufficient condition for existence of the general even order structure**

For the manifold \( M \) equipped with general even order structure, the eigenvalues of \( f \) are given by

\[
 \lambda^n = \frac{-a \pm \sqrt{a^2 - 4b}}{2}
\]

Taking \( a^2 < 4b \) and \( -\frac{a}{2} = \cos \theta \), \( \frac{\sqrt{4b-a^2}}{2} = \sin \theta \). Then \( f \) has \( 2n \) eigenvalues given by

\[
 \lambda = e^{\frac{i\theta}{n}}, \quad n=1,2,3,\ldots,n
\]
Let $P_x$, $x=1,2,\ldots,n$ be eigenvectors of $f$ corresponding to eigenvalue $e^{i\theta}$ and $Q_x$, $x=1,2,\ldots,n$ be eigenvectors for the eigenvalue $e^{-i\theta}$. Then $\{P_x\}$ and $\{Q_x\}$ are linearly independent sets.

For the set $\{P_x,Q_x\}$, suppose that 
\[ a^x P_x + b^x Q_x = 0, \quad x=1,2,\ldots,n \text{ and } a^x, b^x \in R \quad (2.1) \]

Then operating the above equation $(2.1)$ by $f$ and taking into account that $\{P_x,Q_x\}$ are eigenvectors for eigenvalues $e^{i\theta}$ and $e^{-i\theta}$ of $f$, we get 
\[ a^x e^{i\theta} P_x + b^x e^{-i\theta} Q_x = 0 \quad (2.2) \]

In view of the equations $(2.1)$ and $(2.2)$, we get 
\[ b^x (1 - e^{-2i\theta}) Q_x = 0 \Rightarrow b^x = 0, \quad x=1,2,\ldots,n \]

Consequently from $(2.1)$, it follows that $a^x = 0$ as $\{P_x\}$ is linearly independent.

Thus the set $\{P_x,Q_x\}$ is linearly independent. Let us assume that $\pi_1,\pi_2,\ldots,\pi_n$ be tangent sub-bundles spanned by $P_1,P_2,\ldots,P_n$ respectively and $\pi_1,\pi_2,\ldots,\pi_n$ spanned by $Q_1,Q_2,\ldots,Q_n$ respectively.

Then 
\[ \pi_1 \cap \pi_1 = \phi, \pi_2 \cap \pi_2 = \phi, \ldots, \pi_n \cap \pi_n = \phi \]

and 
\[ \pi_1 \cup \pi_2 \cup \ldots \cup \pi_n \cup \pi_1 \cup \pi_2 \cup \ldots \cup \pi_n \]

is a tangent bundle of dimension $2n$.

Thus if the manifold $M$ admits the general even order structure of rank $2n$, it possesses tangent subbundles $\pi_1,\pi_2,\ldots,\pi_n$ each of dimension unit and subbundles $\tilde{\pi}_1,\tilde{\pi}_2,\ldots,\tilde{\pi}_n$ conjugate to $\pi_1,\pi_2,\ldots,\pi_n$, respectively, such that
\( \pi_1, \tilde{\pi}_1, \pi_2, \tilde{\pi}_2, \ldots, \pi_n, \tilde{\pi}_n \) are mutually disjoint and they span together a tangent bundle of dimension \( 2n \).

Suppose conversely that \( M \) admits the general even order structure of rank \( 2n \).
Let \( p^1, p^2, \ldots, p^n, q^1, q^2, \ldots, q^n \) be 1-forms dual to vector fields \( P_1, P_2, \ldots, P_n, Q_1, Q_2, \ldots, Q_n \) respectively. So

\[
p^1 \otimes P_1 + p^2 \otimes P_2 + \ldots + p^n \otimes P_n + q^1 \otimes Q_1 + q^2 \otimes Q_2 + \ldots + q^n \otimes Q_n = I_{2n}
\]
or equivalently \( p^x \otimes P_x + q^x \otimes Q_x = I_{2n} \), \( x \) takes the values \( 1, 2, \ldots, n \) and \( I_{2n} \) denotes the unit tensor field.

Let us now put

\[
f = e^{-\frac{in\theta}{x}} p^x \otimes P_x + e^{-\frac{in\theta}{x}} q^x \otimes Q_x
\]
Then it is easy to show

\[
f^{2n} = e^{-\frac{in\theta}{x}} p^x \otimes P_x + e^{-\frac{in\theta}{x}} q^x \otimes Q_x
\]
and

\[
f^n = p^x \otimes P_x + q^x \otimes Q_x
\]
Thus

\[
f^{2n} + af^n = (a + e^{-\frac{in\theta}{x}}) p^x \otimes P_x + (b + e^{-\frac{in\theta}{x}}) q^x \otimes Q_x \quad (2.3)
\]
It is possible to set

\[
(a + e^{-\frac{in\theta}{x}}) = (b + e^{-\frac{in\theta}{x}}) = -b
\]
Hence the equation (2.3) takes the form

\[
f^{2n} + af^n = -b\{p^x \otimes P_x + q^x \otimes Q_x\}
\]
or

\[
f^{2n} + af^n + bI_{2n} = 0
\]
Thus the manifold \( M \) admits the general even order structure of rank \( 2n \). Thus we have.
Theorem 2.1 In order that the differentiable manifold $M$ admits the general even order structure of rank $2n$, it is necessary and sufficient that it possesses tangent subbundles $\pi_1, \pi_2, \ldots, \pi_n$ each of dimension unit and their respective complex conjugates $\bar{\pi}_1, \bar{\pi}_2, \ldots, \bar{\pi}_n$ such that

$$\pi_1 \cap \bar{\pi}_1 = \phi, \quad \pi_2 \cap \bar{\pi}_2 = \phi, \quad \ldots, \quad \pi_n \cap \bar{\pi}_n = \phi,$$

and they span together a tangent bundle of dimension $2n$.

