On contra $\Lambda_r$–continuous functions

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Abstract

In this paper, we introduce a new class of function called contra $\Lambda_r$-continuous function. Some characterizations and several properties concerning contra $\Lambda_r$-continuity are obtained.

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1 Introduction

$\Lambda_r$-open sets is recently introduced by the authors [6] and studied $\Lambda_r$-$T_0$, $\Lambda_r$-$T_1$ and $\Lambda_r$-$T_2$ spaces, $\Lambda_r$-regular spaces, $\Lambda_r$-normal spaces and variants of continuity
related to this concept in [6, 8, 7]. The purpose of the present paper is to introduce and investigate some of the fundamental properties of contra \( \Lambda_r \)-continuous functions and we obtain characterizations of contra \( \Lambda_r \)-continuous functions.

2 Preliminary Notes

Throughout the paper, \((X, \tau)\) (or simply \(X\)) will always denote a topological space. For a subset \(S\) of a topological space \(X\), \(S\) is called regular-open [10] if \(S = \text{Int } cl\ S\). Then the complement \(S^c = X \setminus S\) of a regular-open set \(S\) is called the regular-closed set. The family of all regular-open sets (resp. regular-closed sets) in \((X, \tau)\) will be denoted by \(RO(X, \tau)\) (resp. \(RC(X, \tau)\)). A subset \(S\) of a topological space \((X, \tau)\) is called \(\Lambda_r\)-set [6] if \(S = \Lambda_r(S)\), where

\[
\Lambda_r(S) = \bigcap \{G / G \in RO(X, \tau) \text{ and } S \subseteq G\}.
\]

The collection of all \(\Lambda_r\)-sets in \((X, \tau)\) is denoted by \(\Lambda_r(X, \tau)\).

Throughout this paper, we adopt the notations and terminology of [6]. Let \(A\) be a subset of a space \((X, \tau)\). Then \(A\) is called a \(\Lambda_r\)-closed set if \(A = S \cap C\) where \(S\) is a \(\Lambda_r\)-set and \(C\) is a closed set. The complement of a \(\Lambda_r\)-closed set is called \(\Lambda_r\)-open. The collection of all \(\Lambda_r\)-open (resp. \(\Lambda_r\)-closed) sets in \((X, \tau)\) is denoted by \(\Lambda_rO(X, \tau)\) (resp. \(\Lambda_rC(X, \tau)\)). Also note that every open set is \(\Lambda_r\)-open; arbitrary union of \(\Lambda_r\)-open sets is \(\Lambda_r\)-open and arbitrary intersection of \(\Lambda_r\)-closed sets is \(\Lambda_r\)-closed; and intersection of two open sets is \(\Lambda_r\)-open.

A point \(x \in X\) is called a \(\Lambda_r\)-cluster point of \(A\) if for every \(\Lambda_r\)-open set \(U\) containing \(x\), \(A \cap U \neq \emptyset\). The set of all \(\Lambda_r\)-cluster points of \(A\) is called the \(\Lambda_r\)-closure of \(A\) and it is denoted by \(\Lambda_r-cl(A)\). Then \(\Lambda_r-cl(A)\) is the intersection of all \(\Lambda_r\)-closed sets containing \(A\) and it is the smallest \(\Lambda_r\)-closed set containing \(A\). Also \(A\) is \(\Lambda_r\)-closed if and only if \(A = \Lambda_r-cl(A)\). The union of \(\Lambda_r\)-open sets contained in \(A\) is called \(\Lambda_r\)-interior of \(A\) and it is denoted by \(\Lambda_r-int(A)\). Before we enter into our work, we recall the following definitions.
Definition 2.1 A function $f : X \to Y$ is called

(i) contra-continuous [3], if $f^{-1}(V)$ is closed in $X$ for each open set $V$ of $Y$
(ii) $\Lambda_r$-continuous [7], if $f^{-1}(V)$ is a $\Lambda_r$-open set in $X$ for each open set $V$ in $Y$
(iii) $\Lambda_r$-irresolute [7], if $f^{-1}(V)$ is a $\Lambda_r$-open set in $X$ for each $\Lambda_r$-open set $V$ in $Y$
(iv) $\Lambda_r^*$-open [7], if the image of each $\Lambda_r$-open set in $X$ is a $\Lambda_r$-open set in $Y$
(v) $\Lambda_r^*$-closed [7], if the image of each $\Lambda_r$-closed set in $X$ is a $\Lambda_r$-closed set in $Y$

Definition 2.2 A topological space $X$ is said to be

(i) Urysohn space [11], if for each pair of distinct points $x$ and $y$ in $X$, there exists two open sets $U$ and $V$ in $X$ such that $x \in U$, $y \in V$ and $cl(U) \cap cl(V) = \emptyset$.
(ii) ultra normal [9], if each pair of nonempty disjoint closed sets can be separated by disjoint closed sets.

3 Contra $\Lambda_r$-continuous function

In this section, we introduce contra $\Lambda_r$-continuous functions, contra $\Lambda_r$-irresolute functions and perfectly contra $\Lambda_r$-irresolute functions and study their properties.

Definition 3.1 A function $f : (X, \tau) \to (Y, \sigma)$ is called contra $\Lambda_r$-continuous, if $f^{-1}(V)$ is $\Lambda_r$-closed in $X$ for each open set $V$ in $Y$.

