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Half Logistic-Topp-Leone Lomax Distribution: Properties and Statistical Inference

Mavis Pararai[1], Fastel Chipepa[2] and Bright Forkenberg[1]

Abstract

A new generalization of the Topp-Leone Lomax distribution was developed. The new distribution is a half logistic transformation of the Topp-Leone Lomax distribution. Some important statistical properties of this new distribution were explored that include raw moments, moment generating function, probability weighted moments, distribution of order statistics, R´enyi entropy, and Shannon entropy. The maximum likelihood estimation (mle) technique was used to estimate the unknown model parameters estimates. Consistency of the mles was assessed via Monte Carlo simulation studies. The mle technique produced consistent estimates based on the simulation studies results and applications to real datasets as demonstrated by means of log-likelihood profile plots. The usefulness the half logistic-Topp Leone Lomax distribution was assessed by means of applications to real world datasets. The model was compared to other generalizations of the Lomax distribution. The new distribution emerged as a good contender to the other generalizations involving the Lomax distribution.

JEL classification numbers: E18, HO, I1, J64, J88.
Keywords: Half Logistic Transformation, Topp-Leone Lomax Distribution, Maximum Likelihood Estimation, Log-Likelihood Profile Plots, Lomax Distribution.

[1] Department of Mathematical and Computer Sciences, Indiana University of Pennsylvania, Pennsylvania, USA.
[2] Department of Mathematical and Statistical Sciences, Botswana International University of Science and Technology Palapye, Botswana.

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1. Introduction

Statisticians worked hard in the past two decades to introduce methodologies that enhance flexibility of classical and existing probability distributions. The motivation being to provide an effective data modelling framework. The demand for flexible distributions is rising across many fields of research such as medical sciences, engineering, environmental science, and finance. Proffered methodologies offer remarkable improvements on classical models and some existing models. Notable methodologies developed for generalizing distributions are the beta-G generator (Eugene et al., 2002), the T-X family (Alzaatreh et al., 2013), the exponentiated half logistic-G (EHL-G) (Cordeiro et al., 2014), the Topp-Leone-G (TL-G) (Al-Shomrani et al., 2016), the Topp-Leone odd exponential half logistic-G (Chipepa and Oluyede, 2021), and A New Power Generalized Weibull-G (Oluyede et al., 2020), among others.

The new distribution is a significant contribution to the field of statistics, since it offers more advantages in data modelling compared to the other generalizations of the Lomax distribution (Lomax, 1954). Its flexibility in modelling heavily skewed data and data with non-monotonic failure rates, gives it a more competitive edge compared to some existing models.

The paper is structured as follows: The new model is introduced in Section 2. Some statistical properties are presented in Section 3. Maximum likelihood estimation is discussed in Section 4. Section 5 presents simulation studies results. Section 6 represents applications to real datasets, and conclusions in Section 7.

2. The Model

In this section, we derive the new distribution by transforming the TLLx distribution via the half logistic generator. The Topp-Leone Lomax (TLLx) distribution (Oguntunde et al., 2019) has cumulative distribution function (cdf) and probability density function (pdf) defined by

$$F(x;\beta,\delta,c) = \left[1-(1+cx)^{\{-2\beta\}}\right]^{\delta} \tag{1}$$

and

$$f(x;\beta,\delta,c) = 2\delta\beta c\,(1+cx)^{-2\beta-1}\left[1-(1+cx)^{\{-2\beta\}}\right]^{\delta-1} \tag{2}$$

respectively, for $\beta,\ \delta,\ c > 0$. (Cordeiro et al., 2017) developed the half logistic generator whose cdf and pdf are defined by

$$G(x;\phi) = \frac{F(x;\phi)}{1+\bar{F}(x;\phi)}$$

and

$$g(x;\phi) = \frac{2f(x;\phi)}{\left(1+\bar{F}(x;\phi)\right)^2},$$

where $\bar{F}(x;\phi) = 1 - F(x;\phi)$ is the survival function of the baseline distribution. The Half-Logistic-G transformation modifies any given distribution $F(x)$ to enhance its flexibility. We, therefore, use this transformation to derive the new distribution, namely, half logistic-Topp-Leone Lomax (HL-TLLx) whose cdf and pdf are defined by

