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On the fractional Black-Scholes market with transaction costs

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Abstract

We consider fractional Black-Scholes market with proportional transaction costs. When transaction costs are present, one trades periodically i.e. we have the discrete trading with equidistance n^{-1} between trading times. We derive a non trivial hedging error for a class of European options with convex payoff in the case when the transaction costs coefficients decrease as $n^{-(1-H)}$. We study the expected hedging error and asymptotic behavior of the hedge as Hurst parameter H approaches $\frac{1}{2}$.

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1 Introduction

In fractional Black-Scholes market, the noise is modelled by geometric fractional Brownian motion instead to Brownian motion in classical Black-Scholes market. It is known that the market admits arbitrage opportunities without

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transaction costs with continuous trading. Guasoni [10] showed that, if one considers such financial markets with proportional transaction costs, then arbitrage opportunities disappear. In [11], the authors studied a more general formulation of similar financial markets with proportional transaction costs. They also find a super replication price for a class of European options.

The motivation of this study comes from the recent work by Azmoodeh et. al. [2]. There we studied a hedging problem for European options with convex payoff in fractional Black-Scholes market with Hurst parameter $H > \frac{1}{2}$. We assume that the market is frictionless and continuous trading is possible. Then any European option with convex payoff can be hedged perfectly. Moreover, hedging strategy and *hedging cost* (see [3]) are given explicitly. Simply speaking we showed that the classical chain rule holds for convex functionals of geometric fractional Brownian motion. Moreover, it is shown that the wealth process is a limit of Riemann-Stieltjes sums almost surely. This makes our model more interesting from financial point of view (see [6] and [21]).

European call option with strike price K serves as a motivating example for us. In this case the *stop-loss-start-gain* strategy

$$u_t = 1_{\{S_t > K\}}$$

is self-financing replication strategy in our model. Note that this replicating strategy is not self-financing in the classical Black-Scholes market with standard Brownian motion. This strategy is of unbounded variation, and hence it is not practical for our model with proportional transaction costs with continuous trading. One possibility is to trade periodically i.e. trade in discrete and equidistant trading times, and the level of transaction costs is the function

$$k = k_n = k_0 n^{-(1-H)}$$

of the number of trading intervals. This is similar to Leland [15] in the case of classical Black-Scholes model with transaction costs.

Leland suggests a way to include proportional transaction costs in classical Black-Scholes market. Namely, over each periodical trading subinterval, the trader follows the delta hedging strategy computed at the left point of the trading subinterval with a modified volatility. The modified volatility depends on the original volatility and the number of trading intervals (see [14]). Leland remarked that when transaction cost coefficients k_n decrease as

$$k_n = k_0 n^{-\alpha}, \qquad \alpha \in (0, \frac{1}{2}],$$

then price of the modified strategy approximately hedges the option payoff at terminal date as the length of trading intervals tends to zero (see [16] and [14] for more details).

The paper is organized as follows. Section 2 includes main results. Also the description of the market model in precise way is given. Section 3 contains discussion and conclusion with emphasis on specific example of European call option. The paper ends with proofs and appendixes in the sections 4 and 5 respectively.

2 Main Results

Consider the following fractional Black-Scholes market, i.e. the two-assets market model consists of :

(i) Riskless asset (bond), $B_t = 1; \quad t \in [0, T]$ which corresponds to zero interest rate.

(ii) Risky asset (stock) whose price is modeled by geometric fractional Brownian motion

$$S_t = S_0 e^{B_t^H}; \quad t \in [0, T],$$

where $B^H = \{B_t^H\}_{t \in [0,T]}$ is a fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$.

Motivated by [2], we define the class $\mathcal{L}_{con}$ of payoff functions consists of all finite linear combination of convex or concave functions, i.e.

$$\mathcal{L}_{con} = \Big\{ f = \sum_{i=1}^{n} a_i f_i \ : \ f_i \text{ are convex or concave functions }, 1 \leq i \leq n, n \in \mathbb{N} \Big\}.$$

Then the main result of [2] says that the following Ito formula

$$f(S_T) = f(S_0) + \int_0^T f_-^{'}(S_t)dS_t, \qquad f \in \mathcal{L}_{con} \tag{1}$$

holds almost surely. Inspired by (1), we define the following class of stochastic processes

$$\mathcal{A}_{con} = \left\{ u = \{u_t\}_{t\in[0,T]} \ : \ u_t = f_-^{'}(S_t) \text{ for some function } f \in \mathcal{L}_{con} \right\}.$$

In language of stochastic finance, Ito formula (1) means that all European options with terminal payoff $f(S_T)$ can be hedged exactly with *delta hedging* strategy $f_-^{'}(S_t)$. Moreover one should pay the *hedging cost* $f(S_0)$ for such exact hedge.