Theorem 3.2 For a function $f : (X, \tau) \to (Y, \sigma)$, the following are equivalent:

(a) $f$ is contra $\Lambda_r$-continuous
(b) For every closed subset $F$ of $Y$, $f^{-1}(F)$ is $\Lambda_r$-open in $X$
(c) For each $x \in X$ and each closed subset $F$ of $Y$ with $f(x) \in F$, there exists a $\Lambda_r$-open set $U$ of $X$ with $x \in U$, $f(U) \subseteq F$

Proof. (a) $\leftrightarrow$ (b) Obvious.
(b) → (c) Let $F$ be any closed subset of $Y$ and let $f(x) \in F$ where $x \in X$. Then by (b), $f^{-1}(F)$ is $\Lambda_r$-open in $X$. Also $x \in f^{-1}(F)$. Take $U = f^{-1}(F)$. Then $U$ is a $\Lambda_r$-open set containing $x$ and $f(U) \subseteq F$.

(c) → (b) Let $F$ be any closed subset of $Y$. If $x \in f^{-1}(F)$, then $f(x) \in F$. Hence by (c), there exists a $\Lambda_r$-open set $U_x$ of $X$ with $x \in U_x$ such that $f(U_x) \subseteq F$. Then

$$f^{-1}(F) = \bigcup \{ U_x : x \in f^{-1}(F) \},$$

and hence $f^{-1}(F)$ is $\Lambda_r$-open in $X$. \(\square\)

**Lemma 3.3** [1] The following properties hold for subsets $A$, $B$ of a space $X$:

(a) $x \in \text{ker}(A)$ if and only if $A \cap F \neq \emptyset$ for any $F \in C(X,x)$

(b) $A \subseteq \text{ker}(A)$ and $A = \text{ker}(A)$ if $A$ is open in $X$

(c) If $A \subseteq B$, then $\text{ker}(A) \subseteq \text{ker}(B)$.

**Theorem 3.4** Let $f : X \rightarrow Y$ be a bijective function. Then the following are equivalent:

(a) $f$ is contra $\Lambda_r$-continuous

(b) $f(\Lambda_r\text{-cl}(A)) \subseteq \text{ker}(f(A))$ for every subset $A$ of $X$

(c) $\Lambda_r\text{-cl}(f^{-1}(B)) \subseteq f^{-1}(\text{ker}(B))$ for every subset $B$ of $Y$

**Proof.** (a) → (b) Let $A$ be any subset of $X$. Suppose $y \notin \text{ker}(f(A))$. By Lemma 3.3(a), there exists $F \in C(Y, f(x))$ such that $f(A) \cap F = \emptyset$. Then $A \cap f^{-1}(F) = \emptyset$. Since $f^{-1}(F)$ is $\Lambda_r$-open by (a), $\Lambda_r\text{-cl}(A) \cap f^{-1}(F) = \emptyset$. That implies $f(\Lambda_r\text{-cl}(A)) \cap F = \emptyset$ and so $y \notin f(\Lambda_r\text{-cl}(A))$. This shows that

$$f(\Lambda_r\text{-cl}(A)) \subseteq \text{ker}(f(A)).$$

(b) → (c) Let $B$ be any subset of $Y$. Then by (b),

$$f(\Lambda_r\text{-cl}(f^{-1}(B))) \subseteq f(\text{ker}(f^{-1}(B))) = \text{ker}(B).$$

Therefore, $\Lambda_r\text{-cl}(f^{-1}(B)) \subseteq f^{-1}(\text{ker}(B))$.

(c) → (a) Let $V$ be open in $Y$. Then $\Lambda_r\text{-cl}(f^{-1}(V)) \subseteq f^{-1}(\text{ker}(V)) = f^{-1}(V)$ by (c) and Lemma 3.3(b). But $f^{-1}(V) \subseteq \Lambda_r\text{-cl}(f^{-1}(V))$. So $f^{-1}(V) = \Lambda_r\text{-cl}(f^{-1}(V))$. This means that $f^{-1}(V)$ is $\Lambda_r$-closed in $X$ so that $f$ is contra $\Lambda_r$-continuous. \(\square\)
Remark 3.5 The Examples 3.6 and 3.7 show that the concepts of $\Lambda_r$-continuity and contra $\Lambda_r$-continuity are independent of each other.

Example 3.6 Let $X = \{a,b,c\}$, $Y = \{a,b,c,d\}$, $\tau = \{X,\emptyset,\{c\},\{a,c\},\{b,c\}\}$ and $\sigma = \{Y,\emptyset,\{a\},\{b,c\},\{a,b,c\}\}$. Then $\Lambda_rO(X,\tau) = \tau$ and $\Lambda_rC(X,\tau) = \{X,\emptyset,\{a\},\{b\},\{a,b\}\}$. Define a function $f : (X,\tau) \rightarrow (Y,\sigma)$ by $f(a) = d$, $f(b) = b$ and $f(c) = c$. Then $f$ is $\Lambda_r$-continuous. But $f$ is not contra $\Lambda_r$-continuous since $\{b,c\}$ is open in $(Y,\sigma)$ but $f^{-1}(\{b,c\}) = \{b,c\}$ is not $\Lambda_r$-closed in $(X,\tau)$.

