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Local existence and Blow-up of solutions for the higher-order nonlinear Kirchhoff-type equation

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Abstract

In this paper, we deal with the local existence and blow-up of solutions for the higher-order nonlinear Kirchhoff-type equation:

$u_{tt}+(-\triangle)^m u_t+\phi(\|\nabla^m u\|^2)(-\triangle)^m u=g(u)$, $x \in \Omega$, $t > 0$

in a bounded domain, where $m > 1$ is a positive integer. At first, we prove the existence and uniqueness of the local solution by on the Banach contraction mapping principle, under some conditions and $E(u_0, u_1)$ is negative, we investigate the blow-up of solutions in finite time, and the concavity method is widely used.

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Keywords: Higher-order nonlinear Kirchhoff equation; Local solutions; Blow-up

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1 Introduction

In this paper, we are concerned with the local existence and blow-up of the solutions for the higher-type nonlinear Kirchhoff-type equation:

$$u_{tt} + (-\Delta)^m u_t + \phi(\|\nabla^m u\|^2)(-\triangle)^m u = g(u), x \in \Omega, t > 0, m > 1, \quad (1.1)$$

$$u(x,t) = 0, \tfrac{\partial^i u}{\partial v^i} = 0, i = 1, 2, \ldots, m-1, x \in \partial\Omega, t > 0, \quad (1.2)$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega, \quad (1.3)$$

Where Ω is a bounded domain of R^n ,with a smooth Dirichlet boundary $\partial\Omega$ and initial data, $m > 1$ is a positive integer, $\phi(s)$ is a positive local Lipschitz function and v is the unit outward normal on $\partial\Omega$.

When $\phi(s) = 1$, $m = 1$, $g(u) = |u|^p u$, the equation (1.1) becomes a nonlinear wave equation:

$$u_{tt} - \Delta u - \Delta u_t = |u|^p u, (x,t) \in \Omega \times [0, +\infty), \quad (1.4)$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega, \quad (1.5)$$

$$u(x,t) = 0, (x,t) \in \partial\Omega \times [0, +\infty). \quad (1.6)$$

It has been extensively studied and several results concerning existence and blowing-up have been established [1-3].

In [4], when $\phi(s) \neq 1$, $g(u) = |u|^p u$, the equation (1.1) becomes the following Kirchhoff equation with Lipschitz type continuous coefficient and strong damping:

$$u_{tt} - M(\|\nabla u\|^2)\Delta u - \omega\Delta u_t = |u|^p u, \quad (1.7)$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega, \quad (1.8)$$

$$u(x,t) = 0, (x,t) \in \partial\Omega \times [0, T], \quad (1.9)$$

where $\Omega \in R^N$, $N \geq 1$ is a bounded domain with a smooth boundary $\partial\Omega$. $p > 2$ and $M(s) = m_0 + bs^r$ is a positive local Lipschitz function. Here, $m_0 > 0$, $b \geq 0$, $r \geq 1$, $s \geq 0$. It has been studied and several results concerning existence and blowing-up have been established.

In [5], Perikles G. Papadopoulos-Nikos M. Stavrakakis study the following degenerate nonlocal quasilinear wave equation of Kirchhoff type with a weak dissipative term

$$u_{tt} - \phi(x)\|\nabla u(t)\|^2 \triangle u + \delta u_t = |u|^\alpha u, x \in R^N, t \geq 0, \quad (1.10)$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in R^N, \quad (1.11)$$

with initial conditions u_0, u_1 in appropriate function spaces, $N \geq 3$, and $\delta \geq 0$. Throughout the paper they assume that the functions ϕ and $g : R^N \to R$ satisfy the following condition:

$\phi(x) > 0$, for all $x \in R^N$ and $(\phi(x))^{-1} =: g(x) \in L^{N/2}(R^N) \cap L^\infty(R^N)$.

Their conclusion show that when the initial energy $E(u_0, u_1)$ is non-negative and small, there exists a unique global solution in time, when the initial energy $E(u_0, u_1)$ is negative, the solution blows-up in finite time. In their work, a combination of the modified potential well method and the concavity method is widely used.