Remark 2.1. *It is worth to mention that the stochastic integral in the right-hand side (1) is understood as a limit of Riemann–Stieltjes sums almost surely, i.e.*

$$\sum_{i=0}^{n} f_-^{'}(S_{t_{i-1}^n})(S_{t_i^n} - S_{t_{i-1}^n}) \xrightarrow{a.s.} \int_0^T f_-^{'}(S_t)dS_t, \qquad t_i^n = \frac{iT}{n}. \tag{2}$$

2.1 Optimality of the class $\mathcal{A}_{con}$

Fractional Brownian motion is not a semimartingale, when $H \neq \frac{1}{2}$. Therefore fractional Black-Scholes market admits arbitrage with continuous trading.

Example 2.2. [***Shiryaev arbitrage***] *It is well known that fractional Brownian motion has zero quadratic variation, when $H > \frac{1}{2}$. Then the classical change of variables formula implies that*

$$(S_T - 1)^2 = \int_0^T 2(S_t - 1)dS_t$$

Therefore the strategy $u_t = 2(S_t - 1) \in \mathcal{A}_{con}$ is a admissible strategy that is arbitrage.

Remark 2.3. *We remark that the class* $\mathcal{A}_{con}$ *of trading strategies is relatively big so that contains the strategies of unbounded variation. A typical example is the stop-loss-start-gain strategy*

$$f^{'}_{-}(S_t) = 1_{\{S_t > K\}}$$

for $f(x) = (x - K)^+$ *corresponding to European call option with strike price* K.

For trading strategy $u \in \mathcal{A}_{con}$, we define the process

$$V_t(u) = \int_0^t u_s dS_s \qquad t \in (0, T],$$

where according to remark (2.1), stochastic integral is understood in pathwise manner as limit of Riemann–Stieltjes sums. The financial meaning of $V_T(u)$ is the total loses or gains of the trading strategy u.

The main result of the subsection deals with super replication price of European options in the fractional Black-Scholes market.

Theorem 2.4. *Consider an European option with payoff* $f(S_T)$ *where* $f \in \mathcal{L}_{con}$ *in the fractional Black-Scholes market. Then super replication price* $p\big(f(S_T)\big)$, *in the class* $\mathcal{A}_{con}$ *of possible trading strategies equals to* $f(S_0)$. *More precisely*

$$\begin{aligned} p\big(f(S_T)\big) &= \inf\big\{x \ : \ x + V_T(u) \geq f(S_T) \text{ for some } u \in \mathcal{A}_{con}\big\} \\ &= f(S_0). \end{aligned}$$

2.2 Limit behavior under transaction costs

In this subsection we assume that the terminal trading time $T = 1$ and $f : \mathbb{R} \to \mathbb{R}$ is a convex function with positive Radon measure μ as its second derivative. For each $n \in \mathbb{N}$, we divide the trading interval $[0, 1]$ to n subintervals $[t^n_{i-1}, t^n_i]$ where

$$t^n_i = \frac{i}{n} = i\Delta_n, \qquad i = 0, 1, ..., n.$$

Consider the *discretized version* of delta hedging strategy given in (1)

$$\theta_t^n = \sum_{i=1}^{n} f_-^{'}(S_{t_{i-1}^n}) 1_{(t_{i-1}^n, t_i^n]}(t); \qquad t \in (0,1].$$

In the presence of proportional transaction costs, the value of this portfolio at terminal date with initial capital $f(S_0)$ is

$$V_1(\theta^n) = f(S_0) + \int_0^1 \theta_t^n dS_t - k \sum_{i=1}^{n} S_{t_{i-1}^n} |f_-^{'}(S_{t_i^n}) - f_-^{'}(S_{t_{i-1}^n})|. \tag{3}$$

Here it is assumed that the transaction costs are "two-sided" i.e. buying and selling are equally charged and transaction costs coefficient k is a function of the length of trading intervals in the shape

$$k = k_n = k_0 n^{-(1-H)}, \quad k_0 > 0.$$

Remark 2.5. *Note that it is assumed that there is no transaction costs at time $t = 0$ when the trader enters to the market.*

The following main result of this subsection studies limit behavior of the payoff of delta hedging strategies under proportional transaction costs.

Theorem 2.6. *Assume the level of transaction costs $k = k_n = k_0 n^{-(1-H)}$, where $k_0 > 0$. Then*

$$\mathbb{P}\text{-}\lim_{n\to\infty} V_1(\theta^n) = f(S_1) - \mathbf{J},$$

where

$$\mathbf{J} = \mathbf{J}(k_0) := \sqrt{\frac{2}{\pi}} k_0 \int_{\mathbb{R}} \int_0^1 S_t l^H(\ln a, dt) \mu(da),$$

and the inner integral in the right hand side is understood as limit of Riemann-Stieltjes sums a.s.