Example 3.7 Let $X = Y = \{a,b,c,d\}$, $\tau = \{X,\emptyset,\{b,d\},\{b,c,d\},\{a,b,d\}\}$ and $\sigma = \{Y,\emptyset,\{a\},\{b\},\{a,b\}\}$. Then $\Lambda_rO(X,\tau) = \tau$ and $\Lambda_rC(X,\tau) = \{X,\emptyset,\{a\},\{c\},\{a,c\}\}$. Define a function $f : (X,\tau) \rightarrow (Y,\sigma)$ by $f(a) = a$, $f(b) = c$, $f(c) = b$ and $f(d) = d$. Then $f$ is contra $\Lambda_r$-continuous. But $f$ is not $\Lambda_r$-continuous since $\{a\}$ is open in $(Y,\sigma)$ but $f^{-1}(\{a\}) = \{a\}$ is not $\Lambda_r$-open in $(X,\tau)$.

Theorem 3.8 If a function $f : (X,\tau) \rightarrow (Y,\sigma)$ is contra $\Lambda_r$-continuous and $Y$ is regular, then $f$ is $\Lambda_r$-continuous.

Proof. Let $x \in X$ and $V$ be an open set in $Y$ with $f(x) \in V$. Since $Y$ is regular, there exists an open set $W$ in $Y$ such that $f(x) \in W$ and $\text{cl}(W) \subseteq V$. Since $f$ is contra $\Lambda_r$-continuous and $\text{cl}(W)$ is a closed subset of $Y$ with $f(x) \in \text{cl}(W)$, by Theorem 3.2 there exists a $\Lambda_r$-open set $U$ of $X$ with $x \in U$ such that $f(U) \subseteq \text{cl}(W)$. That is, $f(U) \subseteq V$. By Theorem 3.4 of [7], $f$ is $\Lambda_r$-continuous. \qed

Recall that a topological space $(X,\tau)$ is said to be $\Lambda_r$-normal [8] if for every pair of disjoint closed sets $A$ and $B$ of $X$, there exists $\Lambda_r$-open sets $U$ and $V$
Theorem 3.9 If $f : (X, \tau) \to (Y, \sigma)$ is closed, injective and contra $\Lambda_r$-continuous and $Y$ is ultra normal, then $X$ is $\Lambda_r$-normal.

Proof. Let $A$ and $B$ be disjoint closed subsets of $X$. Since $f$ is closed and injective, $f(A)$ and $f(B)$ are disjoint closed subsets of $Y$. Since $Y$ is ultra normal, there exists two clopen sets $U$ and $V$ in $Y$ such that $f(A) \subseteq U$, $f(B) \subseteq V$ and $U \cap V = \emptyset$.

Since $f$ is contra $\Lambda_r$-continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are $\Lambda_r$-open sets in $(X, \tau)$. Also $A \subseteq f^{-1}(U)$, $B \subseteq f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. This shows that $X$ is $\Lambda_r$-normal.

Recall that a space $(X, \tau)$ is $\Lambda_r$-T$_2$ [6] if for each pair of distinct points $x$ and $y$ in $X$, there exists a $\Lambda_r$-open set $U$ and a $\Lambda_r$-open set $V$ in $X$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Theorem 3.10 If a function $f : (X, \tau) \to (Y, \sigma)$ is injective, contra $\Lambda_r$-continuous and $Y$ is a Urysohn space, then $X$ is $\Lambda_r$-T$_2$.

Proof. Let $x, y \in X$ with $x \neq y$. Since $f$ is injective, $f(x) \neq f(y)$. Since $Y$ is a Urysohn space, there exists open sets $U$ and $V$ in $Y$ such that $f(x) \in U$, $f(y) \in V$ and $cl(U) \cap cl(V) = \emptyset$.

Since $f$ is contra $\Lambda_r$-continuous, by Theorem 3.2 there exists $\Lambda_r$-open sets $A$ and $B$ in $X$ such that $x \in A$, $y \in B$ and $f(A) \subseteq cl(U)$, $f(B) \subseteq cl(V)$. Then $f(A) \cap f(B) = \emptyset$ and so $f(A \cap B) = \emptyset$. This implies that $A \cap B = \emptyset$ and hence $X$ is $\Lambda_r$-T$_2$.

Remark 3.11 Every contra-continuous function is contra $\Lambda_r$-continuous since every closed set is $\Lambda_r$-closed. But the converse need not be true.

For example, let $X = Y = \{a,b,c,d\}$, $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a,b\}\}$ and $\sigma = \{Y, \emptyset, \{a\}, \{a,b\}, \{a,c\}, \{a,b,c\}\}$. Then the closed sets of $(X, \tau)$ are $X$, $\emptyset$, $\{b,c,d\}$, $\{a,c,d\}$ and $\Lambda_r$-closed sets of $(X, \tau)$ are $X$, $\emptyset$, $\{b,c,d\}$, $\{a,c,d\}$, $\{c,d\}$, $\{a\}, \{b\}$. Define a function

in $X$ such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$. 

**Theorem 3.9** If $f : (X, \tau) \to (Y, \sigma)$ is closed, injective and contra $\Lambda_r$-continuous and $Y$ is ultra normal, then $X$ is $\Lambda_r$-normal.

**Proof.** Let $A$ and $B$ be disjoint closed subsets of $X$. Since $f$ is closed and injective, $f(A)$ and $f(B)$ are disjoint closed subsets of $Y$. Since $Y$ is ultra normal, there exists two clopen sets $U$ and $V$ in $Y$ such that $f(A) \subseteq U$, $f(B) \subseteq V$ and $U \cap V = \emptyset$.

Since $f$ is contra $\Lambda_r$-continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are $\Lambda_r$-open sets in $(X, \tau)$. Also $A \subseteq f^{-1}(U)$, $B \subseteq f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. This shows that $X$ is $\Lambda_r$-normal.