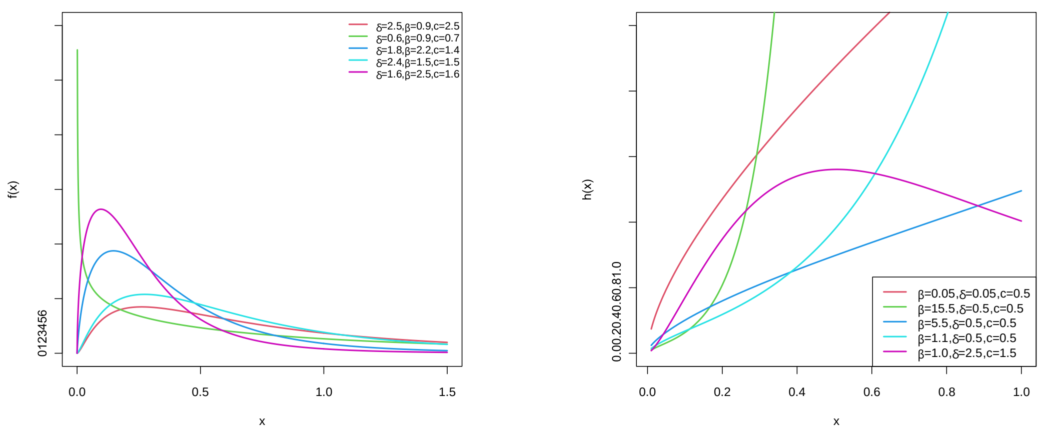
$$G(x;\beta,\delta,c) = \frac{\left[1-(1+cx)^{-2\beta}\right]^{\delta}}{2-\left[1-(1+cx)^{-2\beta}\right]^{\delta}} \qquad (3)$$

and

$$g(x;\beta,\delta,c) = \frac{4\delta\beta c(1+cx)^{-2\beta-1}\left[1-(1+cx)^{-2\beta}\right]^{\delta-1}}{\left(2-\left[1-(1+cx)^{-2\beta}\right]^{\delta}\right)^2} \qquad (4)$$

respectively, for $\beta, \delta, c > 0$.

The HL-TLLx pdf distribution applies to heavily skewed data and data with both monotonically increasing and upside bathtub failure rates as shown in Figures 1(a) and 1 (b).



(a) **(b)**

Figure 1: Some pdf and hrf plots of the HL-TLLx distribution

3. Some Statistical Properties

We present in this section the quantile function, moments and moment generation function, order statistics, R´enyi entropy, and Shannon entropy.

3.1 Quantile Function

To derive the quantile function $Q_X(u)$, we set $F_{HLTLLx}(x;c,\beta,) = u$.

⇒

$$u = \frac{\left[1-(1+cx)^{-2\beta}\right]^{\delta}}{2-\left[1-(1+cx)^{-2\beta}\right]^{\delta}}.$$

Rearranging the equation yields

$$2u - u\left[1-(1+cx)^{-2\beta}\right]^{\delta} = \left[1-(1+cx)^{-2\beta}\right]^{\delta}.$$

⇒

$$\left[1-(1+cx)^{-2\beta}\right]^{\delta} = \frac{2u}{1+u},$$

which further simplifies to

$$x = \frac{\left[1-\left(\frac{2u}{1+u}\right)^{1/\delta}\right]^{-1/(2\beta)} - 1}{c}$$

Therefore, the quantile function for the HL-TLLx distribution is given by

$$Q_X(u) = \frac{\left[1-\left(\frac{2u}{1+u}\right)^{1/\delta}\right]^{-1/(2\beta)} - 1}{c}$$

