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Internal gravity wave tunnelling in a perfectly conducting incompressible fluid with horizontal magnetic field

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Abstract

Internal Alfvén-gravity wave tunnelling through an incompressible, inviscid, Boussinesq, perfectly conducting fluid in presence of a uniform horizontal magnetic field is investigated. The specific cases of wave transmission due to the uniform fluid extending to infinity on both the sides, stratified fluid with discontinuity in Brunt-väisälä frequency N at the interface and of a stratified fluid with discontinuity in density is derived and analysed. The transmission co-efficient of an internal Alfvén-gravity wave crossing over a barrier is computed. The effect of magnetic field on the transmission co-efficient reveal that the magnetic field enhances the transmission of internal Alfvén-gravity waves which is significant in energy transfers in conducting fluids.

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1 Introduction

Internal gravity waves in the upper atmosphere play an important role in the production of certain ionospheric phenomena [1]. A full understanding of their role will depend in part on an understanding of the propagation conditions met at all levels in the atmosphere, and more specifically, of the part played by reflection and ducting [2]. More recent studies of gravity wave ducting in the ocean, the laboratory and the atmosphere have revealed that the formation of ducting of gravity wave activity to be important at smaller scales as well [3].

Internal gravity waves transport energy and momentum in density-stratified fluids on relatively fast time scales [4]. These waves are responsible for a variety of processes in the mesosphere and lower thermosphere, generation of turbulence, formation of general circulation pattern of the atmosphere, deposition of net momentum, eddy conduction of heat, mixing of atmospheric constituents, fluctuation in atmospheric drag and nonlinear interaction with tidal/planetary waves leading to variability of planetary-scale motions and momentum and energy transfer from the troposphere to the middle and upper atmosphere. Gravity waves generated in the troposphere propagate into the stratosphere. Perhaps most important is their role in determining the mean flow of the atmosphere [5]. The wave frequency of internal gravity waves is always be less than the buoyancy frequency. If the wave frequency is greater than buoyancy frequency N, the waves are evanescent and so the amplitude of these waves decrease exponentially with height in the medium.

Gravity waves may be vertically propagating or trapped depending on the background wind, buoyancy characteristics of the atmosphere and their horizontal wavelength. When trapping is essentially complete the waves are quasi-permanent free waves. Henceforward we refer to waves that exhibit a high degree of trapping and propagate horizontally in a trapping layer over many wavelengths as *ducted* waves [6].

Energy sources for gravity wave come from latent heat, deep convection, shear instability and wave ducting. Evanescence of wave in a certain region can also cause wave ducting or tunneling and may provide a means of transporting vertically the energy and momentum associated with wave motions [3].

Earlier studies of gravity wave ducting was confined to three mechanisms namely variation in atmospheric structure [7], the variation in dissipation ([8], [9], [11]) and the variation in the background wind [10] and [11] have shown that there exists another ducting mechanism valid only for gravity waves namely the variation in buoyancy period. The variation in the Brunt frequency could serve as a ducting and tunnelling mechanism which filters out and restricts the higher-frequency components to only the lower altitudes which is referred as Brunt ducting. Addition to above the Studies of the tunnelling of internal gravity wave motions in the atmosphere and ocean have discovered many of the important properties such as potential wave energy transfer in the atmosphere.

The upward propagation of an internal wave packet impinging on a layer of uniform density $(N = 0)$ should reflect. These waves become evanescent in uniform density regions so that the amplitude decreases exponentially with height. When the fluid becomes stratified again at the end of the uniform density layer, the wave will partially transmit across the interface and partially reflect of the layer. This is the process of internal gravity wave tunnelling [12]. Internal wave tunneling between two ducts in the ocean has previously been described theoretically by Eckart [14].

The Sutherland and Yewchuk [12] have derived an analytic theory for internal gravity wave tunneling through a weakly stratified fluid in the atmosphere. They have obtained the transmission coefficient of internal waves crossing a weakly stratified region. This theory provides quantitative predictions of partial reflection and transmission of internal waves incident upon a weakly stratified layer.

