On Solving of Common Fixed Point Problems for Nonexpansive Semigroups in Hilbert Spaces

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Abstract

This paper is concerned with a common fixed point problem of a nonexpansive semigroup in Hilbert spaces. The strong convergence theorem for a nonexpansive semigroup is obtained by a novel general iterative scheme based on the viscosity approximation method and applicability of the results is shown to extend the results of many authors existing in the current literature.

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1 Introduction

Thoughout this paper, we always assume that \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \) with inner product and norm denoted by \( \langle \cdot , \cdot \rangle \) and \( \| \cdot \| \) respectively. Recall that \( P_C \) is the metric projection of \( H \) onto \( C \); that is, for each \( x \in H \) there exists the unique point in \( P_C x \in C \) such that

\[
\| x - P_C x \| = \min_{y \in C} \| x - y \|.
\]

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A mapping $T : C \to C$ is called nonexpansive if
\[ \|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C, \]
and the mapping $f : C \to C$ is called a contraction if there exists a constant $\alpha \in (0, 1)$ such that
\[ \|f(x) - f(y)\| \leq \alpha \|x - y\|, \quad \forall x, y \in C. \]
A point $x \in C$ is a fixed point of $T$ provided $Tx = x$. We denote by $F(T)$ the set of fixed points of $T$; that is, $F(T) = \{x \in C : Tx = x\}$. If $C \subset H$ is bounded, closed and convex and $T$ is a nonexpansive mapping of $C$ into itself, then $F(T)$ is nonempty (see [1]). A family $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ of mappings of $C$ into itself is called a nonexpansive semigroup on $C$ if it satisfies the following conditions:

(i) $T(0)x = x$ for all $x \in C$;
(ii) $T(s + t) = T(s) \circ T(t)$ for all $s, t \geq 0$;
(iii) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \geq 0$;
(iv) for all $x \in C$, $s \mapsto T(s)x$ is continuous.

We denote by $F(\mathcal{S})$ the set of all common fixed points of $\mathcal{S}$, that is, $F(\mathcal{S}) = \{x \in C : T(s)x = x, \ 0 \leq s < \infty\}$. It is known that if $C \subset H$ is bounded, closed and convex, then $F(\mathcal{S})$ is nonempty, closed and convex (see [2]). Construction of fixed points of nonexpansive mappings (and of common fixed points of nonexpansive semigroups) is an important subject in the theory of nonexpansive mappings and finds application in number of applied areas, in particular, in the minimization problem (see, e.g. [3, 4, 5, 6, 7] and the references therein).

A typical problem is to minimize a quadratic function over the set of fixed points of a nonexpansive mapping in a real Hilbert space $H$:
\[ \min_{x \in \Omega} \left\{ \frac{1}{2} \langle Ax, x \rangle - \langle x, b \rangle \right\}, \]
where $A$ is a bounded linear operator on $H$, $\Omega$ is the fixed point set of a nonexpansive mapping $S$ on $H$ and $b$ is a given point in $H$. Recall that $A$
be a strongly positive bounded linear operator on $H$ if there exists a constant $\gamma > 0$ such that
\[
\langle Ax, x \rangle \geq \gamma \|x\|^2, \quad \forall x \in H.
\]

Shimizu and Takahashi [8] introduced an iterative scheme for finding a common fixed points of a nonexpansive semigroup as the following theorem.

**Theorem ST.** Let $C$ be a nonempty closed convex subset of a Hilbert space $H$. Let $S = \{T(s) : s \geq 0\}$ be a nonexpansive semigroup on $C$ such that $F(S) \neq \emptyset$. Suppose that $x_1 = u \in C$ and $\{x_n\}$ is the sequence defined by
\[
x_{n+1} = \alpha_n u + (1 - \alpha_n) \frac{1}{s_n} \int_0^{s_n} T(s) x_n \, ds,
\]
for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset (0, 1)$ and $\{s_n\} \subset (0, \infty)$ satisfying the conditions $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n \to \infty} s_n = \infty$. Then $\{x_n\}$ converges strongly to $P_{F(S)} u$.

Marino and Xu [9] introduced the following an iterative scheme for finding a fixed point of nonexpansive mapping based on the viscosity approximation method introduced by Moudafi [10]:
\[
x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) S x_n, \quad \forall n \in \mathbb{N},
\]
where $x_1 \in H$, $A$ is a strongly positive bounded linear operator on $H$, $f$ is a contraction on $H$ and $S$ is a nonexpansive on $H$. They proved that under some appropriate conditions imposed on the parameters, if $F(S) \neq \emptyset$, then the sequence $\{x_n\}$ generated by (1) converges strongly to the unique solution $z = P_{F(S)}(I - A + \gamma f)z$ of the variational inequality
\[
\langle (A - \gamma f)z, x - z \rangle \geq 0, \quad \forall x \in F(S),
\]
which is the optimality condition for the minimization problem
\[
\min_{x \in F(S)} \left\{ \frac{1}{2} \langle Ax, x \rangle - h(x) \right\},
\]
where $h$ is a potential function for $\gamma f$ (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

Furthermore, Plubtieng and Wangkeeree [11] introduced an iterative scheme for finding a common fixed point of a nonexpansive semigroup as follows:
\[
x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) \frac{1}{s_n} \int_0^{s_n} T(s) x_n \, ds,
\]
for all \( n \in \mathbb{N} \), where \( x_1 \in H \) and \( f \) is a contraction on \( H \). They proved that under some appropriate conditions imposed on the parameters, if \( F(S) \neq \emptyset \), then the sequence \( \{x_n\} \) generated by (2) converges strongly to the unique solution \( z = P_{F(S)}(I - A + \gamma f)z \) of the variational inequality
\[
\langle (A - \gamma f)z, x - z \rangle \geq 0, \quad \forall x \in F(S).
\]