When the number of portfolio revision increases fast enough or equally saying, the transaction costs coefficients k_n decrease faster, we have the perfect replication in the limit, i.e. we have the following result.

Corollary 2.7. *Let the level of transaction costs $k = k_n = k_0 n^{-\alpha}$ where $\alpha > 1 - H$, $k_0 > 0$. Then*

$$\mathbb{P}\text{-}\lim_{n\to\infty} V_1(\theta^n) = f(S_1).$$

Remark 2.8. *Clearly, the hedging error*

$$\begin{aligned}\mathbf{J} = \mathbf{J}(k_0) :&= \sqrt{\frac{2}{\pi}} k_0 \int_{\mathbb{R}} \int_0^1 S_t l^H(\ln a, dt)\mu(da) \\ &= \sqrt{\frac{2}{\pi}} k_0 \int_{\mathbb{R}} a l^H(\ln a, [0,1])\mu(da)\end{aligned}$$

is positive a.s. and strictly positive on a set of positive probability. So with proportional transaction costs, the discretized replication strategy asymptotically subordinates rather than replicate the value of convex European option $f(S_1)$ and the option is always subhedged in the limit.

Remark 2.9. *The limiting hedging error $\mathbf{J} = \mathbf{J}(k_0)$ is small for small values of fixed proportional transaction costs coefficient k_0.*

3 Discussion and conclusion

3.1 The case European call option

Consider European call option with corresponding convex function $f(x) = (x - K)^+$. The approximating wealth process can not hedge perfectly the option payoff $(S_1 - K)^+$ and limiting hedging error takes the form

$$\mathbf{J} = K l^H(\ln K, [0,1]).$$

So, it is interesting to examine the expected hedging error

$$\begin{aligned}\mathbb{E}(\mathbf{J}) &= K\mathbb{E}\big(l^H(\ln K, [0,1])\big) \\ &= \frac{K}{\sqrt{2\pi}} \int_0^1 t^{-H} \exp\{-\frac{1}{2} t^{-2H} \ln^2 K\} dt.\end{aligned}$$

The graph of the expected hedging error as a function with respect to variables strike price K and Hurst parameter H is plotted in below which points out that the strike price $K = 1$ is a critical point. The expected hedging error tends to zero as strike price becomes bigger and bigger.

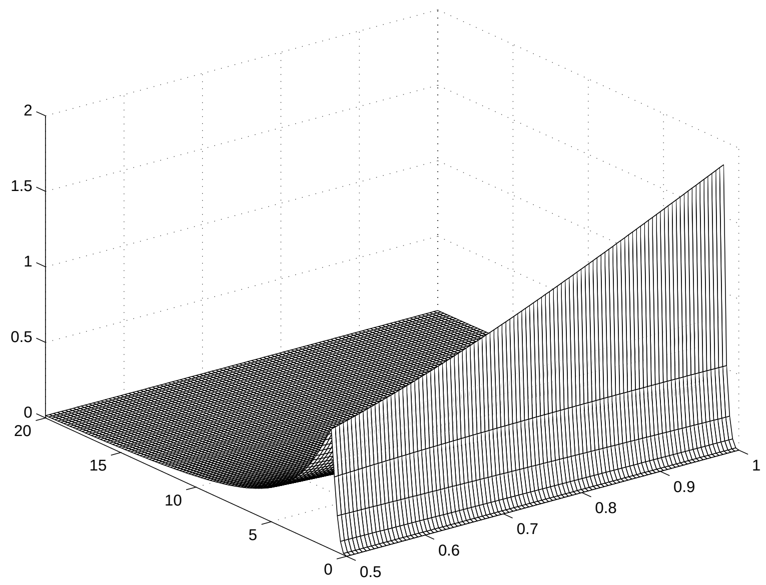

Figure 1: Expected hedging error

3.2 Asymptotic behavior with respect to Hurst parameter H

Assume B^H be a standard fractional Brownian motion with Hurst parameter $H \in (0,1)$. It is straightforward to check that when the Hurst parameter H tends to $H_0 \in (0,1)$, the finite-dimensional distributions of fractional Brownian motions $\{B^H\}$ tend to finite-dimensional distributions of B^{H_0}. It follows because of convergence of covariance functions and by

$$\mathbb{E}|B_t^H - B_s^H|^2 = |t-s|^{2H}$$

and Billingsley criterion (see [5]) we conclude that the family of laws of the fractional Brownian motions $\{B^H\}$ converge in law in the space $C[0,T]$ of continuous function on the interval $[0,T]$ to that of B^{H_0}.