In [6], Fucai Li investigate global existence and blow-up properties of the solution for the following higher-order Kirchhoff-type equation with nonlinear dissipation:

$$u_{tt} + (\int_\Omega |D^m u|^2 dx)^q (-\triangle)^m u + u_t|u_t|^r = |u|^p u, x \in \Omega, t > 0 \tag{1.12}$$

$$u(x,t) = 0, \frac{\partial^i u}{\partial v^i} = 0, i = 1, 2, \ldots, m-1, x \in \partial\Omega, t > 0, \tag{1.13}$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega, \tag{1.14}$$

where $\Omega \subset R^N (N \geq 1)$ is a bounded open domain with smooth boundary, v is the outer norm vector; $m > 1$ is a positive integer, $q, p, r > 0$ are positive constants.

they obtain that the solution exists globally if $p \leq r$, while if $p > maxr, 2q$, then for any initial data with negative initial energy, the solution blows up at finite time in L^{p+2} norm. For results of the same nature, we refer the reader to [7-10] and the references therein.

But, for (1.12)-(1.14) Salim A. Messaoudi and Belkacem Said Houari have different conclusion in [11] ,they improve Li's result and show that certain solutions with positive initial energy also blow up in finite time.

In the present work, we prove the existence and uniqueness of the local solution by on the Banach contraction mapping principle. In particular, in the process of proof about the blow-up solution of (1.1)-(1.3), we first improve the blow-up condition about positive second order continuous differentiable function $\psi(t)$, the blow-up condition was first given by Fajita, about the content of the blow-up condition we can see[12], then we use the new blow-up condition to prove the blow-up of solution for (1.1)-(1.3).

The content of the paper is organized as follows. In section 2, we give some hypotheses and lemmas. In section 3, we prove the existence and uniqueness of the local solution by on the Banach contraction mapping principle. In section 4, we investigate the blow-up properties of solution and estimate the blow-up time T^*.

2 Preliminaries

For convenience, we first introduce the following notations: we define $H_0^m(\Omega) = \{u \in H^m(\Omega) : \frac{\partial^i u}{\partial v^i} = 0, i = 0, 1, \ldots, m-1\}$, $D = \nabla$, $D^{2m} = (-\triangle)^m$, $\|.\|_{H^m} = \|.\|_{H^m(\Omega)}$, $\|.\|_{H_0^m} = \|.\|_{H_0^m(\Omega)}$, $\|.\| = \|.\|_{L^2(\Omega)}$, $\|.\|_p = \|.\|_{L^p(\Omega)}$ for any real number $p > 1$.

Now, we give the general hypotheses

$A(1)$ Let $g(u)$ be a nonlinear C^1-function such that

$$|g(u)| \leq k_0|u|^{p+1}, |f'(u)| \leq k_1|u|^p, \tag{2.1}$$

where $k_0, k_1 \geq 0$.

$A(2)$ For the nonlinearity , we suppose that

$$0 \leq p < +\infty, if 0 \leq n \leq 2m; 0 \leq p \leq \frac{4m}{n-2m}, if n > 2m. \tag{2.2}$$

$(A3)$ we suppose that $\phi(s)$ is a positive local Lipschitz function

$$\phi(s) = a + bs^{\frac{r}{2}}, r > 1, \tag{2.3}$$

for briefness, we suppose $a = b = 1$.

Lemma 2.1(Sobolev-Poincare inequality[12,13]) Let s be a number with $2 \leq s < +\infty$, $n < 2m$ and $2 \leq s \leq \frac{2m}{n-2m}$, $n > 2m$. Then there is a constant K depending on Ω and s such that

$$\|u\|_s \leq K\|(-\triangle)^{\frac{m}{2}}u\|, \forall u \in H_0^m(\Omega). \tag{2.4}$$

3 Local existence of solution

In this section, we will consider the local existence of the solution for the problem (1.1)-(1.3) by the similar arguments as in [14,15].