3.2 Moments and Moment Generating Function

The r^{th} moment is given by

$$E[X^r] = \int_0^\infty X^r f(x)\,dx.$$

Substituting the density function $f(x)$

$$E[X^r] = \sum_{i=1}^{\infty}\left(\frac{1}{2}\right)^{i-1}\delta\int_0^\infty \beta c x^r(1+cx)^{-(2\beta+1)}[1-(1+cx)^{-2\beta}]^{\delta i-1}\,dx.$$

Let $y = (1 + cx)^{-2\beta}$, so

$$dy = -2\beta c(1 + cx)^{-2\beta-1}\, dx,$$

and

$$x = \frac{y^{-\frac{1}{2\beta}} - 1}{c}.$$

The limits of integration transform as follows: when $x = 0$, $y = 1$, when $x = \infty$, $y = 0$.

Substituting these into the integral

$$E[X^r] = \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i \int_0^{\infty} x^r [1 - (1 + cx)^{-2\beta}]^{i-1} 2\beta c(1 + cx)^{-2\beta-1}\, dx.$$

Changing the variable $x \to y$, the moment becomes

$$E[X^r] = \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i \frac{\delta}{c^r} \int_0^1 (y^{-\frac{1}{2\beta}} - 1)^r (1 - y)^{\delta i-1}\, dy.$$

Expanding $(y^{-\frac{1}{2\beta}} - 1)^r$ using the binomial theorem

$$(y^{-\frac{1}{2\beta}} - 1)^r = \sum_{j=0}^{r} \binom{r}{j} (-1)^{r-j} y^{-\frac{j}{2\beta}}.$$

Substituting this expansion

$$E[X^r] = \sum_{i=1}^{\infty} \sum_{j=0}^{r} \left(\frac{1}{2}\right)^i \frac{\delta}{c^r} \binom{r}{j} (-1)^{r+j} \int_0^1 y^{-\frac{j}{2\beta}-1} (1 - y)^{\delta i-1}\, dy.$$

The integral is a beta function

$$\int_0^1 y^{-\frac{j}{2\beta}-1} (1 - y)^{\delta i-1}\, dy = \frac{\Gamma\left(\frac{2\beta-j}{2\beta}\right) \Gamma(\delta i)}{\Gamma\left(\frac{2\beta-j}{2\beta} + \delta i\right)}.$$

Thus, the final expression for the r^{th} moment is

$$E[X^r] = \sum_{i=1}^{\infty}\sum_{j=0}^{r}\left(\frac{1}{2}\right)^i \frac{x\delta}{c^r}\binom{r}{j}(-1)^{r+j}\frac{\Gamma\left(\frac{2\beta-j}{2\beta}\right)\Gamma(\delta i)}{\Gamma\left(\frac{2\beta-j}{2\beta}+\delta i\right)}.$$

The first four moments of the HLTLLx distribution are given by
First Moment (*r* = 1)

$$E[X] = \sum_{i=1}^{\infty}\left(\frac{1}{2}\right)^i \frac{\delta}{c}\left[(-1)^{1+0}\frac{\Gamma(1)\Gamma(\delta i)}{\Gamma(1+\delta i)} + (-1)^{1+1}\frac{\Gamma\left(\frac{2\beta-1}{2\beta}\right)\Gamma(\delta i)}{\Gamma\left(\frac{2\beta-1}{2\beta}+\delta i\right)}\right]$$

$$E[X] = \sum_{i=1}^{\infty}\left(\frac{1}{2}\right)^i \frac{\delta}{c}\left[-\frac{\Gamma(1)\Gamma(\delta i)}{\Gamma(1+\delta i)} + \frac{\Gamma\left(\frac{2\beta-1}{2\beta}\right)\Gamma(\delta i)}{\Gamma\left(\frac{2\beta-1}{2\beta}+\delta i\right)}\right].$$