Moreover, the following theorem shows that the same holds for the family of local times $\{l^H, H \in (0,1)\}$ of the fractional Brownian motions $\{B^H\}$.

Theorem 3.1. *[13] The family $\{l^H, H \in (0,1)\}$ of local times of the fractional Brownian motions $\{B^H, H \in (0,1)\}$ converges in law in the space $C([-D,D] \times [0,T])$, for any $D,T > 0$, to the local time l^{H_0} of fractional Brownian motion B^{H_0}, when H tends to H_0.*

Now consider the family of fractional Brownian motions $\{B^H, H \in (\frac{1}{2}, 1)\}$. Denote Brownian motion B and its local time $l(x,t)$ as the limit in law when H tends to $\frac{1}{2}$ of corresponding fractional Brownian motions $\{B^H\}$ and its local times. Let $\tilde{S} = S_0 \exp\{B\}$ be geometric Brownian motion and $f(\tilde{S})$ be the payoff of European call option in Black–Scholes market model. Then we have that

Corollary 3.2. *Assume the fixed proportional transaction costs coefficient* $k_0 = \sqrt{\frac{\pi}{2}}$. *Then*

$$\lim_{H \downarrow \frac{1}{2}} \big(\lim_{n \to \infty} V_\cdot(\theta^n) \big) = f(\tilde{S}_\cdot) - K l(\ln K, \cdot),$$

which limits take place in law in the space $C[0,T]$.

Corollary 3.3. *Assume the fixed proportional transaction costs coefficient* $k_0 = \frac{1}{2}\sqrt{\frac{\pi}{2}}$. *Then*

$$\lim_{n \to \infty} \big(\lim_{H \downarrow \frac{1}{2}} V_\cdot(\theta^n) \big) = f(\tilde{S}_\cdot) - K l(\ln K, \cdot),$$

which limits again take place in law in the space $C[0,T]$.

4 Conclusion

It is known that geometric fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$ has zero quadratic variation process. Cheridito [7] uses this fact to show pricing model based on it, admits arbitrage. Moreover in our frictionless model in the case of European call option the formula (1) implies that

$$(S_T - K)^+ = (S_0 - K)^+ + \int_0^T 1_{\{S_t > K\}} dS_t.$$

This indicates that out-of-money options are worthless. Moreover the hedging strategy $u_t = 1_{\{S_t > K\}}$ is an arbitrage opportunity. On the other hand, consider a frictionless pricing model with continuous price process

$$X_t = S_0 \exp\{B_t^H + \varepsilon W_t\} \quad \varepsilon > 0,$$

where W is a standard Brownian motion independent of B^H. This process has non-zero quadratic variation and fulfills all conditions of a recent result by Bender et.al. [4]. Their result asserts that, this pricing model does not admit arbitrage opportunities with reasonable trading strategies. This indicates that the existence of non-zero quadratic variation is important for option pricing based on no arbitrage.

Let f be a convex function with positive Radon measure μ as its second derivative. Consider continuous semimartingale $X_t = X_0 e^{W_t}$ with local time l_X and $X_0 \in \mathbb{R}_+$. By *Itô-Tanaka formula* (see [18], page 223) we have

$$\begin{aligned} f(X_1) &= f(X_0) + \int_0^1 f'_-(X_t)dX_t + \frac{1}{2}\int_{\mathbb{R}} l_X(a,[0,1])\mu(da) \\ &= f(X_0) + \int_0^1 f'_-(X_t)dX_t + \frac{1}{2}\int_{\mathbb{R}} a l_W(\ln a,[0,1])\mu(da). \end{aligned}$$

Hence our fractional Black-Scholes model with asymptotic proportional transaction costs has the same effect as the model which the stock price is modelled by semimartingale X. In this sense there is a connection between transaction costs and quadratic variation. We also refer the reader to monograph [20, chapter 6] for the link between existence of non-zero quadratic variation, Itô-Tanaka formula and transaction costs in financial markets.

5 Proofs

Proof of Theorem 2.1. To the contrary assume it is possible to superhedge the European option $f(S_T)$ with a less price and some trading strategy $u \in \mathcal{A}_{con}$. Then there exist a positive real number $\epsilon > 0$ and a function $g \in \mathcal{L}_{con}$ so that we have

$$f(S_0) + \int_0^T f'_-(S_t)dS_t = f(S_T) \overset{a.s.}{\leq} \big(f(S_0) - \epsilon\big) + \int_0^T g'_-(S_t)dS_t.$$

Applying Ito formula (1), we get

$$f(S_T) - f(S_0) < f(S_T) - f(S_0) + \epsilon \overset{a.s.}{\leq} g(S_T) - g(S_0).$$