The theory of momentum transport by gravity waves in a conducting fluid in the presence of a magnetic field is an area of considerable interest in meteorological, geophysical and astrophysical studies [15]. The study of internal Alfvén-gravity wave is significant to study waveforms in noctilucent clouds, creation of turbulence, ionospheric drifts, travelling ionospheric disturbances, because the internal Alfvén-gravity waves account for the ionospheric irregularities. The irregularities of ionization result from the density fluctuations [16] and also he suggested that the reflection imposed by the temperature variation of the middle atmosphere would provide a suitable trapping mechanism. The magnetosphere itself is region of the earth's upper atmosphere and of the ionosphere. Hence it seems appropriate to investigate different reflection and trapping mechanisms due to magnetic field. Internal gravity waves in an unbounded fluid can be trapped to a layer of finite depth by periodic small variations in either the density gradient or in a weak horizontal steady current [17]. In spite of these applications much attention has not been given to the study of internal Alfvén-gravity wave tunnelling in electrically conducting fluids. In the present paper we have studied this problem in an incompressible, inviscid, Boussinesq, perfectly conducting weakly stratified fluid in the presence of a uniform aligned magnetic field.

We study the tunnelling behaviour of internal Alfvén-gravity waves and provides an analytic prediction for the transmission co-efficient of internal Alfvén-gravity waves crossing the tunnelling region, a region in which N^2 is reduced (weakly stratified region). The results are of geophysical and astrophysical importance. The transmission co-efficient of an internal Alfvén-gravity wave crossing over a barrier is computed. The effect of magnetic field on the transmission of internal gravity waves is studied.

2 Mathematical Formulation

We consider an electrically conducting inviscid stratified fluid in which the motion is two dimensional, variation being in the x and z directions (i.e. horizontal and vertical directions respectively). Initially the fluid is assumed to be in the state of rest. It can be shown that under these assumptions the stream function ψ satisfies the following linearized equation.

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\frac{\partial^2\psi}{\partial t^2}+N^2\frac{\partial^2\psi}{\partial x^2}=A^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\frac{\partial^2\psi}{\partial x^2}. \qquad (2.1)$$

Here, the stratification of the mean flow is described in terms of a single parameter which may vary with z, the Brunt-väisälä frequency N, defined by $N=\left(\frac{-g}{\rho_0}\frac{d\rho_0}{dz}\right)^{\frac{1}{2}}=(g\beta)^{\frac{1}{2}}$ and the magnetic field is described by another parameter A, which is the Alfvén velocity defined by $A=\sqrt{\mu H_0^2/\rho_b}$.

We assume that the two-dimensional transient disturbance produced by temporary extraneous forces is horizontally and temporally periodic in the form

$$\psi=\hat{\psi}(z)\exp[i(kx-\omega t)] \quad . \qquad (2.2)$$

3 Transmisson Across Density Barrier

We study this for the following three cases of density stratification:

(i) Uniform density conducting fluid of finite depth $-L/2<z<+L/2$ sandwiched on either side by a stratified conducting fluid extending to infinity, called N^2-barrrier1.

(ii) Weakly stratified conducting fluid of finite depth $-L/2<z<+L/2$ sandwiched on either side by a conducting stratified fluid extending to infinity with density discontinuity at the two interfaces, called N^2-barrrier2.

(iii) Weakly stratified conducting fluid of finite depth $-L/2 < z < +L/2$ sandwiched on either side by a stratified conducting fluid extending to infinity with density being continuous at the interfaces, called locally mixed region.

3.1 Transmission across N^2-barrier1

In this case we have a fluid of uniform density of finite depth L bounded on either side by a stratified conducting fluid extending to infinity. We assume

$$N^2 = \begin{cases} N_0^2, & |z| > \dfrac{L}{2} \\ 0, & |z| \le \dfrac{L}{2} \end{cases} . \tag{3.1}$$

This is called 'N^2-barrier1' of depth L as shown in Figure 1.