In the same way, Plubtieng and Punpaeng [12] introduced an iterative scheme:
\[
x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \beta_n - \alpha_n) \frac{1}{s_n} \int_0^{s_n} T(s)x_n \, ds,
\]
for all \( n \in \mathbb{N} \), where \( x_1 \in C \), \( f \) is a contraction on \( C \) and \( S = \{T(s) : 0 \leq s < \infty\} \) is a nonexpansive semigroup on \( C \). They proved that under some appropriate conditions imposed on the parameters, if \( F(S) \neq \emptyset \), then the sequence \( \{x_n\} \) generated by (3) converges strongly to the unique solution \( z = P_{F(S)} f(z) \) of the variational inequality
\[
\langle (I - f)z, x - z \rangle \geq 0, \quad \forall x \in F(S).
\]

Very recently, Wangkeeree [13] introduced an iterative scheme:
\[
x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)\frac{1}{s_n} \int_0^{s_n} T(s)x_n \, ds,
\]
for all \( n \in \mathbb{N} \), where \( x_1 \in H \) and \( f \) is a contraction on \( H \). We note that their iteration is well defined if we let \( C = H \), and the appropriateness of the control condition \( \alpha_n \) of their iteration should be \( \{\alpha_n\} \subset (0, 1) \) (see Theorem 3.1 in [13]). He proved that under some appropriate conditions imposed on the parameters, if \( F(S) \neq \emptyset \), then the sequence \( \{x_n\} \) generated by (4) converges strongly to the unique solution \( z = P_{F(S)}(I - A + \gamma f)z \).

In this paper, we introduce a novel general iterative scheme by the viscosity approximation method to find a common fixed point of a nonexpansive semigroup in Hilbert space as follows:
\[
x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n Bx_n + ((1 - \epsilon_n)I - \beta_n B - \alpha_n A)\frac{1}{s_n} \int_0^{s_n} T(s)x_n \, ds,
\]
for all \( n \in \mathbb{N} \), where \( x_1 \in H \), \( A \) and \( B \) are two mappings of the strongly positive linear bounded self-adjoint operator mappings, and \( f : H \to H \) be a contraction mapping.

As special cases of the iterative scheme (5), we have the following.
(i) If \( S = \{ T(s) : 0 \leq s < \infty \} = T \), then (5) is reduced to iterative scheme:

\[
x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n Bx_n + ((1 - \epsilon_n)I - \beta_n B - \alpha_n A)Tx_n.
\] (6)

(ii) If \( \epsilon_n = 0 \) for all \( n \in \mathbb{N} \), then (5) is reduced to iterative scheme:

\[
x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n Bx_n + (I - \beta_n B - \alpha_n A) \frac{1}{s_n} \int_0^{s_n} T(s)x_n \, ds.
\] (7)

(iii) If \( B \equiv I \), then (5) is reduced to iterative scheme:

\[
x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n Bx_n + ((1 - \epsilon_n - \beta_n)I - \alpha_n A) \frac{1}{s_n} \int_0^{s_n} T(s)x_n \, ds.
\] (8)

(iv) If \( \beta_n = 0 \) for all \( n \in \mathbb{N} \), then (8) is reduced to iterative scheme:

\[
x_{n+1} = \alpha_n \gamma f(x_n) + ((1 - \epsilon_n)I - \alpha_n A) \frac{1}{s_n} \int_0^{s_n} T(s)x_n \, ds.
\] (9)

(v) If \( \epsilon_n = 0 \) for all \( n \in \mathbb{N} \), then (8) is reduced to iterative scheme of Wangkeeree [13]:

\[
x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A) \frac{1}{s_n} \int_0^{s_n} T(s)x_n \, ds.
\] (10)

(vi) If \( \epsilon_n = 0 \) for all \( n \in \mathbb{N} \), then (9) is reduced to iterative scheme of Plubtieng and Wangkeeree [11]:

\[
x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) \frac{1}{s_n} \int_0^{s_n} T(s)x_n \, ds.
\] (11)

(vii) If \( \gamma = 1 \) and \( A \equiv I \), then (10) is reduced to iterative scheme of Plubtieng and Punpaeng [12]:

\[
x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \beta_n - \alpha_n) \frac{1}{s_n} \int_0^{s_n} T(s)x_n \, ds.
\] (12)

(viii) If \( S = \{ T(s) : 0 \leq s < \infty \} = T \), then (11) is reduced to iterative scheme of Marino and Xu [9]:

\[
x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n.
\] (13)
(ix) If $\beta_n = 0$ for all $n \in \mathbb{N}$ and for $u \in C$ chosen arbitrarily, define the mapping $f : C \to C$ by $f(x) = u$ for all $x \in C$ then (12) is reduced to iterative scheme of Shimizu and Takahashi [8]:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) \frac{1}{s_n} \int_0^{s_n} T(s)x_n \, ds.$$  \hspace{1cm} (14)

We suggest and analyze the iterative scheme (5) above under some appropriate conditions imposed on the parameters, the strong convergence theorem for a common fixed point of a nonexpansive semigroup is obtained and applicability of the results is shown to extend the results of many authors existing in the current literature.

2 Preliminary Notes

We collect the following definition and lemmas which be used in the proof for the main results in the next section.