On the other hand, by continuity property of functions f and g at S_0, there exists a $\delta = \delta(S_0)$ such that when $|x - S_0| < \delta$,

$$|f(x) - f(S_0)| < \frac{\epsilon}{2} \quad \text{and} \quad |g(x) - g(S_0)| < \frac{\epsilon}{2}.$$

Now set, $A_\delta = \{\omega \in \Omega \ : \ \ S_T \in (S_0 - \frac{\delta}{2}, S_0 + \frac{\delta}{2})\}$. Then $\mathbb{P}(A_\delta) > 0$ and

$$|f(S_T) - f(S_0)| < \frac{\epsilon}{2} \quad \text{and} \quad |g(S_T) - g(S_0)| < \frac{\epsilon}{2} \ \text{ on } \ A_\delta.$$

Hence on the set A_δ of positive probability, we have

$$-\frac{\epsilon}{2} + \epsilon \overset{a.s.}{\leq} f(S_T) - f(S_0) + \epsilon < \frac{\epsilon}{2}$$

which is a contradiction.

Proof of Theorem 2.2. We can assume that the support of μ is compact otherwise one can consider auxiliary convex functions

$$\tilde{f_n}(x) = \begin{cases} f'_+(0)x + f(0), & \text{if } x < 0, \\ f(x), & \text{if } 0 \leq x \leq n, \\ f'_-(n)(x-n) + f(n), & \text{if } x > n. \end{cases}$$

(see [2] for details). Since

$$\begin{cases} f(S_1) = f(S_0) + \int_0^1 f'_-(S_t) dS_t, \\ V_1(\theta^n) = f(S_0) + \sum_{i=1}^n f'_-(S_{t^n_{i-1}})(S_{t^n_i} - S_{t^n_{i-1}}) - k_n \sum_{i=1}^n S_{t^n_{i-1}} |f'_-(S_{t^n_i}) - f'_-(S_{t^n_{i-1}})|. \end{cases}$$

Therefore, we have

$$V_1(\theta^n) - f(S_1) = I^1_n - k_0 I^2_n,$$

where

$$\begin{cases} I^1_n = \sum_{i=1}^n f'_-(S_{t^n_{i-1}})(S_{t^n_i} - S_{t^n_{i-1}}) - \int_0^1 f'_-(S_t) dS_t, \\ I^2_n = \Delta_n^{1-H} \sum_{i=1}^n S_{t^n_{i-1}} |f'_-(S_{t^n_i}) - f'_-(S_{t^n_{i-1}})|. \end{cases}$$

Note that $I^1_n \to 0$ almost surely by (2.1). It remains to study the behavior of the second term I^2_n. The representation (5) relates the left derivative of convex function in term I_n^2 to Radon measure of its second derivative. We divide the proof in three steps, depending on the supp μ.

Step 1: supp $\mu = \{a\}$.
We can assume that $\mu(a) = 1$ and $f'_-(x) = 1_{\{x>a\}}$. This follows from representation (5).

For any $m \geq n$,

$$\Big|\Delta_m^{1-H}\sum_{j=1}^m S_{t_{j-1}^m}|1_{\{S_{t_j^m}>a\}} - 1_{\{S_{t_{j-1}^m}>a\}}| - \int_0^1 S_t l^H(\ln a, dt)\Big| \leq$$
$$\Delta_m^{1-H}\Big|\sum_{j=1}^m S_{t_{j-1}^m}|1_{\{S_{t_j^m}>a\}} - 1_{\{S_{t_{j-1}^m}>a\}}| - \sum_{i=1}^n S_{t_{i-1}^n}\sum_{j\in I(i)}|1_{\{S_{t_j^m}>a\}} - 1_{\{S_{t_{j-1}^m}>a\}}|\Big|$$
$$+\Big|\Delta_m^{1-H}\sum_{i=1}^n S_{t_{i-1}^n}\sum_{j\in I(i)}|1_{\{S_{t_j^m}>a\}} - 1_{\{S_{t_{j-1}^m}>a\}}| - \sum_{i=1}^n S_{t_{i-1}^n} l^H(\ln a, (t_{i-1}^n, t_i^n])\Big|$$
$$+\Big|\sum_{i=1}^n S_{t_{i-1}^n} l^H(\ln a, (t_{i-1}^n, t_i^n]) - \int_0^1 S_t l^H(\ln a, dt)\Big|$$
$$= A_{n,m} + B_{n,m} + C_n,$$

where for each $i = 1, 2, .., n, I(i) = \{j : t_j^m \in (t_{i-1}^n, t_i^n]\}$.
Obviously, $\lim_{n\to\infty} C_n = 0$, since $l^H(\ln a, \cdot)$ is increasing in t and

$$l^H(x, (s, t]) = l^H(x, t) - l^H(x, s), \quad s < t.$$

$$|B_{n,m}| \;\leq\; \sum_{i=1}^n S_{t_{i-1}^n}\Big|\Delta_m^{1-H}\sum_{j\in I(i)}|1_{\{S_{t_j^m}>a\}} \;-\; 1_{\{S_{t_{j-1}^m}>a\}}| \;-\; l^H(\ln a(t_{i-1}^n, t_i^n])\Big|.$$