With this the solution of (2.1) takes the form

$$\hat{\psi} = \begin{cases} A_3 e^{i\eta z} & z > \dfrac{L}{2} \\ A_2 e^{\frac{z}{\delta}} + B_2 e^{\frac{-z}{\delta}} & \dfrac{-L}{2} \le z \le \dfrac{L}{2} \\ A_1 e^{i\eta z} + B_1 e^{-i\eta z} & z < \dfrac{L}{2} \end{cases} , \tag{3.2}$$

where $\eta = -k\left(\dfrac{N_0^2}{\omega^2 - k^2 A^2} - 1\right)^{\frac{1}{2}}$, $A = \sqrt{\mu H_0^2/\rho_b}$ and $\delta = \dfrac{1}{k}$, $k(>0)$ is the well defined horizontal wave number, $\omega(\le N)$ is the wave frequency and we assume $\omega > kA$. Applying the boundary conditions [18] to the solution (3.2) we get a system of four linear equations in five unknowns A_1, B_1, A_2, B_2 and A_3.

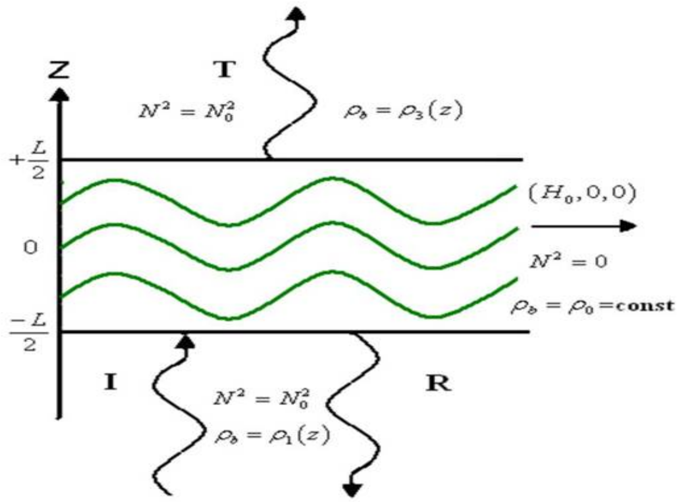

Figure 1: Schematic representation of the system(N^2 -barrier1)

Solving for transmitted amplitude A_3, in terms of the incident amplitude A_1, gives a transmission coefficient $T_m = |A_3/A_1|^2$, which represents the fraction of energy transported across N^2 -barrrier1 and is given by

$$T_m = \left[1 + \frac{\left(1+\eta^2\delta^2\right)^2}{4\eta^2\delta^2}\sinh^2\left(\frac{L}{\delta}\right)\right]^{-1}. \tag{3.3}$$

In the limit $A \to 0$ the above expression reduces to

$$T_s = \left[1 + \frac{\left(1+\gamma^2\delta^2\right)^2}{4\gamma^2\delta^2}\sinh^2\left(\frac{L}{\delta}\right)\right]^{-1}, \tag{3.4}$$

where $\gamma = -k\left(\frac{N_0^2}{\omega^2} - 1\right)^{\frac{1}{2}}$ and $k = \frac{1}{\delta}$ and T_s is the limit of T_m as $A \to 0$.

The maximum value of T_m with respect to A is given by

$$\left(T_m\right)_{\max} = \left[1 + sinh^2\left(L/\delta\right)\right]^{-1} \tag{3.5}$$

at $A = A_{max} = \sqrt{\frac{\omega^2}{k^2} - \frac{N_0^2}{2k^2}}$. A_{max} is real if $2\omega^2 > N_0{}^2$. We also note from expression for η that when $A = \omega/k$ the vertical wave number η tends to ∞ so that the waves become more and more horizontal. Thus the effect of magnetic field is to make the wave to propagate along the magnetic field lines rather than allow it to propagate upwards. When A increase further η becomes complex and the waves becomes evanescent in the region $|z| \geq L/2$ also.

We have plotted the graph of T_m against A in Figures 2a and 2b for various values of N_0 and ω. From these graphs we find that the maximum value of T_m is the same. Thus the maximum value of T_m is independent of N_0 and ω as seen from expression (3.5).