**Definition 2.1.** (see [14]) A space $X$ is said to satisfy Opial’s condition if for each sequence $\{x_n\}$ in $X$ which converges weakly to point $x \in X$, we have

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|, \quad \forall y \in X, \ y \neq x.$$  

**Remark 2.2.** It is well known that Hilbert spaces satisfy Opial’s condition.

**Lemma 2.3.** Let $C$ be a nonempty closed convex subset of a Hilbert space $H$. Then the following inequality holds:

$$\langle x - y, P_C x - P_C y \rangle \geq 0, \quad \forall x \in H, \ y \in C.$$  

**Lemma 2.4.** (see [9]) Let $H$ be a Hilbert space, $f : H \to H$ be a contraction with coefficient $0 < \alpha < 1$, and $A : H \to H$ be a strongly positive linear bounded operator with coefficient $\gamma > 0$. Then,

(1) if $0 < \gamma < \gamma/\alpha$, then

$$\langle x - y, (A - \gamma f)x - (A - \gamma f)y \rangle \geq (\gamma \alpha) \|x - y\|^2, \quad \forall x, y \in H;$$  

(2) if $0 < \rho \leq \|A\|^{-1}$, then $\|I - \rho A\| \leq 1 - \rho \gamma$.  

Lemma 2.5. (see [8]) Assume \( \{a_n\} \) is a sequence of nonnegative real numbers such that
\[
a_{n+1} \leq (1 - \eta_n)a_n + \delta_n, \ n \geq 1,
\]
where \( \{\eta_n\} \) is a sequence in \((0, 1)\) and \( \{\delta_n\} \) is a sequence in \(\mathbb{R}\) such that
\[
(1) \lim_{n \to \infty} \eta_n = 0 \text{ and } \sum_{n=1}^{\infty} \eta_n = \infty;
\]
\[
(2) \limsup_{n \to \infty} (\delta_n/\eta_n) \leq 0 \text{ or } \sum_{n=1}^{\infty} |\delta_n| < \infty.
\]
Then \( \lim_{n \to \infty} a_n = 0. \)

Lemma 2.6. (see [8]) Let \( C \) be a nonempty bounded closed convex subset of a Hilbert space \( H \) and let \( S = \{T(s) : 0 \leq s < \infty\} \) be a nonexpansive semigroup on \( C \). For \( x \in C \) and \( t > 0 \). Then for any \( 0 \leq h < \infty \), we have
\[
\lim_{t \to \infty} \sup_{x \in C} \left\| \frac{1}{t} \int_0^t T(s)x \, ds - T(h) \left( \frac{1}{t} \int_0^t T(s)x \, ds \right) \right\| = 0.
\]

3 Main Results

Theorem 3.1. Let \( H \) be a real Hilbert space. Let \( A, B : H \to H \) be two mappings of the strongly positive linear bounded self-adjoint operator mappings with coefficients \( \overline{\delta}, \overline{\beta} \in (0, 1] \) such that \( \overline{\delta} \leq \|A\| \leq 1 \) and \( \|B\| = \overline{\beta} \), respectively, and let \( f : H \to H \) be a contraction mapping with coefficient \( \delta \in (0, 1) \). Let \( S = \{T(s) : 0 \leq s < \infty\} \) be a nonexpansive semigroup on \( H \). Assume that \( F(S) \neq \emptyset \) and \( 0 < \gamma < \overline{\delta}/\delta \). For \( x_1 = u \in H \), suppose that \( \{x_n\} \) be generated iteratively by
\[
x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n Bx_n + ((1 - \epsilon_n)I - \beta_n B - \alpha_n A) \frac{1}{s_n} \int_0^{s_n} T(s)x_n \, ds, \quad (15)
\]
for all \( n \in \mathbb{N} \), where \( \{\alpha_n\} \subset (0, 1) \), \( \{\beta_n\} \), \( \{\epsilon_n\} \subset [0, 1) \) such that \( \epsilon_n \leq \alpha_n \) and \( \{s_n\} \subset (0, \infty) \) satisfying the following conditions:
\[
(C1) \lim_{n \to \infty} \alpha_n = \lim_{n \to \infty} \beta_n = \lim_{n \to \infty} (\epsilon_n/\alpha_n) = 0;
\]
\[
(C2) \lim_{n \to \infty} s_n = \sum_{n=1}^{\infty} \alpha_n = \infty.
\]
Then the sequence \( \{x_n\} \) converges strongly to \( w \in F(S) \) where \( w = P_{F(S)}(I - A + \gamma f)w \) is a unique solution of the variational inequality
\[
\langle (A - \gamma f)w, y - w \rangle \geq 0, \quad \forall y \in F(S). 
\]
Proof. From \( \|B\| = \beta \in (0, 1] \), \( \{\beta_n\} \subset [0, 1) \), \( \epsilon_n \leq \alpha_n \) for all \( n \in \mathbb{N} \), (C1), we have \( \alpha_n \to 0 \), \( \epsilon_n \to 0 \) and \( \beta_n \to 0 \) as \( n \to \infty \). Thus, we may assume without loss of generality that \( \alpha_n < (1 - \epsilon_n - \beta_n \|B\|)^{-1} \) for all \( n \in \mathbb{N} \). Since \( A \) and \( B \) are two mappings of the linear bounded self-adjoint operators, we have

\[
\|A\| = \sup\{\|Ax\| : x \in H, \|x\| = 1\}
\]

and

\[
\|B\| = \sup\{\|Bx\| : x \in H, \|x\| = 1\}.
\]

Observe that

\[
\langle ((1 - \epsilon_n)I - \beta_n B - \alpha_n A)x, x \rangle = (1 - \epsilon_n) \langle x, x \rangle - \beta_n \langle Bx, x \rangle - \alpha_n \langle Ax, x \rangle \geq 1 - \epsilon_n - \beta_n \|B\| - \alpha_n \|A\| > 0.
\]