**Recall** that a space $(X, \tau)$ is $\Lambda_r$-T$_2$ [6] if for each pair of distinct points $x$ and $y$ in $X$, there exists a $\Lambda_r$-open set $U$ and a $\Lambda_r$-open set $V$ in $X$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

**Theorem 3.10** If a function $f : (X, \tau) \to (Y, \sigma)$ is injective, contra $\Lambda_r$-continuous and $Y$ is a Urysohn space, then $X$ is $\Lambda_r$-T$_2$.

**Proof.** Let $x, y \in X$ with $x \neq y$. Since $f$ is injective, $f(x) \neq f(y)$. Since $Y$ is a Urysohn space, there exists open sets $U$ and $V$ in $Y$ such that $f(x) \in U$, $f(y) \in V$ and $cl(U) \cap cl(V) = \emptyset$.

Since $f$ is contra $\Lambda_r$-continuous, by Theorem 3.2 there exists $\Lambda_r$-open sets $A$ and $B$ in $X$ such that $x \in A$, $y \in B$ and $f(A) \subseteq cl(U)$, $f(B) \subseteq cl(V)$. Then $f(A) \cap f(B) = \emptyset$ and so $f(A \cap B) = \emptyset$. This implies that $A \cap B = \emptyset$ and hence $X$ is $\Lambda_r$-T$_2$.

**Remark 3.11** Every contra-continuous function is contra $\Lambda_r$-continuous since every closed set is $\Lambda_r$-closed. But the converse need not be true.

For example, let $X = Y = \{a,b,c,d\}$, $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a,b\}\}$ and $\sigma = \{Y, \emptyset, \{a\}, \{a,b\}, \{a,c\}, \{a,b,c\}\}$. Then the closed sets of $(X, \tau)$ are $X$, $\emptyset$, $\{b,c,d\}$, $\{a,c,d\}$ and $\Lambda_r$-closed sets of $(X, \tau)$ are $X$, $\emptyset$, $\{b,c,d\}$, $\{a,c,d\}$, $\{c,d\}$, $\{a\}, \{b\}$. Define a function

in $X$ such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$. 

**Theorem 3.9** If $f : (X, \tau) \to (Y, \sigma)$ is closed, injective and contra $\Lambda_r$-continuous and $Y$ is ultra normal, then $X$ is $\Lambda_r$-normal.

**Proof.** Let $A$ and $B$ be disjoint closed subsets of $X$. Since $f$ is closed and injective, $f(A)$ and $f(B)$ are disjoint closed subsets of $Y$. Since $Y$ is ultra normal, there exists two clopen sets $U$ and $V$ in $Y$ such that $f(A) \subseteq U$, $f(B) \subseteq V$ and $U \cap V = \emptyset$.

Since $f$ is contra $\Lambda_r$-continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are $\Lambda_r$-open sets in $(X, \tau)$. Also $A \subseteq f^{-1}(U)$, $B \subseteq f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. This shows that $X$ is $\Lambda_r$-normal.

**Recall** that a space $(X, \tau)$ is $\Lambda_r$-T$_2$ [6] if for each pair of distinct points $x$ and $y$ in $X$, there exists a $\Lambda_r$-open set $U$ and a $\Lambda_r$-open set $V$ in $X$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

**Theorem 3.10** If a function $f : (X, \tau) \to (Y, \sigma)$ is injective, contra $\Lambda_r$-continuous and $Y$ is a Urysohn space, then $X$ is $\Lambda_r$-T$_2$.

**Proof.** Let $x, y \in X$ with $x \neq y$. Since $f$ is injective, $f(x) \neq f(y)$. Since $Y$ is a Urysohn space, there exists open sets $U$ and $V$ in $Y$ such that $f(x) \in U$, $f(y) \in V$ and $cl(U) \cap cl(V) = \emptyset$.

Since $f$ is contra $\Lambda_r$-continuous, by Theorem 3.2 there exists $\Lambda_r$-open sets $A$ and $B$ in $X$ such that $x \in A$, $y \in B$ and $f(A) \subseteq cl(U)$, $f(B) \subseteq cl(V)$. Then $f(A) \cap f(B) = \emptyset$ and so $f(A \cap B) = \emptyset$. This implies that $A \cap B = \emptyset$ and hence $X$ is $\Lambda_r$-T$_2$.

**Remark 3.11** Every contra-continuous function is contra $\Lambda_r$-continuous since every closed set is $\Lambda_r$-closed. But the converse need not be true.

For example, let $X = Y = \{a,b,c,d\}$, $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a,b\}\}$ and $\sigma = \{Y, \emptyset, \{a\}, \{a,b\}, \{a,c\}, \{a,b,c\}\}$. Then the closed sets of $(X, \tau)$ are $X$, $\emptyset$, $\{b,c,d\}$, $\{a,c,d\}$ and $\Lambda_r$-closed sets of $(X, \tau)$ are $X$, $\emptyset$, $\{b,c,d\}$, $\{a,c,d\}$, $\{c,d\}$, $\{a\}, \{b\}$. Define a function
f : \( (X, \tau) \rightarrow (Y, \sigma) \) by \( f(a) = d, f(b) = a, f(c) = c \) and \( f(d) = c \).
Then \( f \) is contra \( \Lambda_r \)-continuous but not contra-continuous since \( \{a\} \) is open in \( (Y, \sigma) \) but \( f^{-1}(\{a\}) = \{b\} \) is not closed in \( (X, \tau) \).