Theorem 3.1. Suppose that $(A1)-(A3)$ hold, and for any given $(u_0, u_1) \in H^{2m}(\Omega) \cap H_0^m(\Omega) \times L^2(\Omega)$, then there exists $T > 0$ such that the problem (1.1)-(1.3) has a unique local solution satisfying

$$\begin{aligned} &u \in C^0([0,T); H^{2m}(\Omega) \cap H_0^m(\Omega)), \\ &u_t \in C^0([0,T); L^2(\Omega)) \cap L^2(0,T; H_0^m(\Omega)), \end{aligned} \tag{3.1}$$

moreover, at least one of the following statements holds: (1) $T = +\infty$; (2) $\|u_t\|^2 + \|D^{2m}u\|^2 \to +\infty$, as $t \to T^-$.

Proof. To apply Banach contraction mapping principle, we introduce the two parameter space of solutions

$$\begin{aligned} X_{T,R} =:\{&v \in C^0([0,T); H^{2m}(\Omega) \cap H_0^m(\Omega)), v_t \in C^0([0,T); \\ &L^2(\Omega)) \cap L^2(0,T; H_0^m(\Omega)) : \\ &e_1(v(t)) \le R^2, t \in [0,T], v(0) = u_0, v_t(0) = u_1\}, \end{aligned}$$

where $T > 0$ and $R > 0$, $e_1(v(t)) = \|D^{2m}v\|^2 + \|v_t\|^2$, then $X_{T,R}$ is a complete metric space under the distance

$$d(v_1, v_2) = \sup_{0 \le t \le T} e_1(v_1(t) - v_2(t)). \tag{3.2}$$

Next, we define the non-linear mapping S in the following way. Given $v \in X_{T,R}$, $u = Sv$ is the unique solution of the following equation:

$$u_{tt} + (-\Delta)^m u_t + \phi(\|\nabla^m v\|^2)(-\triangle)^m u = g(v), x \in \Omega, t > 0, m > 1, \tag{3.3}$$

$$u(x,t) = 0, \frac{\partial^i u}{\partial v^i} = 0, i = 1, 2, \ldots, m-1, x \in \partial\Omega, t > 0, \tag{3.4}$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega, \tag{3.5}$$

We shall show that there exist $T > 0$ and $R > 0$ such that

(i) S maps $X_{T,R}$ into itself;

(ii) S is a contraction mapping with respect to the metric $d(.,.)$.

first, we shall check (i). We multiply equation (3.4) by $2u_t$, and integrating it over Ω, to get

$$\begin{aligned} & \frac{de_2(u(t))}{dt} + 2\|D^m u_t\|^2 \\ & = \|D^m u\|^2 \frac{d}{dt}(1 + \|D^m v\|^r) + (g(v), 2u_t) \\ & = I_1 + I_2, \end{aligned} \tag{3.6}$$

where $e_2(u(t)) = \|u_t\|^2 + (1 + \|D^m v\|^r)\|D^m u\|^2$. According to Lemma 2.1, for $v \in X_{T,R}$, we have

$$\begin{aligned} I_1 & = \|D^m u\|^2 \frac{d}{dt}(1 + \|D^m v\|^r) \\ & = \|D^m u\|^2 \frac{d}{dt}\|D^m v\|^r \\ & = r\|D^m u\|^2\|D^m v\|^{r-2}\int_\Omega D^{2m}v.v_t dx \\ & \leq r\|D^m u\|^2\|D^m v\|^{r-2}\|D^{2m}v\|\|v_t\| \\ & \leq rK^{r-2}R^r\|D^m u\|^2 \\ & \leq rK^{r-2}R^r e_2(u(t)). \end{aligned} \tag{3.7}$$

According to $A(1)$ and $A(2)$, then

$$\begin{aligned} I_2 & = (g(v), 2u_t) \\ & \leq 2k_0|\int_\Omega |v|^{p+1}u_t dx| \\ & \leq 2k_0\|u_t\|\|v\|_{2p+2}^{p+1} \\ & \leq 2k_0K^{2p+2}R^{p+1}\|u_t\| \\ & \leq 2k_0K^{2p+2}R^{p+1}(e_2(u(t)))^{\frac{1}{2}}. \end{aligned} \tag{3.8}$$