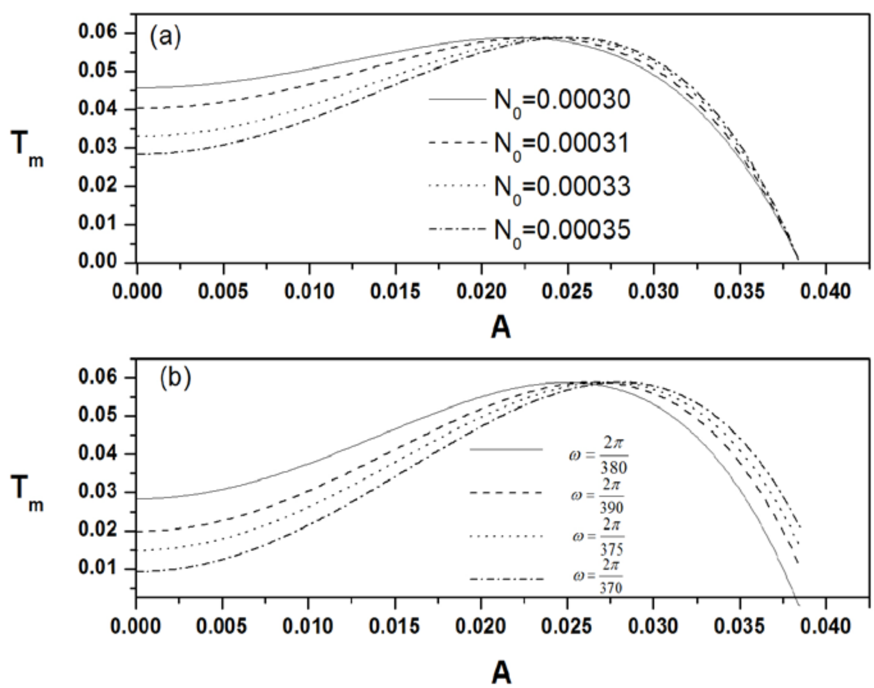

Figure 2: Variation of Transmission coefficient T_m for $2\omega^2 > N_0{}^2$ and (a) $\omega = 2\pi/390$, $k = 2\pi/15$ and (b) $N_0 = 0.0004$, $k = 2\pi/15$.

However, when $2\omega^2 < N_0{}^2$, $A_{\max}$ is not real. Thus the maximum for T_m does not exist. As A increases T_m continuously decreases and becomes 0 when $A = \omega/k$. This is shown in Figure 3. From (3.3) we find that when $A = 0$, $T_m = T_s$ where T_s is the transmission coefficient obtained in the hydrodynamic case of Sutherland and Yewchuk [13].

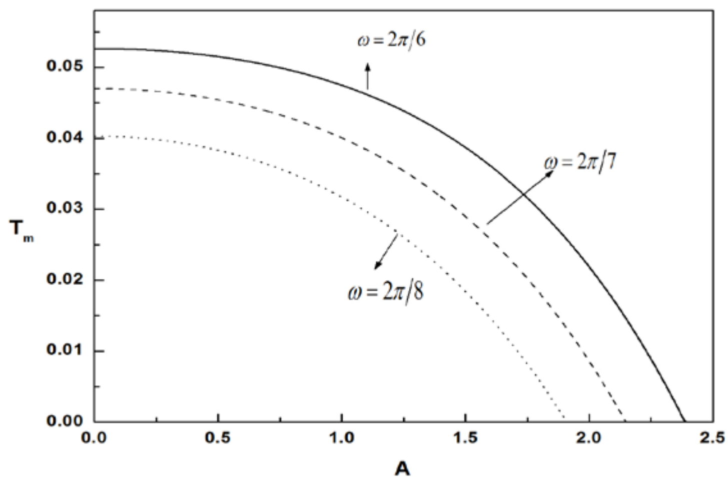

Figure 3: Variation of transmission coefficient T_m with A for $2\omega^2 < N_0{}^2$ and $N_0 = 3$ $k = 2\pi/15$, $N_1 = 0.00002$.