**Definition 3.12** A topological space \( X \) is said to be \( \Lambda_r \)-connected, if \( X \) cannot be written as a disjoint union of two nonempty \( \Lambda_r \)-open sets.
A subset \( B \) of a topological space \( X \) is \( \Lambda_r \)-connected, if \( B \) is \( \Lambda_r \)-connected as a subspace of \( X \).

**Theorem 3.13** For a topological space \( X \), the following are equivalent:

(i) \( X \) is \( \Lambda_r \)-connected
(ii) The only subsets of \( X \) which are both \( \Lambda_r \)-open and \( \Lambda_r \)-closed are the sets \( X \) and \( \emptyset \)
(iii) Each \( \Lambda_r \)-continuous function of \( X \) into a discrete space \( Y \) with at least two points is a constant function

**Proof.** (i) \( \rightarrow \) (ii) Let \( U \) be a both \( \Lambda_r \)-open and \( \Lambda_r \)-closed subset of \( X \). Then \( X \setminus U \) is both \( \Lambda_r \)-open and \( \Lambda_r \)-closed. Since \( X \) is \( \Lambda_r \)-connected and \( X \) is the disjoint union of \( \Lambda_r \)-open sets \( U \) and \( X \setminus U \), one of these must be empty.
Hence either \( U = \emptyset \) or \( U = X \).

(ii) \( \rightarrow \) (i) Suppose that \( X \) is not \( \Lambda_r \)-connected. Then \( X = A \cup B \) where \( A \) and \( B \) are nonempty \( \Lambda_r \)-open sets such that \( A \cap B = \emptyset \). Since \( B = X \setminus A \) is \( \Lambda_r \)-open, \( A \) is both \( \Lambda_r \)-open and \( \Lambda_r \)-closed. By (ii), \( A = \emptyset \) or \( X \). That is, either \( A = \emptyset \) or \( B = \emptyset \), which is a contradiction. Therefore \( X \) is \( \Lambda_r \)-connected.

(ii) \( \rightarrow \) (iii) Let \( f : X \rightarrow Y \) be a \( \Lambda_r \)-continuous function from a topological space \( X \) into a discrete topological space \( Y \). Then for each \( y \in Y \), \( \{y\} \) is both open and closed in \( Y \). Since \( f \) is \( \Lambda_r \)-continuous, \( f^{-1}(y) \) is both \( \Lambda_r \)-open and \( \Lambda_r \)-closed in \( X \).
Hence \( X \) is covered by \( \Lambda_r \)-open and \( \Lambda_r \)-closed covering \( \{f^{-1}(y) : y \in Y\} \).
By (ii), \( f^{-1}(y) = \emptyset \) or \( X \) for each \( y \in Y \). If \( f^{-1}(y) = \emptyset \) for each \( y \in Y \), then \( f \) fails to be a map. Hence there exists only one point \( y \in Y \) such that \( f^{-1}(y) = X \), which shows that \( f \) is a constant function.
(iii) → (ii) Let \( U \) be both \( \Lambda_r \)-open and \( \Lambda_r \)-closed in \( X \). Suppose \( U \neq \emptyset \). Let \( f : X \to Y \) be a \( \Lambda_r \)-continuous function from a topological space \( X \) into a discrete topological space \( Y \) defined by \( f(U) = \{y\} \) and \( f(X \setminus U) = \{w\} \), where \( y, w \in Y \) and \( y \neq w \). By (iii), \( f \) is constant so that \( U = X \).

**Theorem 3.14** Let \((X, \tau)\) be a \( \Lambda_r \)-connected space and \((Y, \sigma)\) be any topological space. If \( f : X \to Y \) is surjective and contra \( \Lambda_r \)-continuous, then \( Y \) is not a discrete space.

**Proof.** If possible, let \( Y \) be a discrete space. Let \( A \) be any proper nonempty subset of \( Y \). Then \( A \) is both open and closed in \((Y, \sigma)\). Since \( f \) is contra \( \Lambda_r \)-continuous, \( f^{-1}(A) \) is \( \Lambda_r \)-closed and \( \Lambda_r \)-open in \((X, \tau)\). Since \( X \) is \( \Lambda_r \)-connected, by Theorem 3.13, the only subsets of \( X \) which are both \( \Lambda_r \)-open and \( \Lambda_r \)-closed are the sets \( X \) and \( \emptyset \). Hence \( f^{-1}(A) \) is either \( X \) or \( \emptyset \). If \( f^{-1}(A) = \emptyset \), then it contradicts the fact that \( A \neq \emptyset \) and \( f \) is surjective. If \( f^{-1}(A) = X \), then \( f \) fails to be a map. Hence \( Y \) is not a discrete space.

**Theorem 3.15** If \( f : (X, \tau) \to (Y, \sigma) \) is surjective, contra \( \Lambda_r \)-continuous and \( X \) is \( \Lambda_r \)-connected, then \( Y \) is connected.

**Proof.** Assume that \( Y \) is not connected. Then \( Y = A \cup B \) where \( A \) and \( B \) are nonempty open sets in \( Y \) such that \( A \cap B = \emptyset \). Set \( U = Y \setminus A \) and \( V = Y \setminus B \). Then \( U \) and \( V \) are nonempty closed sets in \( Y \). Since \( f \) is surjective and contra \( \Lambda_r \)-continuous, \( f^{-1}(U) \) and \( f^{-1}(V) \) are nonempty \( \Lambda_r \)-open sets in \((X, \tau)\).