Therefore for each fixed n by Theorem $B.1$ as $m \to \infty$,

$$\mathbb{P}\text{-}\lim_{m\to\infty}\Big|\Delta_m^{1-H}\sum_{j\in I(i)}|1_{\{S_{t_j^m}>a\}} - 1_{\{S_{t_{j-1}^m}>a\}}| - l^H(\ln a, (t_{i-1}^n, t_i^n])\Big|$$
$$= \mathbb{P}\text{-}\lim_{m\to\infty}\Big|\Delta_m^{1-H} N_{\Delta_m}^{\ln a}(B^H, (t_{j-1}^n, t_j^n]) - l^H(\ln a, (t_{i-1}^n, t_i^n])\Big| = 0.$$

Hence $B_{n,m}$ converges in probability to zero as m tends to infinity.

$$
\begin{aligned}
|A_{n,m}| &\leq \sum_{i=1}^{n} \Delta_m^{1-H} \sum_{j\in I(i)} |S_{t_{i-1}^n} - S_{t_{j-1}^m}||1_{\{S_{t_j^m}>a\}} - 1_{\{S_{t_{j-1}^m}>a\}}| \\
&\leq \sum_{i=1}^{n} \sup_{u\in(t_{i-1}^n,t_i^n)} |S_{t_{i-1}^n} - S_u|\Delta_m^{1-H} \sum_{j\in I(i)} |1_{\{S_{t_j^m}>a\}} - 1_{\{S_{t_{j-1}^m}>a\}}| \\
&\xrightarrow{\mathbb{P}} \sum_{i=1}^{n} \sup_{u\in(t_{i-1}^n,t_i^n)} |S_{t_{i-1}^n} - S_u| l^H(\ln a, (t_{i-1}^n, t_i^n]) = A_n.
\end{aligned}
$$

Fix $\varepsilon > 0$. Then there exists (see [8], page 21) a partition $\pi_\varepsilon := \{0 = t'_0 < t'_1 < \cdots < t'_m = 1\}$ of interval $[0,1]$ so that

$$
Osc\big(S, (t'_{i-1}, t'_i)\big) := \sup_{s,t\in(t'_{i-1},t'_i)} \big|S_t - S_s\big| \leq \varepsilon \quad \text{for each } i = 1, 2, \cdots, m.
$$

For $n \in \mathbb{N}$ and $\pi_n = \{t_i^n; i = 0, 1, \cdots, n\}$, set $\pi = \pi_\varepsilon \cup \pi_n = \{0 = t_0 < \cdots < t_l = 1\}$. Therefore

$$
Osc\big(S, (t_{i-1}, t_i)\big) = \sup_{s,t\in(t_{i-1},t_i)} |S_t - S_s| \leq \varepsilon, \qquad \text{for } t_i \in \pi.
$$

Hence,

$$
\begin{aligned}
A_n &\leq \sum_{t_i\in\pi} \sup_{u\in(t_{i-1},t_i)} |S_{t_{i-1}} - S_u| l^H(\ln a, (t_{i-1}, t_i]) \\
&\leq \varepsilon l^H(\ln a, [0,1]).
\end{aligned}
$$

So A_n converges to zero almost surely as n tends to infinity.

Step 2: supp $\mu = \{a_1, a_2, ..., a_l\}$.

Before to show the convergence in this case we need the following simple lemma.

Lemma 5.1. *For $x, y \in \mathbb{R}$ and positive numbers $\alpha_1, \alpha_2, ..., \alpha_l$ we have*

$$
|\sum_{j=1}^{l} (1_{\{y>a_j\}} - 1_{\{x>a_j\}})\alpha_j| = \sum_{j=1}^{l} |1_{\{y>a_j\}} - 1_{\{x>a_j\}}|\alpha_j.
$$

Next we work with I_n^2.

