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Existence results for nonlinear fractional differential equations with integral boundary value problems

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Abstract

In this paper, by using contraction mapping principle and Krasnoselskii's fixed theorem, we study the existence results of solution to the integral boundary value problem(BVP),

$$D^{\alpha}u(t) + f(t, u(t)) = 0, \quad 0 < t < 1,$$
$$u(0) = u'(0) = 0, \quad u(1) = \lambda \int_0^1 u(s)\mathrm{d}s,$$

where $2 < \alpha \leq 3$ is a real number, D^{α}_{0+} is the standard Riemann-Liouville differentiation, and $f : [0,1] \times X \to X$ is continuous, and $\lambda \in R$ is such that $\lambda \neq \alpha$. Here, $(X, \|\cdot\|)$ is a Banach space.

Mathematics Subject Classification: 34B15
Keywords: Riemann-Liouville derivative; Contraction mapping principle; Krasnoselskii's fixed theorem; Integral boundary value problem

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1 Introduction

Fractional differential equations have gained considerable importance due to their application in various sciences, such as physics, mechanics, chemistry, engineering, etc. On this kind of equations the derivatives of fractional order [1–3] are involved. The interest of the study of fractional-order differential equations lies in the fact that fractional-order models are more accurate than integer-order models,that is, there are more degrees of freedom in the fractional-order models. Furthermore, fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes due to the existence of a "memory" term in a model. This memory term insures the history and its impact to the present and future, see[4]. In consequence, the subject of fractional differential equations is gaining much importance and attention. For details, see [5-12] and the references therein.

Integral boundary conditions have various applications in applied fields such as blood flow problems,chemical engineering, thermo-elasticity, underground water flow, population dynamics, and so forth. For a detailed description of the integral boundary conditions, we refer the reader to some recent papers [13-17] and the references therein.

A. Cabada, Wang [18] considered the positive solutions of nonlinear fractional differential equations with integral boundary value conditions

$$\begin{aligned} &{}^{C}D^{\alpha}u(t) + f(t, u(t)) = 0, \quad 0 < t < 1,\\ &u(0) = u''(0) = 0, \quad u(1) = \lambda \int_0^1 u(s)\mathrm{d}s, \end{aligned}$$

where $2 < \alpha < 3, 0 < \lambda < 2$, ${}^{C}D^{\alpha}$ is the Caputo fractional derivative and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is a continuous function. The author obtained at least one positive solution by Guo-Krasnoselskii's fixed point theorem.

Ahmad et al. [19] considered the existence and uniqueness of solutions for a boundary value problem of nonlinear fractional differential equations of order $q \in (1, 2]$ with three-point integral boundary conditions:

$$\begin{aligned} &{}^{C}D^{q}x(t) = f(t, x(t), \quad 0 < t < 1,\\ &x(0) = 0, \quad x(1) = \alpha \int_0^{\eta} x(s)\mathrm{d}s, \quad 0 < \eta < 1, \end{aligned}$$

where ${}^{C}D^{q}$ denotes the Caputo fractional derivative of order $q, f : [0,1]\times X \to X$ is continuous and $\alpha \neq \frac{2}{\eta^2}$. The author obtained at least one solution by Krasnoselskii's fixed point theorem.

Motivated by papers [18] and [19], in this paper, we deal with the following nonlinear fractional differential equations with integral boundary value problem

$$D^{\alpha}u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \tag{1}$$

$$u(0) = u'(0) = 0, \quad u(1) = \lambda \int_0^1 u(s)\mathrm{d}s, \tag{2}$$

where D^{α}denotes Riemann-Liouville fractional derivative of order α, $2 < \alpha \leq 3$, $f : [0,1] \times X \to X$ is continuous, and $\lambda \in R$ is such that $\lambda \neq \alpha$. Here, $(X, \|\cdot\|)$ is a Banach space and $\mathcal{C} = C([0,1], X)$ denotes the Banach space of all continuous functions from $[0,1] \to X$ endowed with a topology of uniform convergence with the norm denoted by $\|\cdot\|$. In paper, by using contraction mapping principle and Krasnoselskii's fixed theorem, we study the existence results of solution to the integral boundary value problem (1) and (2).