Combining (3.7)-(3.9), we get

$$\frac{de_2(u(t))}{dt} + 2\|D^m u_t\|^2 \leq rK^{r-2}R^r e_2(u(t)) + 2k_0K^{2p+2}R^{p+1}(e_2(u(t)))^{\frac{1}{2}}. \tag{3.9}$$

Integrating (3.10) over $[0,T]$ and by Gronwall inequality, to get

$$e_2(u(t)) + 2\int_0^T \|D^m u_t(s)\|^2 ds \leq [(e_2(u(0)))^{\frac{1}{2}} + 2k_0K^{2p+2}R^{p+1}T]^2 e^{rK^{r-2}R^r T}, \tag{3.10}$$

where $e_2(u(0)) = \|u_1\|^2 + (1 + \|D^m u_0\|^r)\|D^m u_0\|^2$. So, for all $t \in [0, T]$, we have

$$\begin{aligned} & e_1(u(t)) + 2\int_0^T \|D^m u_t(s)\|^2 ds \\ & \leq e_2(u(t)) + 2\int_0^T \|D^m u_t(s)\|^2 ds \\ & \leq [(\|u_1\|^2 + (1 + \|D^m u_0\|^r)\|D^m u_0\|^2)^{\frac{1}{2}} + 2k_0 K^{2p+2} R^{p+1} T]^2 e^{rK^{r-2}R^r T}. \end{aligned} \tag{3.11}$$

Therefore, in order to that the map S verifies (1), it will be enough that the parameters T and R satisfy:

$$[(\|u_1\|^2 + (1 + \|D^m u_0\|^r)\|D^m u_0\|^2)^{\frac{1}{2}} + 2k_0 K^{2p+2} R^{p+1} T]^2 e^{rK^{r-2}R^r T} \leq R^2 \tag{3.12}$$

Moreover, it follows from (3.12) that $u_t \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^m(\Omega))$ and $u \in L^\infty(0, t; H^{2m}(\Omega) \cap H_0^m(\Omega))$. It implies

$$\begin{aligned} & u \in C^0([0, T]; H^{2m}(\Omega) \cap H_0^m(\Omega)), u_t \in C^0([0, T]; \\ & L^2(\Omega)) \cap L^2(0, T; H_0^m(\Omega)). \end{aligned} \tag{3.13}$$

Next, (ii) will be proved. Suppose that (3.13) holds. We take $v_1, v_2 \in X_{T,R}$, let $u_1 = Sv_1, u_2 = Sv_2$, and set $\omega = u_1 - u_2$. Then ω satisfies

$$\begin{aligned} & \omega_{tt} + (-\triangle)^m \omega_t + (1 + \|D^m v_1\|^r)(-\triangle)^m \omega = \\ & \qquad = -(\|D^m v_1\|^r - \|D^m v_2\|^r)(-\triangle)^m u_2 + g(v_1) - g(v_2) \end{aligned} \tag{3.14}$$

$$\omega(x, t) = \frac{\partial^i \omega}{\partial v^i} = 0, \quad i = 1, 2, \ldots, m-1, (x, t) \in \partial\Omega \times [0, T], \tag{3.15}$$

$$\omega(x, 0) = 0, \omega_t(x, 0) = 0, x \in \Omega. \tag{3.16}$$

Multiplying (3.15) by $2\omega_t$ and integrating it over Ω and using Green's formula, we obtain

$$\begin{aligned} & \frac{d}{dt}[\|\omega_t\|^2 + (1 + \|D^m v_1\|^r)\|D^m \omega\|^2] + 2\|D^m \omega_t\|^2 = \\ & = \|D^m \omega\|^2 \frac{d}{dt}(1 + \|D^m v_1\|^r) - \\ & \quad - 2(\|D^m v_1\|^r - \|D^m v_2\|^r)(D^{2m} u_2, \omega_t) + 2(g(v_1) - g(v_2), \omega_t) \\ & = I_3 + I_4 + I_5. \end{aligned} \tag{3.17}$$