$$\begin{aligned}
I_n^2 &= \Delta_n^{1-H}\sum_{i=1}^{n} S_{t_{i-1}^n}|f'_-(S_{t_i^n}) - f'_-(S_{t_{i-1}^n})| \\
&= \Delta_n^{1-H}\sum_{i=1}^{n} S_{t_{i-1}^n}\Big|\frac{1}{2}\sum_{j=1}^{l}\big[(1_{\{S_{t_i^n}>a_j\}} - 1_{\{S_{t_i^n}<a_j\}}) - (1_{\{S_{t_{i-1}^n}>a_j\}} - 1_{\{S_{t_{i-1}^n}<a_j\}})\big]\mu(a_j)\Big| \\
&= \Delta_n^{1-H}\sum_{i=1}^{n} S_{t_{i-1}^n}\Big|\frac{1}{2}\sum_{j=1}^{l}(1_{\{S_{t_i^n}>a_j\}} - 1_{\{S_{t_{i-1}^n}>a_j\}})\mu(a_j)\Big| \\
&+ \Delta_n^{1-H}\sum_{i=1}^{n} S_{t_{i-1}^n}\Big|\frac{1}{2}\sum_{j=1}^{l}(1_{\{S_{t_{i-1}^n}<a_j\}} - 1_{\{S_{t_i^n}<a_j\}})\mu(a_j)\Big| \\
&= A_n + B_n
\end{aligned}$$

By lemma 5.1 we see that

$$\begin{aligned}
A_n &= \frac{1}{2}\Delta_n^{1-H}\sum_{i=1}^{n} S_{t_{i-1}^n}\sum_{j=1}^{l}\big|1_{\{S_{t_i^n}>a_j\}} - 1_{\{S_{t_{i-1}^n}>a_j\}}\big|\mu(a_j) \\
&= \frac{1}{2}\Delta_n^{1-H}\sum_{j=1}^{l}\mu(a_j)\sum_{i=1}^{n} S_{t_{i-1}^n}\big|1_{\{S_{t_i^n}>a_j\}} - 1_{\{S_{t_{i-1}^n}>a_j\}}\big| \\
&\xrightarrow{\mathbb{P}} \frac{1}{2}\sum_{j=1}^{l}\mu(a_j)\int_0^1 S_t l^H(\ln a_j, dt) = \frac{1}{2}\int_{\mathbb{R}}\int_0^1 S_t l^H(\ln a, dt)\mu(da).
\end{aligned}$$

By a similar argument for the term B_n we conclude that

$$\mathbb{P}\text{-}\lim_{n\to\infty} I_n^2 = \int_{\mathbb{R}}\int_0^1 S_t l^H(\ln a, dt)\mu(da).$$

Step 3: general case.

Let compact interval $[a,b]$ contain the supp μ and let P_m be the convex linear approximation of convex function f on the interval $[a,b]$ based on equidistant partition of interval $[a,b]$ i.e. polygonal with vertices $\{\big(a+i\Delta_m(b-a), f(a+i\Delta_m(b-a))\big)\}_{i=0}^m$. Define P_m's the same as f outside the interval $[a,b]$. Note that outside of the interval $[a,b]$ the convex function f is linear. Then for $m,n\in\mathbb{N}$ we have

$$|I_n^2 - \mathbf{J}| \le |I_n^2 - I_{n,m}| + |I_{n,m} - I_m| + |I_m - \mathbf{J}|,$$

where

$$\begin{cases} I_{n,m} = \Delta_n^{1-H} \sum_{i=1}^{n} S_{t_{i-1}^n} \big| (P_m)'_-(S_{t_i^n}) - (P_m)'_-(S_{t_{i-1}^n}) \big|, \\ I_m = \int_{\mathbb{R}} \int_0^1 S_t l^H(\ln a, dt) \mu_m(da). \end{cases}$$

By the elementary inequality $||a| - |b|| \leq |a - b|$, we have that

$$|I_n^2 - I_{n,m}| \leq \Delta_n^{1-H} \sum_{i=1}^{n} \Big| \big(P_m\big)'_-(S_{t_i^n}) - f'_-(S_{t_i^n})\big) - \big(P_m\big)'_-(S_{t_{i-1}^n}) - f'_-(S_{t_{i-1}^n})\big) \Big|.$$

Now for fixed n, by Theorem A.4 the right-hand side converges to zero almost surely as m tends to infinity. By simple calculations

$$\begin{aligned} \int_0^1 S_t l^H(\ln a, dt) &= \int_0^1 S_t \mathbf{1}_{\{B_t^H = \ln a\}} dt \\ &= \int_0^1 e^{\ln a} \mathbf{1}_{\{B_t^H = \ln a\}} dt \\ &= a l^H(\ln a, [0,1]). \end{aligned}$$

It follows that

$$|I_m - \mathbf{J}| \leq \Big| \int_{\mathbb{R}} a l^H(\ln a, [0,1]) \mu_m(da) - \int_{\mathbb{R}} a l^H(\ln a, [0,1]) \mu(da) \Big|.$$

By Theorem A.4 the right-hand side converges to zero almost surely as m tends to infinity, since support of μ is compact.