The variation of the transmission coefficient T_m with A further is analysed by expanding T_m in terms of A^2 for $A << 1$ using asymptotic expansion. When $A << 1$, the transmission coefficient T_m given by (3.3) can be expanded in the form

$$\left(T_m\right)_{small\,A} = \left(T_s\right) + \frac{N_0^4 b_3 \sinh^2\left(\dfrac{L}{\delta}\right)}{4\omega^2\left(N_0^2 - \omega^2\right)} T_s^2 A^2 + \cdots + \infty\,, \tag{3.6}$$

where $T_s = \left(1+\frac{\left(1+\gamma^2\delta^2\right)^2}{4\gamma^2\delta^2}\sinh^2\left(\frac{L}{\delta}\right)\right)^{-1}$, $b_3 = \frac{k^2}{\left(N_0^2-\omega^2\right)} - \frac{k^2}{\omega^2}$. Here A^2 is positive when $2\omega^2 > N_0{}^2$. Thus when A is small $\left(T_m\right)_{small\,A}$ increases as seen in Figure 2 and figure 3 for small A. In the limit $A \to 0$ the transmission coefficient $\left(T_m\right)_{small\,A} \to T_s$. When $2\omega^2 < N^2$ the coefficient of A^2 is negative and hence the transmission coefficient decreases with A as seen in Figure 3 for small values of A .

3.2 Transmission across N^2-barrier2

In the second case we assume a weakly stratified fluid of finite depth L bounded on either side by a strongly stratified fluid extending to infinity on either side. In this case we have

$$N^2 = \begin{cases} N_0^2, & |z| > \frac{L}{2} \\ N_1^2, & |z| \le \frac{L}{2} \end{cases}, \tag{3.7}$$

The transmission coefficient is given by

$$T_{mb} = \left[1+\frac{\left(\xi^2+\eta^2\right)^2}{4\eta^2\xi^2}\sinh^2\left(\xi L\right)\right]^{-1}. \tag{3.8}$$

where $\eta = -k\left(\frac{N_0^2}{\omega^2-k^2A^2}-1\right)^{\frac{1}{2}}$, $\xi = k\left(1-\frac{N_1^2}{\omega^2-k^2A^2}\right)^{\frac{1}{2}}$.

To understand how the transmission coefficient changes from N^2-barrier1 to N^2-barrier 2 we have plotted graphs of T_m and T_{mb} against A in Figure 4.

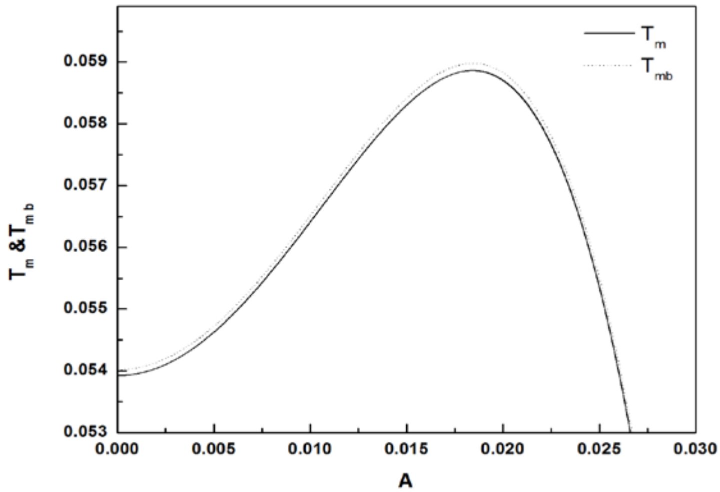

Figure 4: Comparison of transmission coefficients T_m and T_{mb} for N^2- barrier1 and N^2- barrier2 for $N_1 = 0.00002$, $k = 2\pi/15$, $\omega = 2\pi/390$, $L = 5$.

From the Figure 4 we find that the transmission coefficient T_{mb} is higher for N^2-barrier2. This is because in region $|z| \le L/2$ the waves propagate in N^2-barrier2 and evanescent in N^2-barrier1.