Therefore, we obtain

\[
(1 - \epsilon_n)I - \beta_n B - \alpha_n A
\]

is positive. Thus, by the strong positively of \( A \) and \( B \), we get

\[
\|((1 - \epsilon_n)I - \beta_n B - \alpha_n A)x, x \rangle : x \in H, \|x\| = 1\}
\]

\[
= \sup\{(1 - \epsilon_n) \langle x, x \rangle - \beta_n \langle Bx, x \rangle - \alpha_n \langle Ax, x \rangle : x \in H, \|x\| = 1\}
\]

\[
\leq 1 - \epsilon_n - \beta_n \|B\| - \alpha_n \|A\| < 0.
\]

Therefore, we obtain \( (1 - \epsilon_n)I - \beta_n B - \alpha_n A \) is positive. Thus, by the strong positively of \( A \) and \( B \), we get

\[
\|((1 - \epsilon_n)I - \beta_n B - \alpha_n A)x, x \rangle : x \in H, \|x\| = 1\}
\]

\[
= \sup\{(1 - \epsilon_n) \langle x, x \rangle - \beta_n \langle Bx, x \rangle - \alpha_n \langle Ax, x \rangle : x \in H, \|x\| = 1\}
\]

\[
\leq 1 - \epsilon_n - \beta_n \|B\| - \alpha_n \|A\| < 0.
\]

(16)

Define the sequence of mappings \( \{P_n : H \to H\} \) as follows:

\[
P_n x = \alpha_n \gamma f(x) + \beta_n Bx + ((1 - \epsilon_n)I - \beta_n B - \alpha_n A) \frac{1}{s_n} \int_0^{s_n} T(s)x \, ds, \forall x \in H,
\]

for all \( n \in \mathbb{N} \). Firstly, we prove that \( P_n \) has a unique fixed point in \( H \). Note that for all \( x, y \in H \), by (16), the contraction of \( f \), the nonexpansiveness of \( T(s) \), and the linearity of \( A \) and \( B \), we have

\[
\|P_n x - P_n y\| \leq \alpha_n \gamma \|f(x) - f(y)\| + \beta_n \|B\| \|x - y\|
\]

\[
+ \|((1 - \epsilon_n)I - \beta_n B - \alpha_n A) \frac{1}{s_n} \int_0^{s_n} T(s)x - T(s)y \| \, ds
\]

\[
\leq \alpha_n \gamma \|x - y\| + \beta_n \|x - y\| + (1 - \beta_n \|B\| - \alpha_n \|A\|) \|x - y\|
\]

\[
= (1 - (\delta - \gamma \alpha_n)) \|x - y\|.
\]
Hence, $P_n$ is a contraction with coefficient $1 - (\bar{\delta} - \gamma \delta)\alpha_n \in (0, 1)$. Therefore, by Banach contraction principle guarantees that $P_n$ has a unique fixed point in $H$, and so the iteration (15) is well defined.

Next, we prove that $\{x_n\}$ is bounded. Pick $p \in F(S) = \bigcap_{s \geq 0} F(T(s))$ and setting $y_n = \frac{1}{s_n} \int_0^{s_n} T(s)x_n \, ds$. By the nonexpansiveness of $T(s)$, we have

$$
\|y_n - p\| = \left\| \frac{1}{s_n} \int_0^{s_n} T(s)x_n \, ds - \frac{1}{s_n} \int_0^{s_n} T(s)p \, ds \right\|
\leq \frac{1}{s_n} \int_0^{s_n} \|T(s)x_n - T(s)p\| \, ds
\leq \|x_n - p\|.
$$

By (16), (17), the contraction of $f$, and the linearity of $A$ and $B$, we have

$$
\|x_{n+1} - p\| = \|\alpha_n \gamma f(x_n) + \beta_n Bx_n + ((1 - \epsilon_n)I - \beta_n B - \alpha_n A)x_n - p\|
\leq \alpha_n \|\gamma f(x_n) - Ap\| + \beta_n \|B\| \|x_n - p\|
+ (1 - \epsilon_n)\|I - \beta_n B - \alpha_n A\| \|y_n - p\| + \epsilon_n \|p\|
\leq \alpha_n \|\gamma f(x_n) - f(p)\| + \alpha_n \|\gamma f(p) - Ap\| + \beta_n \|A\| \|x_n - p\|
+ (1 - \beta_n \bar{\beta} - \alpha_n \bar{\delta}) \|x_n - p\| + \alpha_n \|p\|
\leq \max \left\{ \|x_n - p\|, \frac{\|\gamma f(p) - Ap\| + \|p\|}{\bar{\delta} - \gamma \delta} \right\}.
$$

It follows from induction that

$$
\|x_{n+1} - p\| \leq \max \left\{ \|x_1 - p\|, \frac{\|\gamma f(p) - Ap\| + \|p\|}{\bar{\delta} - \gamma \delta} \right\},
$$

for all $n \in \mathbb{N}$. Hence, $\{x_n\}$ is bounded, and so are $\{y_n\}$ and $\{f(x_n)\}$.