Now, \( f^{-1}(U) \cap f^{-1}(V) = \emptyset \) and \( f^{-1}(U) \cup f^{-1}(V) = X \). This contradicts the fact that \( X \) is \( \Lambda_r \)-connected and so \( Y \) is connected.

**Theorem 3.16** A space \( X \) is \( \Lambda_r \)-connected if every contra \( \Lambda_r \)-continuous function from a space \( X \) into any \( T_0 \)-space \( Y \) is constant.

**Proof.** Suppose that \( X \) is not \( \Lambda_r \)-connected and every contra \( \Lambda_r \)-continuous function from \( X \) into a \( T_0 \)-space \( Y \) is constant. Since \( X \) is not \( \Lambda_r \)-connected, by Theorem 3.13, there exists a proper nonempty subset \( A \) of \( X \) such that \( A \) is both
Let $Y = \{a,b\}$ and $\sigma = \{Y, \emptyset, \{a\}, \{b\}\}$ be a topology for $Y$. Let $f : X \to Y$ be a function such that $f(A) = \{a\}$ and $f(X \setminus A) = \{b\}$. Then $f$ is non constant and contra $\Lambda_r$-continuous such that $Y$ is $T_0$, which is a contradiction. This shows that $X$ must be $\Lambda_r$-connected.

**Theorem 3.17** If $f : (X, \tau) \to (Y, \sigma)$ is contra $\Lambda_r$-continuous and $g : (Y, \sigma) \to (Z, \gamma)$ is continuous, then $g \circ f : (X, \tau) \to (Z, \gamma)$ is contra $\Lambda_r$-continuous.

**Proof.** It directly follows from the definitions.

**Theorem 3.18** Let $f : (X, \tau) \to (Y, \sigma)$ be surjective, $\Lambda_r$-irresolute and $\Lambda_r^*$-open and $g : (Y, \sigma) \to (Z, \gamma)$ be any function. Then $g \circ f$ is contra $\Lambda_r$-continuous if and only if $g$ is contra $\Lambda_r$-continuous.

**Proof.** Suppose $g \circ f$ is contra $\Lambda_r$-continuous. Let $F$ be any closed set in $(Z, \gamma)$. Then $(g \circ f)^{-1}(F) = f^{-1}(g^{-1}(F))$ is $\Lambda_r$-open in $(X, \tau)$. Since $f$ is $\Lambda_r^*$-open and surjective, $f(f^{-1}(g^{-1}(F))) = g^{-1}(F)$ is $\Lambda_r$-open in $(Y, \sigma)$ and we obtain that $g$ is contra $\Lambda_r$-continuous.

For the converse, suppose $g$ is contra $\Lambda_r$-continuous. Let $V$ be closed in $(Z, \gamma)$. Then $g^{-1}(V)$ is $\Lambda_r$-open in $(Y, \sigma)$. Since $f$ is $\Lambda_r$-irresolute, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is $\Lambda_r$-open in $(X, \tau)$ and so $g \circ f$ is contra $\Lambda_r$-continuous.

**Theorem 3.19** Let $f : X \to Y$ be a function and $g : X \to X \times Y$ the graph function of $f$, defined by $g(x) = (x, f(x))$ for every $x \in X$. If $g$ is contra $\Lambda_r$-continuous, then $f$ is contra $\Lambda_r$-continuous.

**Proof.** Let $U$ be an open set in $Y$. Then $X \times U$ is open in $X \times Y$. Since $g$ is contra $\Lambda_r$-continuous, $g^{-1}(X \times U) = f^{-1}(U)$ is $\Lambda_r$-closed in $X$. This shows that $f$ is contra $\Lambda_r$-continuous.

**Theorem 3.20** If $f : X \to Y$ is contra-continuous, $g : X \to Y$ is contra-continuous and $Y$ is Urysohn, then $E = \{x \in X : f(x) = g(x)\}$ is $\Lambda_r$-closed in $X$. 
Proof. Let $x \in X \setminus E$. Then $f(x) \neq g(x)$. Since $Y$ is Urysohn, there exists open sets $V$ and $W$ in $Y$ such that $f(x) \in V$, $g(x) \in W$ and $\text{cl}(V) \cap \text{cl}(W) = \emptyset$. Since $f$ is contra-continuous, $f^{-1}(\text{cl}(V))$ is open in $X$. Since $g$ is contra-continuous, $g^{-1}(\text{cl}(W))$ is open in $X$. Let $G = f^{-1}(\text{cl}(V))$ and $H = g^{-1}(\text{cl}(W))$ and set $A = G \cap H$. Then $A$ is a $\Lambda_r$-open set containing $x$ in $X$. Now,

$$f(A) \cap g(A) \subseteq f(G) \cap g(H) \subseteq \text{cl}(V) \cap \text{cl}(W) = \emptyset.$$  

This implies that $A \cap E = \emptyset$ where $A$ is $\Lambda_r$-open. So $x$ is not a $\Lambda_r$-cluster point of $E$. Hence $x \notin \Lambda_r\text{cl}(E)$ and this completes the proof.

**Definition 3.21** A subset $A$ of a topological space $X$ is said to be $\Lambda_r$-dense in $X$ if $\Lambda_r\text{cl}(A) = X$.

**Theorem 3.22** Let $f : X \to Y$ be a contra-continuous function and $g : X \to Y$ be a contra-continuous function. If $Y$ is Urysohn and $f = g$ on a $\Lambda_r$-dense set $A \subseteq X$, then $f = g$ on $X$.