To the authors' knowledge, no one has studied the existence of solutions for fractional boundary value problems (1) and (2). The goal of present paper is by using some fixed-point theorems, we obtain sufficient conditions for the existence of integral boundary value problem.

2 Preliminary Notes

In this section, we present some definitions and establish some lemmas.

Definition 2.1. ([2]) *The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (0, +\infty) \to \mathbb{R}$ is given by*

$$D^{\alpha}_{0^+}f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{(n)} \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}}\,\mathrm{d}s,$$

where $n = [\alpha] + 1$ denotes the integer part of number α, provided that the right side is pointwise defined on $(0, +\infty)$.

Definition 2.2. ([2]) *The Riemann-Liouville fractional integral of order* $\alpha > 0$ *of a function* $f : (0, +\infty) \to \mathbb{R}$ *is given by*

$$\mathrm{I}_{0+}^{\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\mathrm{d}s,$$

provided that the right side is point wise defined on $(0, +\infty)$.

Lemma 2.3. ([2]) *Let* $\alpha > 0$. *If we assume* $u \in C(0,1) \bigcap L(0,1)$, *then the fractional differential equation*

$$D_{0+}^{\alpha} u(t) = 0$$

has $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \ldots + c_N t^{\alpha-N}, c_i \in \mathbb{R}, i = 1, 2, \ldots, N$, *as unique solutions, where* N *is the smallest integer greater than or equal to* α.

Lemma 2.4. ([2]) *Assume that* $u \in C(0,1) \bigcap L(0,1)$ *with a fractional derivative of order* $\alpha > 0$ *that belongs to* $C(0,1) \bigcap L(0,1)$. *Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \ldots + c_N t^{\alpha-N}$$

for some $c_i \in \mathbb{R}, i = 1, 2, \ldots, N$, *where* N *is the smallest integer greater than or equal to* α.

Lemma 2.5. *Given* $y \in C[0,1]$ *and* $2 < \alpha < 3$, *the unique solution of*

$$D^{\alpha} u(t) + y(t) = 0, \quad 0 < t < 1, \tag{3}$$

$$u(0) = u'(0) = 0, \quad u(1) = \lambda \int_0^1 u(s)\mathrm{d}s, \tag{4}$$

is

$$u(t) = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\mathrm{d}s + \int_0^1 \frac{(\alpha - \lambda + s)t^{\alpha-1}(1-s)^{\alpha-1}}{(\alpha-\lambda)\Gamma(\alpha)} y(s)\mathrm{d}s.$$

Proof. We may apply Lemma 2.1 to reduce Eq. (3) to an equivalent integral equation

$$u(t) = I_{0+}^{\alpha} y(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + C_3 t^{\alpha-3},$$

for some $C_1, C_2, C_3 \in R$. Consequently, the general solution of Eq. (3) is

$$u(t) = -\int_0^t \frac{(t-s)^{\alpha-1} y(s)}{\Gamma(\alpha)} \mathrm{d}s + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + C_3 t^{\alpha-3},$$

The boundary condition $u(0) = 0, u'(0) = 0$ implies that $C_3 = 0, C_2 = 0$, We get

$$u(t) = -\int_0^t \frac{(t-s)^{\alpha-1}y(s)}{\Gamma(\alpha)}\mathrm{d}s + C_1 t^{\alpha-1}.$$

Finally, condition $u(1) = \lambda\int_0^1 u(s)\mathrm{d}s$ implies that

$$u(1) = -\int_0^1 \frac{(1-s)^{\alpha-1}y(s)}{\Gamma(\alpha)}\mathrm{d}s + C_1 = \lambda\int_0^1 u(s)\mathrm{d}s,$$
$$C_1 = \int_0^1 \frac{(1-s)^{\alpha-1}y(s)}{\Gamma(\alpha)}\mathrm{d}s + \lambda\int_0^1 u(s)\mathrm{d}s.$$