Put $z_1 = P_{F(S)} x_1$ and set

$$
D = \{z \in H : \|z - z_1\| \leq \|x_1 - z_1\| + \frac{\|\gamma f(z_1) - Az_1\| + \|z_1\|}{\bar{\delta} - \gamma \delta} \}.
$$

Then $D$ is a nonempty bounded closed convex subset of $H$ which is $T(s)$-invariant for each $s \in [0, \infty)$ and $\{x_n\}, \{y_n\} \subset D$. Without loss of generality,
we may assume that $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ is a nonexpansive semigroup on $D$. By (C2) and Lemma 2.6, we get
\[
\lim_{n \to \infty} \|y_n - T(h)y_n\| = 0,
\]
for every $h \in [0, \infty)$. For all $x, y \in H$, by Lemma 2.4(2), the nonexpansiveness of $P_{F(S)}$, the contraction of $f$ and the linearity of $A$, we have
\[
\|P_{F(S)}(I - A + \gamma f)x - P_{F(S)}(I - A + \gamma f)y\|
\leq \|(I - A + \gamma f)x - (I - A + \gamma f)y\|
\leq \gamma\|f(x) - f(y)\| + \|I - A\|\|x - y\|
\leq \gamma \delta \|x - y\| + (1 - \delta)\|x - y\|
= (1 - (\delta - \gamma \delta))\|x - y\|.
\]
Therefore, $P_{F(S)}(I - A + \gamma f)$ is a contraction with coefficient $1 - (\delta - \gamma \delta) \in (0, 1)$, by Banach contraction principle guarantees that $P_{F(S)}(I - A + \gamma f)$ has a unique fixed point, say $w \in H$, that is, $w = P_{F(S)}(I - A + \gamma f)w$. Hence, by Lemma 2.3, we obtain
\[
\langle (A - \gamma f)w, y - w \rangle \geq 0, \quad \forall y \in F(S).
\]
Next, we claim that
\[
\limsup_{n \to \infty} \langle \gamma f(w) - Aw, y_n - w \rangle \leq 0.
\]
To show this inequality, we choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that
\[
\limsup_{n \to \infty} \langle \gamma f(w) - Aw, y_n - w \rangle = \lim_{i \to \infty} \langle \gamma f(w) - Aw, y_{n_i} - w \rangle.
\]
Since $\{y_{n_i}\} \subset D$ is bounded, there exists a subsequence $\{y_{n_{i_j}}\}$ of $\{y_{n_i}\}$ which converges weakly to $\overline{w}$. Without loss of generality, we can assume that $y_{n_i} \rightharpoonup \overline{w}$ as $i \to \infty$.

Next, we prove that $\overline{w} \in F(S) = \bigcap_{s \geq 0} F(T(s))$. Suppose that $\overline{w} \notin F(S)$, that is, $T(h)\overline{w} \neq \overline{w}$ for some $h \in [0, \infty)$. Since $\|y_{n_i} - T(h)y_{n_i}\| \to 0$ as $i \to \infty$ by (18), therefore, by the nonexpansiveness of $T(h)$ and the Opial’s condition, we have
\[
\liminf_{i \to \infty} \|y_{n_i} - \overline{w}\| < \liminf_{i \to \infty} \|y_{n_i} - T(h)\overline{w}\|
\leq \liminf_{i \to \infty} \left(\|y_{n_i} - T(h)y_{n_i}\| + \|T(h)y_{n_i} - T(h)\overline{w}\|\right)
\leq \liminf_{i \to \infty} \|y_{n_i} - \overline{w}\|.
\]
This is a contradiction. So, we obtain \( \bar{w} \in F(S) \). Therefore, from (19) and (20), we obtain

\[
\limsup_{n \to \infty} \langle \gamma f(w) - Aw, y_n - w \rangle = \lim_{i \to \infty} \langle \gamma f(w) - Aw, y_i - w \rangle = \langle (\gamma f - A)w, \bar{w} - w \rangle \leq 0. \tag{21}
\]

Next, we prove that \( x_n \to w \) as \( n \to \infty \). Since \( w \in F(S) \), the same as in (17), we have

\[
\|y_n - w\| \leq \|x_n - w\|. \tag{22}
\]

Therefore, by (16), (22), the contraction of \( f \), and the linearity of \( A \) and \( B \), we have

\[
\|x_{n+1} - w\|^2 = \|\alpha_n \gamma f(x_n) + \beta_n Bx_n + ((1 - \epsilon_n)I - \beta_n B - \alpha_n A)y_n - w\|^2
\]
\[
= \|\alpha_n (\gamma f(x_n) - Aw) + \beta_n B(x_n - w)
+ ((1 - \epsilon_n)I - \beta_n B - \alpha_n A)(y_n - w) - \epsilon_n w\|^2
\]
\[
\leq \left( \|\alpha_n (\gamma f(x_n) - Aw) + \beta_n B(x_n - w)
+ ((1 - \epsilon_n)I - \beta_n B - \alpha_n A)(y_n - w)\| + \epsilon_n \|w\| \right)^2
\]
\[
= \|\alpha_n (\gamma f(x_n) - Aw) + \beta_n B(x_n - w)
+ ((1 - \epsilon_n)I - \beta_n B - \alpha_n A)(y_n - w)\|^2 + M_n^{(1)}
\]
\[
= \|\beta_n B(x_n - w) + ((1 - \epsilon_n)I - \beta_n B - \alpha_n A)(y_n - w)\|^2
+ 2\alpha_n ((1 - \epsilon_n)I - \beta_n B - \alpha_n A)(y_n - w, \gamma f(x_n) - Aw)
+ M_n^{(1)} + M_n^{(2)}
\]
\[
\leq \left( \beta_n \|B\| \|x_n - w\| + \|((1 - \epsilon_n)I - \beta_n B - \alpha_n A\|y_n - w\| \right)^2
+ 2\alpha_n \gamma \langle y_n - w, f(x_n) - f(w) \rangle + M_n^{(1)} + M_n^{(2)} + M_n^{(3)}
\]
\[
\leq \left( \beta_n \|x_n - w\| + (1 - \beta_n \beta - \alpha_n \delta) \|x_n - w\| \right)^2
+ 2\alpha_n \gamma \delta \|x_n - w\|^2 + M_n^{(1)} + M_n^{(2)} + M_n^{(3)}
\]
\[
= (1 - 2(\delta - \gamma \delta)\alpha_n) \|x_n - w\|^2 + \alpha_n^2 \delta^2 \|x_n - w\|^2
+ M_n^{(1)} + M_n^{(2)} + M_n^{(3)}
\]
\[
\leq (1 - \eta_n) \|x_n - w\|^2 + \delta_n,
\]
where