By Lemma 2.1 observe that

$$I_3 = \|D^m \omega\|^2 \tfrac{d}{dt}(1 + \|D^m v_1\|^r) \leq rK^{r-2} R^r e_1(\omega(t)). \tag{3.18}$$

$$\begin{aligned} I_4 &= -2(\|D^m v_1\|^r - \|D^m v_2\|^r)(D^{2m}u_2, \omega_t) \\ &\le r[\theta\|D^m v_1\|^2 + (1-\theta)\|D^m v_2\|^2]^{\frac{r}{2}-1}\times \\ &(\|D^m v_1\| + \|D^m v_2\|)\|D^m(v_1 - v_2)\|\|D^{2m}u_2\|\|\omega_t\| \\ &\le 2rK^rR^r[e_1(v_1(t)-v_2(t))]^{\frac{1}{2}}[e_1(\omega(t))]^{\frac{1}{2}}, \end{aligned} \tag{3.19}$$

where $0 < \theta < 1$.

$$\begin{aligned} I_5 &= 2(g(v_1) - g(v_2), \omega_t)) \\ &\le 2k_0 \int_\Omega (|v_1|^p + |v_2|^p)|v_1 - v_2||\omega_t| dx \\ &\le 2k_0(\|v_1\|^p_{np} + \|v_2\|^p_{np})\|v_1 - v_2\|_{\frac{2n}{n-2}}\|\omega_t\| \\ &\le 4k_0K^{2p+2}R^p\|D^{2m}(v_1 - v_2)\|\|\omega_t\| \\ &\le 4k_0K^{2p+2}R^p[e_1(v_1(t)-v_2(t))]^{\frac{1}{2}}[e_1(\omega(t))]^{\frac{1}{2}}. \end{aligned} \tag{3.20}$$

Inserting I_3, I_4 and I_5 into (3.18), we get

$$\begin{aligned} \frac{de_2(\omega(t))}{dt} &\le rK^rR^re_1(\omega(t)) + (2rK^rR^r + 4k_0K^{2p+2}R^p)[e_1(v_1(t)-v_2(t))]^{\frac{1}{2}}[e_1(\omega(t))]^{\frac{1}{2}} \\ &\le rK^rR^re_2(\omega(t)) + (2rK^rR^r + 4k_0K^{2p+2}R^p)[e_2(v_1(t)-v_2(t))]^{\frac{1}{2}}[e_2(\omega(t))]^{\frac{1}{2}}, \end{aligned} \tag{3.21}$$

where $e_2(\omega(t)) = \|\omega_t\|^2 + (1 + \|D^m v_1\|^r)\|D^m\omega\|^2$.

Applying the Gronwall inequality, we obtain

$$e_2(\omega(t)) \le (2rK^rR^r + 4k_0K^{2p+2}R^p)T^2e^{rK^rR^rT} \sup_{0\le t\le T} e_1(v_1(t)-v_2(t)). \tag{3.22}$$

So, we have

$$\sup_{0\le t\le T} e_1(u_1(t)-u_2(t)) \le C_{T,R} \sup_{0\le t\le T} e_1(v_1(t)-v_2(t)), \tag{3.23}$$

where $C_{T,R} = (2rK^rR^r + 4k_0K^{2p+2}R^p)T^2e^{rK^rR^rT}$. If $C_{T,R} < 1$, we can see S is a contraction mapping. Then we choose suitable R is sufficiently large and T is sufficiently small, such that (i) and (ii) hold. By applying Banach fixed point theorem, we get the local existence.

The second statement of the theorem is proved by a standard continuation argument(see [14]). The proof of the theorem is now completed. ∎

4 Blow-up of solution

In this section we consider the blowing-up property of the solution of the initial value problem (1.1)-(1.3), we give the following definition and lemma.

Now, we define the energy function $E(u(t))$ of the solution u of (1.1)-(1.3) by

$$E(t) =: E(u(t)) = \tfrac{1}{2}\|u_t\|^2 + \tfrac{1}{2}\|D^m u\|^2 + \tfrac{1}{r+2}\|D^m u\|^{r+2} - \int_\Omega G(u)dx, \quad (4.1)$$

where $G(u) = \int_0^u g(\eta)d\eta$.