Hence, we have the following form

$$u(t) = -\int_0^t \frac{(t-s)^{\alpha-1}y(s)}{\Gamma(\alpha)}\mathrm{d}s + t^{\alpha-1}\int_0^1 \frac{(1-s)^{\alpha-1}y(s)}{\Gamma(\alpha)}\mathrm{d}s + \lambda t^{\alpha-1}\int_0^1 u(s)\mathrm{d}s, \quad (5)$$

Let $\int_0^1 u(s)\mathrm{d}s = A$, then, from the previous equality, we deduce that

$$\begin{aligned} A &= \int_0^1 u(t)\mathrm{d}t \\ &= -\int_0^1\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s\mathrm{d}t + \int_0^1 t^{\alpha-1}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s\mathrm{d}t \\ &\quad +\lambda\int_0^1 t^{\alpha-1}\int_0^1 u(s)\mathrm{d}s\mathrm{d}t \\ &= -\int_0^1 \frac{(1-s)^{\alpha}}{\alpha\Gamma(\alpha)}y(s)\mathrm{d}s + \frac{1}{\alpha}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s + \frac{\lambda A}{\alpha}. \end{aligned} \quad (6)$$

So, expression (6) implies that

$$A = -\frac{1}{\alpha-\lambda}\int_0^1 \frac{(1-s)^{\alpha}y(s)}{\Gamma(\alpha)}\mathrm{d}s + \frac{1}{\alpha-\lambda}\int_0^1 \frac{(1-s)^{\alpha-1}y(s)}{\Gamma(\alpha)}.$$

Replacing this value in (5), we arrive at the following expression for the function u:

$$\begin{aligned} u(t) &= -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s + t^{\alpha-1}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s \\ &\quad -\frac{1}{\alpha-\lambda}\int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha}}{\Gamma(\alpha)}y(s)\mathrm{d}s + \frac{1}{\alpha-\lambda}\int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s \\ &= -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s + \int_0^1 \frac{(\alpha-\lambda+s)t^{\alpha-1}(1-s)^{\alpha-1}}{(\alpha-\lambda)\Gamma(\alpha)}y(s)\mathrm{d}s. \end{aligned}$$

This completes the proof. □

In view of Lemma (2.3), we define an operator $F:\mathcal{C}\to\mathcal{C}$ by

$$\begin{aligned}(Fu)(t) &= -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))\mathrm{d}s \\ & q+\int_0^1 \frac{(\alpha-\lambda+s)t^{\alpha-1}(1-s)^{\alpha-1}}{(\alpha-\lambda)\Gamma(\alpha)} f(s,u(s))\mathrm{d}s. \end{aligned} \tag{7}$$

To prove the main results, we need the following assumptions:

(A_1) $\|f(t,x)-f(t,y)\|\le L\|x-y\|$, for all $t\in[0,1], L>0, x,y\in X$;

(A_2) $\|f(t,u)\|\le \mu(t)$, for all $(t,u)\in[0,1]\times X$, and $\mu\in L^1([0,1],R^+)$.

For convenience, let us set

$$\Lambda=\frac{1}{\Gamma(\alpha+1)}\left(1+\frac{|\alpha-\lambda|(\alpha+1)+1}{|\alpha-\lambda|(\alpha+1)}\right). \tag{8}$$

3 Main Results

Theorem 3.1. *Assume that $f:[0,1]\times X\to X$ is a jointly continuous function and satisfies the assumption (A_1) with $L<\frac{1}{\Lambda}$, where Λ is given by (8). Then the boundary value problem (1) and (2) has a unique solution.*