\[ M_n^{(1)} := 2\epsilon_n \alpha_n (\gamma f(x_n) - Aw) + \beta_n B(x_n - w) \]
\[ + ((1 - \epsilon_n)I - \beta_n B - \alpha_n A)(y_n - w)\|\|w\| + \epsilon_n^2 \|w\|^2, \]
\[ M_n^{(2)} := \alpha_n^2 \|\gamma f(x_n) - Aw\|^2 + 2\alpha_n\beta_n \langle B(x_n - w), \gamma f(x_n) - Aw \rangle, \]
\[ M_n^{(3)} := 2\alpha_n \langle y_n - w, \gamma f(w) - Aw \rangle \]
\[ - 2\alpha_n \langle (\epsilon_n I + \beta_n B + \alpha_n A)(y_n - w), \gamma f(x_n) - Aw \rangle, \]
\[ \eta_n := (\bar{\delta} - \gamma \delta)\alpha_n \in (0, 1), \]
\[ \delta_n := \alpha_n^2 \bar{\delta}^2 \|x_n - w\|^2 + M_n^{(1)} + M_n^{(2)} + M_n^{(3)}. \]

By (C1), (C2), \( \lim_{n \to \infty} \epsilon_n = 0 \) and (21), we can found that \( \lim_{n \to \infty} \eta_n = 0 \), \( \sum_{n=1}^{\infty} \eta_n = \infty \) and \( \limsup_{n \to \infty} (\delta_n/\eta_n) \leq 0 \). Therefore, by Lemma 2.5, we obtain \( \{x_n\} \) converges strongly to \( w \). This completes the proof. 

Remark 3.2. The iteration (15) is the difference with many others as the following.

1. Two mappings \( A \) and \( B \) of the strongly positive linear bounded self-adjoint operator mappings are used in the iteration of \( \{x_n\} \), which be used only one mapping \( A \) by many others.

2. Three parameters \( \alpha_n, \beta_n \) and \( \epsilon_n \) are used in the iteration of \( \{x_n\} \), which be used only two parameters \( \alpha_n \) and \( \beta_n \) by many others.

4 Applications

Theorem 4.1. Let \( H \) be a real Hilbert space. Let \( A, B : H \to H \) be two mappings of the strongly positive linear bounded self-adjoint operator mappings with coefficients \( \bar{\delta}, \bar{\beta} \in (0, 1] \) such that \( \bar{\delta} \leq \|A\| \leq 1 \) and \( \|B\| = \bar{\beta} \), respectively, and let \( f : H \to H \) be a contraction mapping with coefficient \( \delta \in (0, 1) \). Let \( T : H \to H \) be a nonexpansive mapping. Assume that \( F(T) \neq \emptyset \) and \( 0 < \gamma < \bar{\delta}/\delta \). For \( x_1 = u \in H \), suppose that \( \{x_n\} \) be generated iteratively by

\[ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n Bx_n + ((1 - \epsilon_n)I - \beta_n B - \alpha_n A)T x_n, \]

for all \( n \in \mathbb{N} \), where \( \{\alpha_n\} \subset (0, 1) \) and \( \{\beta_n\}, \{\epsilon_n\} \subset [0, 1) \) such that \( \epsilon_n \leq \alpha_n \) satisfying the following conditions:
\( (C1) \ \lim_{n \to \infty} \alpha_n = \lim_{n \to \infty} \beta_n = \lim_{n \to \infty} (\epsilon_n/\alpha_n) = 0; \)

\( (C2) \ \sum_{n=1}^{\infty} \alpha_n = \infty. \)

Then the sequence \( \{x_n\} \) converges strongly to \( w \in F(T) \) where \( w = P_{F(T)}(I - A + \gamma f)w \) is a unique solution of the variational inequality

\[
\langle (A - \gamma f)w, y - w \rangle \geq 0, \quad \forall y \in F(T).
\]

**Proof.** It is concluded from Theorem 3.1 immediately, by putting \( \epsilon = 0 \) for all \( n \in \mathbb{N} \).

\( \square \)

**Theorem 4.2.** Let \( H \) be a real Hilbert space. Let \( A, B : H \to H \) be two mappings of the strongly positive linear bounded self-adjoint operator mappings with coefficients \( \tilde{\delta}, \beta \in (0, 1] \) such that \( \tilde{\delta} \leq \|A\| \leq 1 \) and \( \|B\| = \beta \), respectively, and let \( f : H \to H \) be a contraction mapping with coefficient \( \delta \in (0, 1) \). Let \( S = \{T(s) : 0 \leq s < \infty\} \) be a nonexpansive semigroup on \( H \). Assume that \( F(S) \neq \emptyset \) and \( 0 < \gamma < \delta/\tilde{\delta} \). For \( x_1 = u \in H \), suppose that \( \{x_n\} \) be generated iteratively by

\[
x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n Bx_n + (I - \beta_n B - \alpha_n A) \frac{1}{s_n} \int_0^{s_n} T(s)x_n \, ds,
\]

for all \( n \in \mathbb{N} \), where \( \{\alpha_n\} \subset (0, 1), \ \{\beta_n\} \subset [0, 1) \) and \( \{s_n\} \subset (0, \infty) \) satisfying the following conditions:

\( (C1) \ \lim_{n \to \infty} \alpha_n = \lim_{n \to \infty} \beta_n = 0; \)

\( (C2) \ \lim_{n \to \infty} s_n = \sum_{n=1}^{\infty} \alpha_n = \infty. \)

Then the sequence \( \{x_n\} \) converges strongly to \( w \in F(S) \) where \( w = P_{F(S)}(I - A + \gamma f)w \) is a unique solution of the variational inequality

\[
\langle (A - \gamma f)w, y - w \rangle \geq 0, \quad \forall y \in F(S).
\]

**Proof.** It is concluded from Theorem 3.1 immediately, by putting \( \epsilon_n = 0 \) for all \( n \in \mathbb{N} \).

\( \square \)

**Theorem 4.3.** Let \( H \) be a real Hilbert space. Let \( A : H \to H \) be a strongly positive linear bounded self-adjoint operator mapping with coefficient \( \tilde{\delta} \in (0, 1] \) such that \( \tilde{\delta} \leq \|A\| \leq 1 \), and let \( f : H \to H \) be a contraction mapping with
coefficient $\delta \in (0, 1)$. Let $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on $H$. Assume that $F(\mathcal{S}) \neq \emptyset$ and $0 < \gamma < \frac{\delta}{\delta}$. For $x_1 = u \in H$, suppose that $\{x_n\}$ be generated iteratively by

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \epsilon_n - \beta_n)I - \alpha_n A) \frac{1}{s_n} \int_0^{s_n} T(s)x_n \, ds,$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset (0, 1)$, $\{\epsilon_n\} \subset [0, 1)$ such that $\epsilon_n \leq \alpha_n$ and $\{s_n\} \subset (0, \infty)$ satisfying the following conditions:

(C1) $\lim_{n \to \infty} \alpha_n = \lim_{n \to \infty} \beta_n = \lim_{n \to \infty} (\epsilon_n / \alpha_n) = 0$;

(C2) $\lim_{n \to \infty} s_n = \sum_{n=1}^{\infty} \alpha_n = \infty$.

Then the sequence $\{x_n\}$ converges strongly to $w \in F(\mathcal{S})$ where $w = P_{F(\mathcal{S})}(I - A + \gamma f)w$ is a unique solution of the variational inequality

$$\langle (A - \gamma f)w, y - w \rangle \geq 0, \quad \forall y \in F(\mathcal{S}).$$

Proof. It is concluded from Theorem 3.1 immediately, by putting $B \equiv I$. \hfill $\square$

**Theorem 4.4.** Let $H$ be a real Hilbert space. Let $A : H \to H$ be a strongly positive linear bounded self-adjoint operator mapping with coefficient $\delta \in (0, 1]$ such that $\delta \leq \|A\| \leq 1$, and let $f : H \to H$ be a contraction mapping with coefficient $\delta \in (0, 1)$. Let $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on $H$. Assume that $F(\mathcal{S}) \neq \emptyset$ and $0 < \gamma < \frac{\delta}{\delta}$. For $x_1 = u \in H$, suppose that $\{x_n\}$ be generated iteratively by

$$x_{n+1} = \alpha_n \gamma f(x_n) + ((1 - \epsilon_n - \beta_n)I - \alpha_n A) \frac{1}{s_n} \int_0^{s_n} T(s)x_n \, ds,$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset (0, 1)$, $\{\epsilon_n\} \subset [0, 1)$ such that $\epsilon_n \leq \alpha_n$ and $\{s_n\} \subset (0, \infty)$ satisfying the following conditions:

(C1) $\lim_{n \to \infty} \alpha_n = \lim_{n \to \infty} \beta_n = \lim_{n \to \infty} (\epsilon_n / \alpha_n) = 0$;

(C2) $\lim_{n \to \infty} s_n = \sum_{n=1}^{\infty} \alpha_n = \infty$.

Then the sequence $\{x_n\}$ converges strongly to $w \in F(\mathcal{S})$ where $w = P_{F(\mathcal{S})}(I - A + \gamma f)w$ is a unique solution of the variational inequality

$$\langle (A - \gamma f)w, y - w \rangle \geq 0, \quad \forall y \in F(\mathcal{S}).$$
Proof. It is concluded from Theorem 4.3 immediately, by putting $\beta_n = 0$ for all $n \in \mathbb{N}$. \hfill \Box

**Theorem 4.5.** (Wangkeeree [13]) Let $H$ be a real Hilbert space. Let $A : H \to H$ be a strongly positive linear bounded self-adjoint operator mapping with coefficient $\delta \in (0, 1]$ such that $\delta \leq \|A\| \leq 1$, and let $f : H \to H$ be a contraction mapping with coefficient $\delta \in (0, 1)$. Let $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on $H$. Assume that $F(\mathcal{S}) \neq \emptyset$ and $0 < \gamma < \delta/\delta$. For $x_1 = u \in H$, suppose that $\{x_n\}$ be generated iteratively by

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A) \frac{1}{s_n} \int_0^{s_n} T(s)x_n \, ds,$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset (0, 1)$, $\{\beta_n\} \subset [0, 1)$ and $\{s_n\} \subset (0, \infty)$ satisfying the following conditions:

(C1) $\lim_{n \to \infty} \alpha_n = \lim_{n \to \infty} \beta_n = 0$;

(C2) $\lim_{n \to \infty} s_n = \sum_{n=1}^{\infty} \alpha_n = \infty$.