Finally, for fixed m by Step 2, we have that

$$\mathbb{P}\text{-} \lim_{n \to \infty} I_{n,m} = I_m.$$

A Auxiliary results on convex functions

We recall some results on convex functions. First, recall that every convex function $f : \mathbb{R} \to \mathbb{R}$ has a left-derivative f'_- and a right-derivative f'_+.

The next theorem gives information about the left-derivative f'_- and right-derivative f'_+.

Theorem A.1. *[18] The functions f'_- and f'_+ are increasing, respectively left and right-continuous and the set $\{x : f'_-(x) \neq f'_+(x)\}$ is at most countable.*

Moreover, the second derivative of a convex function f exists as a distribution, and first derivative can be represented in terms of the second derivative.

Theorem A.2. *[18] The second derivative f'' of convex function f exists in the sense of distributions, and it is a positive Radon measure; conversely, for any Radon measure μ on $\mathbb{R}$, there is a convex function f such that $f'' = \mu$ and for any interval I and $x \in \operatorname{int}(I)$ we have the equality*

$$f'_-(x) = \frac{1}{2}\int_I \operatorname{sgn}(x-a)\mu(da) + \alpha_I, \tag{4}$$

where α_I is a constant and $\operatorname{sgn} x = 1$ if $x > 0$ and -1 if $x \leq 0$.

Remark A.3. *If the supp(μ) is compact, then one can globally state that*

$$f'_-(x) = \frac{1}{2}\int \operatorname{sgn}(x-a)\mu(da) \tag{5}$$

up to a constant term.

Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function. For each interval $[a, b]$, let $\pi = \{a = a_0 < a_1 < ... < a_n = b\}$ be a partition of the interval and

$$\|\pi\| := \max_{1\leq i\leq n}(a_i - a_{i-1}).$$

A piecewise linear function through points $(a_i, f(a_i))$ is called a *convex linear approximation* of convex function f on the interval $[a, b]$ based on the partition π.

Theorem A.4. *Let $[a, b]$ be a closed interval and $\{\pi_m\}$ be a sequence of partitions of the interval $[a, b]$ such that*

$$\|\pi_m\| \to 0 \quad as \quad m \to \infty$$

where $\pi_m = \{a_1, a_2, ..., a_{n(m)}\}$.

Let P_m be a convex linear approximation of convex function f on the interval $[a, b]$ based on partitions π_m. Then we have

(i) On the interval (a, b) as $m \to \infty$

$$P_m \to f \qquad and \qquad (P_m)'_- \to f'_- \qquad pointwise.$$

(ii) For any bounded continuous function g we have

$$\int_{[a,b]} g d\mu_m \to \int_{[a,b]} g d\mu \qquad as \quad m \to \infty,$$

where μ_m stands for Radon measure corresponding to the second derivative of P_m.

B Local time of fractional Brownian motion

The *occupation measure* related to fractional Brownian motion is defined by

$$\Gamma_{B^H}(I \times U) = \lambda\{t \in I : B_t^H \in U\} = \int_I \mathbf{1}_{\{B_t^H \in U\}} dt$$

where I and U are Borel sets on time interval $[0,T]$ and the real line respectively and λ stands for Lebesgue measure. It is well-known that the occupation measure has a jointly continuous density (*local time*) which is denoted by $l^H(x,t) := l^H(x,[0,t])$ and is Hölder continuous in t of any order $\alpha < 1-H$ and in x of any order $\beta < \frac{1-H}{2H}$ (for a survey article on the subject see Geman and Horowitz [9]).

Let $B_\Delta^H = \{B_\Delta^H(t)\}_{t\in[0,T]}$ be polygonal approximation of size Δ of B^H i.e. B_Δ^H is the polygonal lines which connect points $\{?(i\Delta, B^H(i\Delta))\}$ for suitable running index i. Set

$$C_\Delta^a(B^H,[0,T]) = \{t \in [0,T] : B_\Delta^H(t) = a \quad \text{and} \quad t? \neq i\Delta \quad \text{for each index } \ i\}$$

and

$$N_\Delta^a(B^H,[0,T]) = \# \, C_\Delta^a(B^H,[0,T]),$$

i.e. the number of level a crossing of B_Δ^H over interval $[0,T]$. Then we have the following approximation for the local time $l^H(a,t)$.

Theorem B.1. *Assume B^H be a standard fractional Brownian motion with $H \in (0,1)$ and $N_\Delta^a(B^H,[0,T])$ be the number of level a crossing of size $\Delta-$ polygonal approximation of B^H. Then*

$$\sqrt{\frac{\pi}{2}}\Delta^{1-H}N_\Delta^a(B^H,[0,T]) \to l^H(a,[0,T]) \quad in \quad L^2 \qquad as \ \ \Delta \to 0.$$

Proof. See [1], Theorem 5.

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