Proof. Setting $\sup_{t\in[0,1]}|f(t,0)|=M$ and choosing $r\ge\frac{\Lambda M}{(1-L\Lambda)}$, we show that $FB_r\subset B_r$, where $B_r=\{u\in\mathcal{C}:\|u\|\le r\}$. For $u\in B_r$, we have

$$\begin{aligned}\|Fu(t)\| &\le \frac{1}{\Gamma(\alpha)}\int_0^1 (t-s)^{\alpha-1}\|f(s,u(s))\|\mathrm{d}s \\ &+ \left|\frac{1}{(\alpha-\lambda)\Gamma(\alpha)}\right|\int_0^1(\alpha-\lambda+s)t^{\alpha-1}(1-s)^{\alpha-1}\|f(s,u(s))\|\mathrm{d}s \\ &\le \frac{1}{\Gamma(\alpha)}\int_0^1 (t-s)^{\alpha-1}(\|f(s,u(s))-f(s,0)\|+\|f(s,0)\|)\mathrm{d}s \\ &\quad +\left|\frac{t^{\alpha-1}}{(\alpha-\lambda)\Gamma(\alpha)}\right|\int_0^1(1-s)^{\alpha-1}(\alpha-\lambda+s)\cdot \\ &\quad \cdot(\|f(s,u(s))-f(s,0)\|+\|f(s,0)\|)\mathrm{d}s \\ &\le (Lr+M)\left[\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mathrm{d}s\right. \\ &\quad \left.+\left|\frac{t^{\alpha-1}}{(\alpha-\lambda)\Gamma(\alpha)}\right|\int_0^1(1-s)^{\alpha-1}(\alpha-\lambda+s)\mathrm{d}s\right]\end{aligned}$$

$$
\begin{aligned}
&\leq \ (Lr+M)\left(\frac{1}{\Gamma(\alpha+1)}+\left|\frac{t^{\alpha-1}}{(\alpha-\lambda)\Gamma(\alpha)}\right|\left[\frac{\alpha-\lambda}{\alpha}+\frac{1}{\alpha(\alpha+1)}\right]\right)\\
&\leq \ \frac{(Lr+M)}{\Gamma(\alpha+1)}\left(1+\frac{|\alpha-\lambda|(\alpha+1)+1}{|\alpha-\lambda|(\alpha+1)}\right)\\
&\leq \ (Lr+M)\Lambda\leq r. \qquad (9)
\end{aligned}
$$

Now, for $x, y \in \mathcal{C}$ and for each $t \in [0,1]$, we obtain

$$
\begin{aligned}
\|F(u_1(t))-F(u_2(t))\| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,u_1(s))-f(s,u_2(s))\|\mathrm{d}s\\
&+\left|\frac{t^{\alpha-1}}{(\alpha-\lambda)\Gamma(\alpha)}\right|\int_0^1(\alpha-\lambda+s)(1-s)^{\alpha-1}\cdot\\
&\cdot\|f(s,u_1(s))-f(s,u_2(s))\|\mathrm{d}s\\
&\leq L\|u_1-u_2\|\left[\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\mathrm{d}s\right.\\
&\left.+\left|\frac{t^{\alpha-1}}{(\alpha-\lambda)\Gamma(\alpha)}\right|\int_0^1(\alpha-\lambda+s)(1-s)^{\alpha-1}\mathrm{d}s\right]\\
&\leq \frac{L}{\Gamma(\alpha+1)}\left(1+\frac{|\alpha-\lambda|(\alpha+1)+1}{|\alpha-\lambda|(\alpha+1)}\right)\|u_1-u_2\|\\
&= L\Lambda\|u_1-u_2\| \qquad (10)
\end{aligned}
$$

where Λ is given by (8). Observe that Λ depends only on the parameters involved in the problem. As $L < \frac{1}{\Lambda}$, therefore F is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle (Banach fixed point theorem). □

Theorem 3.2. ([20]) (Krasnoselskii's fixed point theorem) *Let M be a closed convex and nonempty subset of a Banach space X. Let A, B be the operators such that*

(1) *$Ax + By \in M$ whenever $x, y \in M$;*

(2) *A is compact and continuous;*

(3) *B is a contraction mapping.*

Then there exists $z \in M$ such that $z = Az + Bz$.