Then the sequence $\{x_n\}$ converges strongly to $w \in F(\mathcal{S})$ where $w = P_{F(\mathcal{S})}(I - A + \gamma f)w$ is a unique solution of the variational inequality

$$\langle (A - \gamma f)w, y - w \rangle \geq 0, \quad \forall y \in F(\mathcal{S}).$$

Proof. It is concluded from Theorem 4.3 immediately, by putting $\epsilon_n = 0$ for all $n \in \mathbb{N}$. \hfill \Box

**Theorem 4.6.** (Plubtieng and Wangkeeree [11]) Let $H$ be a real Hilbert space. Let $A : H \to H$ be a strongly positive linear bounded self-adjoint operator mapping with coefficient $\delta \in (0, 1]$ such that $\delta \leq \|A\| \leq 1$, and let $f : H \to H$ be a contraction mapping with coefficient $\delta \in (0, 1)$. Let $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on $H$. Assume that $F(\mathcal{S}) \neq \emptyset$ and $0 < \gamma < \delta/\delta$. For $x_1 = u \in H$, suppose that $\{x_n\}$ be generated iteratively by

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) \frac{1}{s_n} \int_0^{s_n} T(s)x_n \, ds,$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset (0, 1)$ and $\{s_n\} \subset (0, \infty)$ satisfying the following conditions:
(C1) \( \lim_{n \to \infty} \alpha_n = 0; \)

(C2) \( \lim_{n \to \infty} s_n = \sum_{n=1}^\infty \alpha_n = \infty. \)

Then the sequence \( \{x_n\} \) converges strongly to \( w \in F(S) \) where \( w = P_{F(S)}(I - A + \gamma f)w \) is a unique solution of the variational inequality

\[
\langle (A - \gamma f)w, y - w \rangle \geq 0, \quad \forall y \in F(S).
\]

Proof. It is concluded from Theorem 4.4 immediately, by putting \( \epsilon_n = 0 \) for all \( n \in \mathbb{N} \).

Theorem 4.7. (Plubtieng and Punpaeng [12]) Let \( C \) be a nonempty closed convex of a real Hilbert space \( H \). Let \( f : C \to C \) be a contraction mapping with coefficient \( \delta \in (0, 1) \) and let \( S = \{T(s) : 0 \leq s < \infty\} \) be a nonexpansive semigroup on \( C \). Assume that \( F(S) \neq \emptyset \). For \( x_1 = u \in C \), suppose that \( \{x_n\} \) be generated iteratively by

\[
x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \beta_n - \alpha_n) \frac{1}{s_n} \int_0^{s_n} T(s)x_n \, ds,
\]

for all \( n \in \mathbb{N} \), where \( \{\alpha_n\} \subset (0, 1), \{\beta_n\} \subset [0, 1) \) and \( \{s_n\} \subset (0, \infty) \) satisfying the following conditions:

(C1) \( \lim_{n \to \infty} \alpha_n = \lim_{n \to \infty} \beta_n = 0; \)

(C2) \( \lim_{n \to \infty} s_n = \sum_{n=1}^\infty \alpha_n = \infty. \)

Then the sequence \( \{x_n\} \) converges strongly to \( w \in F(S) \) where \( w = P_{F(S)}(I - f)w \) is a unique solution of the variational inequality

\[
\langle (I - f)w, y - w \rangle \geq 0, \quad \forall y \in F(S).
\]

Proof. It is concluded from Theorem 4.5 immediately, by putting \( \gamma = \delta = 1 \) and \( A \equiv I \).

Theorem 4.8. (Marino and Xu [9]) Let \( H \) be a real Hilbert space. Let \( A : H \to H \) be a strongly positive linear bounded self-adjoint operator mapping with coefficient \( \delta \in (0, 1] \) such that \( \delta \leq \|A\| \leq 1 \), and let \( f : H \to H \) be a contraction mapping with coefficient \( \delta \in (0, 1) \). Let \( T : H \to H \) be a nonexpansive mapping. Assume that \( F(T) \neq \emptyset \) and \( 0 < \gamma < \delta/\delta \). For \( x_1 = u \in H \), suppose that \( \{x_n\} \) be generated iteratively by

\[
x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n,
\]

for all \( n \in \mathbb{N} \), where \( \{\alpha_n\} \subset (0, 1) \) satisfying the following conditions:
Then the sequence $\{x_n\}$ converges strongly to $w \in F(T)$ where $w = P_{F(T)}(I - A + \gamma f)u$ is a unique solution of the variational inequality

$\langle (A - \gamma f)w, y - w \rangle \geq 0, \quad \forall y \in F(T)$. 

**Proof.** It is concluded from Theorem 4.6 immediately, by putting $S = \{T(s) : 0 \leq s < \infty\} = T$. □

**Theorem 4.9.** (Shimizu and Takahashi [8]) Let $C$ be a nonempty closed convex of a real Hilbert space $H$. Let $S = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on $C$. Assume that $F(S) \neq \emptyset$. For $x_1 = u \in C$, suppose that $\{x_n\}$ be generated iteratively by

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) \frac{1}{s_n} \int_0^{s_n} T(s)x_n \, ds,$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset (0, 1)$ and $\{s_n\} \subset (0, \infty)$ satisfying the following conditions:

(C1) $\lim_{n \to \infty} \alpha_n = 0$;

(C2) $\lim_{n \to \infty} s_n = \sum_{n=1}^{\infty} \alpha_n = \infty$.

Then the sequence $\{x_n\}$ converges strongly to $w \in F(S)$ where $w = P_{F(S)}u$ is a unique solution of the variational inequality

$\langle w - u, y - w \rangle \geq 0, \quad \forall y \in F(S)$. 

**Proof.** It is concluded from Theorem 4.7 immediately, by putting $\beta_n = 0$ for all $n \in \mathbb{N}$ and define the mapping $f : C \to C$ by $f(x) = u$ for all $x \in C$. □

**References**