**Proof.** Let $E = \{x \in X : f(x) = g(x)\}$. Since $f$ is contra-continuous, $g$ is contra-continuous and $Y$ is Urysohn, by Theorem 3.20, $E$ is $\Lambda_r$-closed in $X$. By assumption, we have $f = g$ on $A$ where $A$ is $\Lambda_r$-dense in $X$. Since $A \subseteq E$, $A$ is $\Lambda_r$-dense and $E$ is $\Lambda_r$-closed, we have

$$X = \Lambda_r\text{cl}(A) \subseteq \Lambda_r\text{cl}(E) = E.$$  

Hence $f = g$ on $X$.

**Definition 3.23** A space $(X, \tau)$ is said to be

(i) $\Lambda_r$-space, if every $\Lambda_r$-open set is open in $X$

(ii) locally $\Lambda_r$-indiscrete, if every $\Lambda_r$-open set is closed in $X$.

**Theorem 3.24** Let $f : X \to Y$ be a contra $\Lambda_r$-continuous function. Then

(i) $f$ is contra-continuous, if $X$ is a $\Lambda_r$-space

(ii) $f$ is continuous, if $X$ is locally $\Lambda_r$-indiscrete

**Proof.** (i) and (ii) are directly follows from the definitions.
Theorem 3.25 Let \( f : X \to Y \) be surjective, closed and contra \( \Lambda_r \)-continuous. If \( X \) is \( \Lambda_r \)-space, then \( Y \) is locally indiscrete.

**Proof.** Let \( V \) be open in \( Y \). Since \( f \) is contra \( \Lambda_r \)-continuous, \( f^{-1}(V) \) is \( \Lambda_r \)-closed in \( X \) and hence closed in \( X \) since \( X \) is \( \Lambda_r \)-space. Since \( f \) is closed and surjective, \( f(f^{-1}(V)) = V \) is closed in \( Y \) and so \( Y \) is locally indiscrete. \( \square \)

Recall that a function \( f : X \to Y \) is said to be contra \( \lambda \)-continuous [2] (resp., contra \( \alpha \)-continuous [5], contra-precontinuous [4] ), if \( f^{-1}(V) \) is \( \lambda \)-closed (resp., \( \alpha \)-closed, pre-closed ) in \( X \) for each open set of \( Y \).

**Remark 3.26** Since every \( \Lambda_r \)-closed set is \( \lambda \)-closed, every contra \( \Lambda_r \)-continuous function is contra \( \lambda \)-continuous. But the converse need not be true which is shown by the following example.

Let \( X = Y = \{a,b,c\}, \ \tau = \{X, \emptyset, \{a\}, \{b\}, \{a,b\}\} \) and \( \sigma = \{X, \emptyset, \{a\}\} \). Then the function \( f : (X, \tau) \to (Y, \sigma) \) defined by \( f(a) = a, f(b) = a \) and \( f(c) = c \) is contra \( \lambda \)-continuous but not contra \( \Lambda_r \)-continuous.

The following examples show that contra \( \Lambda_r \)-continuous and contra-precontinuous functions (resp., contra-\( \alpha \)-continuous) are independent notions.

The function which is defined in Remark 3.11 is contra \( \Lambda_r \)-continuous but not contra-precontinuous and not contra-\( \alpha \)-continuous.

Let \( X = Y = \{a,b,c\}, \ \tau = \{X, \emptyset, \{a\}\} \) and \( \sigma = \{Y, \emptyset, \{a\}, \{b\}, \{a,b\}\} \). Then \( \Lambda_r(O(X,\tau)) = \tau, \ PO(X,\tau) = \{X, \emptyset, \{a\}, \{a,b\}, \{a,c\}\} \) and \( \alpha(X,\tau) = \{X, \emptyset, \{a\}, \{a,b\}, \{a,c\}\} \).

Define a function

\[ f : (X, \tau) \to (Y, \sigma) \text{ by } f(a) = c, \ f(b) = b \text{ and } f(c) = a. \]

Then \( f \) is contra-pre continuous and contra-\( \alpha \)-continuous but not contra \( \Lambda_r \)-continuous.
On contra $\Lambda_r$–continuous functions

In this diagram,

“$A \rightarrow B$” means $A$ implies $B$ but not conversely

“$A \leftrightarrow B$” means $A$ and $B$ are independent of each other.

**Definition 3.27** A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called contra $\Lambda_r$-irresolute, if $f^{-1}(V)$ is $\Lambda_r$-closed in $(X, \tau)$ for each $\Lambda_r$-open set $V$ in $(Y, \sigma)$.

**Remark 3.28** The following examples show that the concepts of $\Lambda_r$-irresolute and contra $\Lambda_r$-irresolute are independent of each other.

**Example 3.29** Let $X = \{a,b,c,d\}$, $Y = \{a,b,c,d,e\}$, $\tau = \{X, \emptyset, \{a\}, \{a,c\}, \{a,b,d\}\}$ and $\sigma = \{Y, \emptyset, \{a\}, \{a,b\}, \{a,b,e\}, \{a,c,d\}, \{a,b,c,d\}\}$. Then $\Lambda_rO(X, \tau) = \tau$, $\Lambda_rC(X, \tau) = \{X, \emptyset, \{c\}, \{b,d\}, \{b,c,d\}\}$ and $\Lambda_rO(Y, \sigma) = \sigma$.

Define a function

$f : (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = a$, $f(b) = e$, $f(c) = c$ and $f(d) = e$.