3.3 Transmission across N^2-barrier3

In this case, $\rho_b(z)$ can be assumed to vary continuously below even though its slope is discontinuous at $z = \pm L/2$. However, it is more realistic to consider mixed regions within a stratified fluid with discontinuous density profile in the form:

$$\rho_b = \begin{cases} \rho_0\left(1-\frac{z}{H_1}\right) & |z| \le \frac{L}{2} \\ \rho_0\left(1-\frac{z}{H_0}\right) & |z| > \frac{L}{2} \end{cases}, \tag{3.9}$$

where $H_0 \equiv g/N_0^2$ *and* $H_1 \equiv g/N_1^2$ measure the strength of stratification respectively outside and within a partially mixed region of depth L. The corresponding squared buoyancy frequency is the same as that for the generalization of the N^2-barrier except for infinite spikes at $z = \pm L/2$ where the density changes discontinuously by $\Delta\rho_b = \rho_0\left[\frac{\left(N_0^2 - N_1^2\right)}{g}\right]\left(\frac{L}{2}\right)$ and using the prescribed N^2-barrier in (3.7) we have computed the transmission coefficient for the case $N_1 \le \omega \le N_0$, we use the solution $\hat{\psi}(z)$ given by equation (3.2), requiring that the $\hat{\psi}(z)$, $\frac{d\hat{\psi}(z)}{dz}$ and $\bar{\rho}\left(\hat{\psi}' - \left(k^2 g/\omega^2 - k^2A^2\right)\hat{\psi}\right)$ are continuous across the interface which is equivalent to velocity and pressure continuity. Then the transmission coefficient is given by

$$T_{Amix} = \left[1 + \left(\frac{\left(\xi^2+\eta^2\right)^2}{4\eta^2\xi^2}\sinh^2\left(\xi L\right)\tau_A\right)^2\right]^{-1}, \tag{3.10}$$

in which $\tau_A = \left[1 + \frac{k^2L^2N_0^2\left(1-N_1^2/N_0^2\right)}{4\left(\omega^2-k^2A^2\right)} - L\xi\coth\left(L\xi\right)\right]$, $N_1 \le \omega \le N_0$.

The changes from N^2-barrier2 to N^2-barrier3 is observed through the plotted graphs of T_{Amix} and T_{mb} against A in Figure 5.

From the Figure 5 we find that the transmission coefficient in N^2-barrier3 is higher when A is small and lower when A is large. This is because as A increases Alfvén-gravity waves propagate more and more horizontally (i.e. along the direction of the magnetic field).

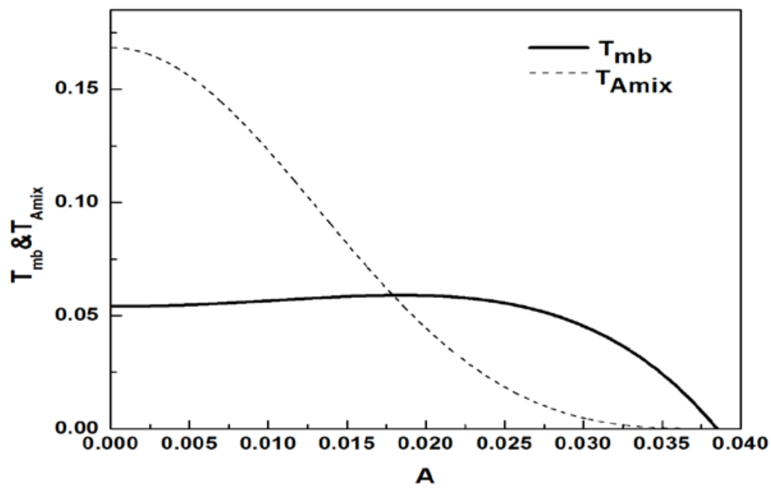

Figure 5: Comparison of transmission coefficients T_{mb} and T_{Amix} for N^2- barrier2 and N^2- barrier3 for $N_1 = 0.00002$, $k = 2\pi/15$, $\omega = 2\pi/390$, $L = 5$.