3 General even order structure when $b = 0$

Suppose the manifold $M$ admits the general even order structure for $b = 0$. Hence we have

$$f^{2n} + af^n = 0 \quad (a \neq 0)$$

If we take the operators

$$l = -af^{-n} \quad \text{and} \quad m = l + af^{-n} \quad (3.1)$$

Then it is easy to show

$$l^2 = l, \quad m^2 = m, \quad l + m = I, \quad lm = ml = 0.$$ 

Thus for general even order structure for $b = 0$, the operators $l$ and $m$ defined by (3.1) when applied to the tangent space of $M$ at a point are complementary projection operators. Corresponding to projection operators $l$ and $m$, we get complementary distributions $L^*$ and $M^*$ respectively. If rank of $f$ is constant every where and equal to $r$, the dimensions of $L^*$ and $M^*$ are $r$ and $(n-r)$ respectively.

Let us now introduce in the manifold $M$ a local coordinate system and denote by

$$f^h_i, \quad l^h_i, \quad m^h_i$$

the local components of $f$, $l$ and $m$ respectively.
Let \( \mathbf{u}^h_a \) be \( r \) mutually orthogonal unit vectors in \( L^* \) and \( (2n-r) \) such vectors in \( M^* \) denoted by \( \mathbf{u}^h_B \). Thus we have

\[
\begin{align*}
I^h u^i_b &= u^h_b, \quad I^h u^i_B = 0 \\
m^h_i u^i_b &= 0, \quad m^h_i u^i_B = u^h_B
\end{align*}
\]  
(3.2)

If \( (v^a_i, v^A_i) \) be the matrix inverse to \( (u^h_b, u^h_B) \), then we can write

\[
\begin{align*}
v^a_i u^i_b &= \delta^a_b, \quad v^a_i u^i_B = 0 \\
v^A_i u^i_b &= 0, \quad v^A_i u^i_B = \delta^A_B
\end{align*}
\]  
(3.3)

\( \delta^a_b \) denotes the Kroneker delta. Also

\[ v^a_i u^i_b + v^A_i u^i_A = \delta^h_i \]

In view of the equations (3.2) and (3.3), we have

\[
\begin{align*}
(I^h v^a_i) u^i_b &= \delta^a_b, \quad (I^h v^a_i) u^i_B = 0 \\
(m^h_i v^A_i) u^i_b &= 0, \quad (m^h_i v^A_i) u^i_B = \delta^A_B
\end{align*}
\]

Thus we have

\[
\begin{align*}
I^h v^a_i &= v^a_i, \quad I^h v^A_i = 0 \\
m^h_i v^A_i &= 0, \quad m^h_i v^A_i = v^A_i
\end{align*}
\]  
(3.4)

Since \( fm = 0 \), we have \( f^h_i m^j_i = 0 \). Contracting with \( v^A_j \) and using (3.4), we get

\[ f^h_i v^A_i = 0 \]

Again since \( I^h_j u^j_a = u^h_a \), therefore

\[ I^h_j u^j_a v^a_i = u^h_a v^a_i \]

or

\[ I^h_j (\delta^a_j - u^A_j v^A_j) = u^h_a v^a_i \]

Thus we have

\[ I^h_i = u^h_a v^a_i \]

Similarly we can show that

\[ m^h_i = u^h_a v^A_i \]
Let us now define
\[ g_{ji} = v_j^a v_i^a + v_j^A v_i^A \]
Then \( g_{ji} \) is globally defined positive definite Riemannian metric relative to which \((u_h^h, u_B^h)\) form an orthogonal frame and
\[ v_j^a = g_{ji} u_i^j, v_j^A = g_{ji} u_i^j \]
Let us further put
\[ l_{ji} = v_j^a v_i^a, m_{ji} = v_j^A v_i^A \]
Thus
\[ l_{ji} + m_{ji} = g_{ji} \]
The following equations can be proved easily
\[ l_j^l l_i^s g_{is} = l_{ji} \]
\[ l_j^l m_i^s g_{is} = 0 \]
\[ m_j^l m_i^s g_{is} = m_{ji} \]
If we put
\[ G_{ji} = \frac{1}{2} (g_{ji} + m_{ji} + f_i^s f_j^t g_{ij}) \]
then \( G_{ji} \) is globally defined Riemannian metric and satisfies
\[ v_j^A = G_{ji} u_i^j \quad \text{and} \quad m_{ji} = m_j^l G_{il} \]
Now
\[ G(u_a, u_A) = \frac{1}{2} \{ g(u_a, u_A) + m(u_a, u_A) + f_i^s f_j^t u_a^j u_A^l \} \quad (3.5) \]
Since \( L^* \) and \( M^* \) are orthogonal with respect to Riemannian metric \( g \), hence in view of above equation (3.5), it follows that \( L^* \) and \( M^* \) are also orthogonal with respect to \( G \). Hence we have the theorem.
Theorem 3.1 Let $M$ be a $2n$ dimensional differentiable manifold equipped with general even order structure of rank $2n$. Then there exist complementary distributions $L^*$ and $M^*$ and a positive definite Riemannian metric $G$ with respect to which $L^*$ and $M^*$ are orthogonal.

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