Then $f$ is $\Lambda_r$-irresolute but not contra $\Lambda_r$-irresolute since $\{a\}$ is open in $(Y, \sigma)$, but $f^{-1}(\{a\}) = \{a\}$ is not $\Lambda_r$-closed in $(X, \tau)$.

**Example 3.30** Let $X = Y = \{a,b,c,d\}$, $\tau = \{X, \emptyset, \{a\}, \{b,c\}, \{a,b,c\}\}$ and $\sigma = \{Y, \emptyset, \{a\}, \{a,b\}, \{a,c\}, \{a,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}\}$. Then $\Lambda_rO(X, \tau) = \{X, \emptyset, \{a\}, \{b,c\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}\}$, $\Lambda_rC(X, \tau) = \{X, \emptyset, \{a\}, \{d\}, \{a,d\}, \{b,c\}, \{b,c,d\}\}$ and $\Lambda_rO(Y, \sigma) = \sigma$.

Define a function

$f : (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = f(b) = f(c) = d$ and $f(d) = a$. 


Then \( f \) is contra \( \Lambda_r \)-irresolute but not \( \Lambda_r \)-irresolute since \( \{a\} \) is open in \((Y, \sigma)\), but \( f^{-1}(\{a\}) = \{d\} \) is not \( \Lambda_r \)-open in \((X, \tau)\).

**Remark 3.31** Every contra \( \Lambda_r \)-irresolute function is contra \( \Lambda_r \)-continuous. But the converse need not be true as shown by the following example.

In Example 3.7, \( f \) is contra \( \Lambda_r \)-continuous but not contra \( \Lambda_r \)-irresolute.

**Theorem 3.32** A function \( f : (X, \tau) \to (Y, \sigma) \) is contra \( \Lambda_r \)-irresolute if and only if \( f^{-1}(V) \) is \( \Lambda_r \)-open in \( X \) for each \( \Lambda_r \)-closed set \( V \) in \( Y \).

**Proof.** Obvious.

**Theorem 3.33** Let \( f : (X, \tau) \to (Y, \sigma) \) and \( g : (Y, \sigma) \to (Z, \gamma) \) be two functions.

Then

(a) if \( g \) is \( \Lambda_r \)-irresolute and \( f \) is contra \( \Lambda_r \)-irresolute, then \( g \circ f \) is contra \( \Lambda_r \)-irresolute

(b) if \( g \) is contra \( \Lambda_r \)-irresolute and \( f \) is \( \Lambda_r \)-irresolute, then \( g \circ f \) is contra \( \Lambda_r \)-irresolute

**Proof.** (a) Let \( V \) be \( \Lambda_r \)-open in \( Z \). Since \( g \) is \( \Lambda_r \)-irresolute, \( g^{-1}(V) \) is \( \Lambda_r \)-open in \( Y \).

Since \( f \) is contra \( \Lambda_r \)-irresolute, \( f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V) \) is \( \Lambda_r \)-closed in \( X \). This means that \( g \circ f \) is contra \( \Lambda_r \)-irresolute. (b) is similar to (a).

**Theorem 3.34** If \( f : (X, \tau) \to (Y, \sigma) \) is contra \( \Lambda_r \)-irresolute and \( g : (Y, \sigma) \to (Z, \gamma) \) is \( \Lambda_r \)-continuous, then \( g \circ f \) is contra \( \Lambda_r \)-continuous.

**Proof.** It directly follows from the definitions.

Recall that a subset \( A \) of a topological space \((X, \tau)\) is called \( \Lambda_r \)-clopen [8] if \( A \) is both \( \Lambda_r \)-open and \( \Lambda_r \)-closed in \( X \). The collection of all \( \Lambda_r \)-clopen sets in \((X, \tau)\) is denoted by \( \Lambda_r \text{CO}(X, \tau) \).
**Definition 3.35** A function $f : (X, \tau) \to (Y, \sigma)$ is called perfectly contra $\Lambda_r$-irresolute if $f^{-1}(V)$ is $\Lambda_r$-clopen in $X$ for each $\Lambda_r$-open set $V$ in $Y$.

**Remark 3.36** Every perfectly contra $\Lambda_r$-irresolute function is contra $\Lambda_r$-irresolute and $\Lambda_r$-irresolute. The following two examples show that a contra $\Lambda_r$-irresolute function may not be perfectly contra $\Lambda_r$-irresolute, and a $\Lambda_r$-irresolute function may not be perfectly contra $\Lambda_r$-irresolute.

In Example 3.30, $f$ is contra $\Lambda_r$-irresolute but not perfectly contra $\Lambda_r$-irresolute.

In Example 3.29, $f$ is $\Lambda_r$-irresolute but not perfectly contra $\Lambda_r$-irresolute.

**Theorem 3.37** A function $f : (X, \tau) \to (Y, \sigma)$ is perfectly contra $\Lambda_r$-irresolute if and only if $f$ is contra $\Lambda_r$-irresolute and $\Lambda_r$-irresolute.

**Proof.** It directly follows from the definitions.

We have the following relation for the functions defined above:

```
perfectly contra \Lambda_r-irresolute
     \Lambda_r-irresolute  \equiv  \Lambda_r-continuous  \equiv  contra \Lambda_r-irresolute
           \equiv  contra \Lambda_r-continuous
```

In this diagram,

“$A \rightarrow B$” means $A$ implies $B$ but not conversely

“$A \leftrightarrow B$” means $A$ and $B$ are independent of each other
References