5 Conclusion

The investigation of the internal Alfvén-gravity wave tunnelling through an incompressible, inviscid, Boussinesq, perfectly conducting fluid in the presence of a uniform horizontal magnetic field reveals that in §3.1 we have obtained the transmission coefficient for N^2-barrier1. We have shown that the transmission coefficient varies with A for various values of N_0^2, N_1^2 and ω^2. When the magnetic field when $2\omega^2 < N_0^{\ 2}$ the transmission coefficient increases and attains maximum and then decrease to zero $A = \omega/k$. We also note from expression for η that when $A = \omega/k$ the vertical wave number η tends to ∞ so that the waves become more and more horizontal. Thus the effect of magnetic field is to make the wave to propagate along the magnetic field lines rather than allow it to propagate upwards. Followed by the transmission coefficient in §3.2 based on the vertical wave numbers ξ(in the region $|z| \le L/2$) and η(in the region $|z| > L/2$) and in § 3.3 on the vertical wave numbers ξ(in the region $|z| \le L/2$) and η(in the region $|z| > L/2$) defined below (3.8) are real or imaginary. From the figure 5 we find that the transmission coefficient in N^2-barrier 3 is higher when A is small and lower when A is large. This is because as A increases Alfvén-gravity waves propagate more and more horizontally (i.e. along the direction of the magnetic field) rather than vertically.

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References

[1] C.O. Hines, Internal atmospheric gravity waves at ionospheric heights, *Can. J. Phy.*, **34**, **(**1960), 1441-1481.

[2] M.L.V. Pitteway and C.O. Hines, Reflection and ducting of atmospheric acoustic-gravity waves, *Canadian Journal of Physics*, **43**, (1965), 2222.

[3] D.C. Fritts and Li Yuan, An analysis of gravity wave ducting in the atmosphere: Eckart's resonances In thermal and Doppler ducts, *J. of Geophysical Research*, **94**(18), (1989), 455.

[4] Bruce R. Sutherland, Stuart B. Dailziel,Graham O. Hughes and P.F.L Lindzen, Visualization and measurement of internal wave by 'synthetic schlieren': Part 1. Verticallyoscillating cylinder, *J. Fluid Mech*, **390**, (1999), 93-126.

[5] Anne K. Smith, Interactions between the Lower, Middle and Upper Atmoshere, *Space Sci Rev*., **158**, (2011), doi:10.1007/s11214-011-9791.

[6] R.L. Walterscheid, G. Schubert and D.G. Brinkman, Small scale gravity waves in the upper mesosphere and lower thermosphere generated by deep tropical convection, *J. of Geophysical Research*, **106**, D23, (2001), 31825.

[7] S.H. Francis, Global propagation of atmospheric gravity waves: A review, *J. Atmos. Terr. Phys*., **37**, (1975), 1011.

[8] A.D. Richmond, The nature of gravity of gravity wave ducting in the thermosphere, *J. Geophys. Res*., **83**, (1978), 1385.

[9] L. Yu, T.F. Tuan and H. Tai, On potential well treatment for atmospheric gravity waves, *J. Geophys. Res*., **85**, (1980), 1297.

[10] G. Chimonas and C.O. Hines, Doppler ducting of atmospheric gravity waves, *J. Geophys. Res*., **91**, (1986), 1219.

[11] D.Y. Wang and T.F. Tuan, Brunt-Droppler ducting of small-period gravity waves, *J. Geophys. Res*., **93**(A9), (1988), 9916-9926.

[12] G.L. Brown and B.R. Sutherland, Internal wave tunneling through non-uniformly stratified shear flow, *Atmos. Ocean*, **45**(1), (2004) 47-56.

[13] Bruce R. Sutherland and Kerianne Yewchuk, Internal wave tunneling, *J. Fluid Mechanics*, **511**, (2004), 125.

[14] Carl Eckart, Internal waves in the Ocean, *The physics of fluids*, **4**, (1961), 7.

[15] N. Rudraiah and M. Venkatachappa, Momentum transport by gravity waves in a perfectly conducting shear flow, *J. of Fluid Mechanics*, **54**, (1972), 217.

[16] C.O. Hines, The Formation of Midlatitude Sporadic E Layers (Sun, Upper atmosphere and space), *Research in geophysics*, **1**, (1964).

[17] O.M. Philips, The interaction trapping of internal gravity waves, *J. Fluid Mechanics*, **34**(2), (1968), 407-416.

[18] J.R. Booker and F.P. Bretherton, The critical layer for internal gravity waves in a shear flow, *Quart. J. Met. Society*, **95**, (1967), 313.

[19] S. Chandrasekar, *Hydrodynamic Stability*, Cambridge University Press, Dover Publications Newyork, pp. 197, 1961.