Theorem 3.3. *Let $f : [0,1] \times X \to X$ be a jointly continuous function mapping bounded subsets of $[0,1] \times X$ into relatively compact subsets of X, and the assumptions (A_1) and (A_2) hold with*

$$\frac{L}{\Gamma(\alpha+1)}\left(\frac{|\alpha-\lambda|(\alpha+1)+1}{|\alpha-\lambda|(\alpha+1)}\right) < 1. \tag{11}$$

Then the boundary value problem (1) and (2) has at least one solution on $[0,1]$.

Proof. Letting $\sup_{t\in[0,1]}|\mu(t)| = \|\mu\|$, we fix

$$\overline{r} \geq \frac{\|\mu\|}{\Gamma(\alpha+1)}\left(1+\frac{|\alpha-\lambda|(\alpha+1)+1}{|\alpha-\lambda|(\alpha+1)}\right), \tag{12}$$

and consider $B_{\overline{r}} = \{u \in \mathcal{C}; \|u\| \leq \overline{r}\}$. We define the operators D and Q on $B_{\overline{r}}$ as

$$Du(t) = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))\mathrm{d}s, \tag{13}$$

$$Qu(t) = \int_0^1 \frac{(\alpha-\lambda+s)t^{\alpha-1}(1-s)^{\alpha-1}}{(\alpha-\lambda)\Gamma(\alpha)} f(s,u(s))\mathrm{d}s \tag{14}$$

For $u_1, u_2 \in B_{\overline{r}}$, we find that

$$\|Du_1 + Qu_2\| \leq \frac{\|\mu\|}{\Gamma(\alpha+1)}\left(1+\frac{|\alpha-\lambda|(\alpha+1)+1}{|\alpha-\lambda|(\alpha+1)}\right) \leq \overline{r}. \tag{15}$$

Thus, $Du_1 + Qu_2 \in B_{\overline{r}}$. It follows from the assumption (A_1) together with (3.3) that Q is a continuous mapping. Continuity of f implies that the operator D is continuous. Also, D is uniformly bounded on $B_{\overline{r}}$ as

$$\|Du\| \leq \frac{\|\mu\|}{\Gamma(\alpha+1)}. \tag{16}$$

Now we prove the compactness of the operator D.

In view of (A_1), we define $\sup_{(t,u)\in[0,1]\times B_{\overline{r}}}|f(t,u)| = \overline{f}$, and consequently we have

$$\begin{aligned}
&\|Du(t_1 - Du(t_2))\| \\
= &\left\|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}]f(s,u(s))\mathrm{d}s \right.\\
&\left. + \int_{t_1}^{t_2}(t_2-s)^{\alpha-1}f(s,u(s))\mathrm{d}s\right\| \\
\leq &\frac{\overline{f}}{\Gamma(\alpha+1)}\left|2(t_2-t_1)^\alpha + t_1^\alpha - t_2^\alpha\right|
\end{aligned} \tag{17}$$

which is independent of u. Thus, D is equicontinuous. Using the fact that f maps bounded subsets into relatively compact subsets, we have that $D(\mathcal{A})$ is relatively compact in X for every t, where $\mathcal{A}$ is a bounded subset of $\mathcal{C}$. So D is relatively compact on $B_{\overline{r}}$. Hence, by the Arzela-Ascoli Theorem, D is compact on $B_{\overline{r}}$. Thus all the assumptions of Theorem (3.2) are satisfied. So the conclusion of Theorem (3.2) implies that the boundary value problem (1) and (2) has at least one solution on $[0, 1]$. $\square$

4 Examples

Examples 4.1 Consider the following equation

$$D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad 2 < \alpha \leq 3, \quad 0 < t < 1, \tag{18}$$

$$u(0) = u'(0) = 0, \quad u(1) = \int_0^1 u(s)\mathrm{d}s. \tag{19}$$

where $f(t, u(t)) = -\frac{1}{(t+7)^2} \frac{\|u\|}{1+\|u\|}$, and $\lambda = 1$.

By simple calculation, we know all conditions of Theorem 3.1 are satisfied. By Theorem 3.1, BVP (18) and (19) has a unique solution on $[0, 1]$.

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