Lemma 4.1. Suppose that $\psi(t)$ is a nonnegative $C^2(0,+\infty)$ function, as $t > 0$, satisfying inequality

$$\alpha(\psi'(t))^2 - \psi''(t) - 2C_1\psi'(t) \le C_2, \quad (4.2)$$

where $\alpha > 0$, $C_1, C_2 \ge 0$, then as $\psi(0) > 0, \psi'(0) > -\alpha^{-1}r_2$, $C_1 + C_2 > 0$, we have

$$\psi(t) \to \infty (t \to t_1), \quad (4.3)$$

where $t_1 \le t_2 = \frac{1}{2(C_1^2+\alpha C_2)^{\frac{1}{2}}} \ln \frac{r_1\psi(0)+\alpha\psi'(0)}{r_2\psi(0)+\alpha\psi'(0)}$, $r_1 = -C_1 + (C_1^2 + \alpha C_2)^{\frac{1}{2}}$, $r_2 = -C_1 - (C_1^2 + \alpha C_2)^{\frac{1}{2}}$.
If $\psi(0) > 0$, $\psi'(0) > 0$, $C_1 = C_2 = 0$, then

$$\psi(t) \to \infty (t \to t_1 \le t_2 = \tfrac{\psi(0)}{\alpha\psi'(0)}). \quad (4.4)$$

Proof. Let $\Phi(t) = e^{-\alpha\psi(t)}$, then

$$\Phi'(t) = e^{-\alpha\psi(t)} \cdot (-\alpha\psi'(t))$$

$$\Phi''(t) = e^{-\alpha\psi(t)} \cdot (-\alpha\psi'(t))^2 + e^{-\alpha\psi(t)} \cdot (-\alpha\psi''(t)) = \alpha^2 e^{-\alpha\psi(t)}(\psi'(t))^2 - \alpha e^{-\alpha\psi(t)}\psi''(t)$$

$$\begin{aligned} &\Phi''(t) + 2C_1\Phi'(t) - \alpha C_2\Phi(t) \\ &= \alpha^2 e^{-\alpha\psi(t)}(\psi'(t))^2 - \alpha e^{-\alpha\psi(t)}\psi''(t) + 2C_1 e^{-\alpha\psi(t)} \cdot (-\alpha\psi'(t)) - \alpha C_2 e^{-\alpha\psi(t)} \\ &= \alpha e^{-\alpha\psi(t)}[\alpha(\psi'(t))^2 - \psi''(t) - 2C_1\psi'(t) - C_2]. \end{aligned} \quad (4.5)$$

Let $f(t) = \alpha(\psi'(t))^2 - \psi''(t) - 2C_1\psi'(t) - C_2$, so we have

$$\Phi''(t) + 2C_1\Phi'(t) - \alpha C_2\Phi(t) \equiv f(t) \le 0. \quad (4.6)$$

For $C_1+C_2>0$ this situation, we can calculate the solution of (4.6)

$$\begin{aligned}\Phi(t) &= \beta_1 e^{r_1 t}+\beta_2 e^{r_2 t}+\frac{1}{r_1-r_2}\int_0^t f(s)[e^{r_1(t-s)}-e^{r_2(t-s)}]ds \\ &\le \beta_1 e^{r_1 t}+\beta_2 e^{r_2 t},\end{aligned} \tag{4.7}$$

where r_1, r_2 are the eigenvalues of $\Phi''(t)+2C_1\Phi'(t)-\alpha C_2\Phi(t)=0$, and β_1, β_2 satisfying

$$\begin{cases}\beta_1+\beta_2=\Phi(0)\\ \beta_1 r_1+\beta_2 r_2=\Phi'(0).\end{cases}$$

Hence,

$$\beta_1=(r_1-r_2)^{-1}[\Phi'(0)-r_2\Phi(0)]=-(r_1-r_2)^{-1}[\alpha\psi'(0)+r_2]e^{-\alpha\psi(0)}$$
$$\beta_2=[1+(r_1-r_2)^{-1}(\alpha\psi'(0)+r_2)]e^{-\alpha\psi(0)}$$

At the same time, we let $\Phi(t_2)=0$, then $t_2=\frac{1}{2(C_1^2+\alpha C_2)^{\frac{1}{2}}}\ln\frac{r_1\psi(0)+\alpha\psi'(0)}{r_2\psi(0)+\alpha\psi'(0)}$.

To sum up, we can know, $\Phi(t)\to 0$, as $t_1\le t_2$, $t\to t_1$, i.e., $\psi(t)\to\infty$. (4.4) still hold when $C_1=C_2=0$. ∎

Theorem 4.1. Suppose that $(A_1)-(A_3)$ hold, then for any initial data $(u_0,u_1)\in H^{2m}(\Omega)\cap H_0^m(\Omega)\times H_0^m(\Omega)$, the solution of (1.1) blows up provided that $E(0)<0$.

Proof. ¿From (4.1), we can get

$$E'(t)=-2\|D^m u_t\|^2, \tag{4.8}$$

i.e., $E(t)$ is a decreasing function satisfying

$$E(t)\le E(0)<0. \tag{4.9}$$

Next,we introducing the function $F(t)=\frac{1}{2}\|u\|^2$ for any solution u, then we have that

$$F'(t)=\int_\Omega uu_t dx. \tag{4.10}$$

Now, we define

$$H(t)=\int_0^t(-E(t))dt+F(t). \tag{4.11}$$

Next, our goal is to prove $H(t)$ satisfying (4.2), and using the proof of Lemma 4.1 to estimate the lifespan T^*

$$\begin{aligned} H'(t) &= -E(t) + F'(t) \\ &= -E(t) + \int_\Omega u u_t dx \le -E(t) + \frac{1}{2}\|u\|^2 + \frac{1}{2}\|u_t\|^2. \end{aligned} \tag{4.12}$$

$$\begin{aligned} H''(t) &= -E'(t) + F''(t) \\ &= -E'(t) + \int_\Omega u u_{tt} dx + \int_\Omega u_t u_t dx \qquad (4.13) \\ &= 2\|D^m u_t\|^2 + \|u_t\|^2 + \int_\Omega u u_{tt} dx, \qquad (4.14) \end{aligned}$$

where

$$\begin{aligned} \int_\Omega u u_{tt} dx &= \int_\Omega u[g(u) - (1 + \|D^m u\|^r)(-\triangle)^m u - (-\triangle)^m u_t] dx \\ &= (u, g(u)) - \|D^m u\|^2 - \|D^m u\|^{r+2} - \int_\Omega u(-\triangle)^m u_t dx \qquad (4.15) \\ &\le k_0 \|u\|^{p+2} - \|D^m u\|^2 - \|D^m u\|^{r+2} - \int_\Omega u(-\triangle)^m u_t dx, \end{aligned}$$

where

$$\begin{aligned} \left|\int_\Omega u(-\triangle)^m u_t dx\right| &\le \|D^m u\| \|D^m u_t\| \\ &\le \frac{1}{4(2+K)}\|D^m u\|^2 + (2+K)\|D^m u_t\|^2. \end{aligned} \tag{4.16}$$

So, adding (4.14),(4.15) into (4.13), we can get

$$H''(t) \le k_0 \|u\|^{p+2} - \frac{9+4K}{8+4K}\|D^m u\|^2 - \|D^m u\|^{r+2}. \tag{4.17}$$

As $t \in [0, T]$, $H'(t), H''(t)$ are bounded values, and $H'(t) \le C_1(t)$, $H''(t) \le C_2(t)$, so, we have

$$\alpha(H'(t))^2 - H''(t) - 2C_1 H'(t) \le C_2. \tag{4.18}$$

Then according to Lemma 4.1, we can get the lifespan T^*

$$T^* = \frac{1}{2(C_1^2 + \alpha C_2)^{\frac{1}{2}}} \ln \frac{r_1 H(0) + \alpha H'(0)}{r_2 H(0) + \alpha H'(0)}$$

i.e., the solution of (1.1)-(1.3) blows up, as $t \le T^*$. ∎

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