Second Moment (*r* = 2)

$$E[X^2] = \frac{\delta}{c^2}\sum_{i=1}^{\infty}\left(\frac{1}{2}\right)^i \left[\frac{1}{\delta i} - 2\frac{\Gamma\left(\frac{2\beta-1}{2\beta}\right)\Gamma(\delta i)}{\Gamma\left(\frac{2\beta-1}{2\beta}+\delta i\right)} + \frac{\Gamma\left(\frac{2\beta-2}{2\beta}\right)\Gamma(\delta i)}{\Gamma\left(\frac{2\beta-2}{2\beta}+\delta i\right)}\right].$$

Third Moment (*r* = 3)

$$E[X^3] = \sum_{i=1}^{\infty}\sum_{j=0}^{3}\binom{3}{j}(-1)^{3+j}\left(\frac{1}{2}\right)^i \frac{\delta}{c^3}\frac{\Gamma\left(\frac{2\beta-j}{2\beta}\right)\Gamma(\delta i)}{\Gamma\left(\frac{2\beta-j}{2\beta}+\delta i\right)}.$$

Fourth Moment (*r* = 4)

$$E[X^4] = \sum_{i=1}^{\infty}\sum_{j=0}^{4}\binom{4}{j}\left(\frac{1}{2}\right)^i \frac{\delta}{c^4}(-1)^{4+j}\frac{\Gamma\left(\frac{2\beta-j}{2\beta}\right)\Gamma(\delta i)}{\Gamma\left(\frac{2\beta-j}{2\beta}+\delta i\right)}.$$

The mean (μ), variance (σ^2), coefficient of variation (CV), coefficient of skewness (CS), and coefficient of kurtosis (CK) are given by

$$\mu = E[X] = \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i \frac{\delta}{c} \left[-\frac{\Gamma(1)\Gamma(\delta i)}{\Gamma(1+\delta i)} + \frac{\Gamma\left(\frac{2\beta-1}{2\beta}\right)\Gamma(\delta i)}{\Gamma\left(\frac{2\beta-1}{2\beta}+\delta i\right)} \right]$$

$$^2 = \mu_2' - \mu^2,$$

$$CV = \frac{}{\mu} = \frac{\sqrt{(\mu_2^1 - \mu^2)}}{\mu} = \sqrt{\frac{\mu_2'}{\mu^2} - 1},$$

$$CS = \frac{E[(X-\mu)^3]}{[E(X-\mu)^2]^{3/2}} = \frac{\mu_3' - 3\mu\mu_2' + 2\mu^3}{(\mu_2' - \mu^2)^{3/2}},$$

and

$$CK = \frac{E[(X-\mu)^4]}{[E(X-\mu)^2]^2} = \frac{\mu_4' - 4\mu\mu_3' + 6\mu^2\mu_2' - 3\mu^4}{(\mu_2{}' - \mu^2)^2}.$$

Note that

$$e^{tx} = \sum_{k=0}^{\infty} \frac{(tx)^k}{k!} = \sum_{k=0}^{\infty} \frac{t^k x^k}{k!}.$$

The moment generating function $M_X(t)$ is

$$M_x(t) = E[e^{tx}] = E\left[\sum_{k=0}^{\infty} \frac{t^k x^k}{k!}\right] = \sum_{k=0}^{\infty} \frac{t^k}{k!} E[x^k].$$

Substituting k for r in $E[X^k]$, we have

$$M_x(t) = \sum_{i=1}^{\infty} \sum_{j=0}^{k} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{1}{2}\right)^i \frac{\delta}{c^k} \binom{k}{j} \frac{\Gamma\left(\frac{2\beta-j}{2\beta}\right)\Gamma(\delta i)}{\Gamma\left(\frac{2\beta-j}{2\beta}+\delta i\right)}.$$

3.3 Order Statistics

Order statistics are essential in various fields, providing insights into data distributions by identifying key values like minimum, maximum, median, and quantiles. They are foundational to descriptive statistics and serve as a basis for non-parametric statistical methods, including rank tests and confidence intervals, enhancing the analysis and interpretation of empirical data. The distribution of the i^{th} order statistics of the HL-TLLx distribution is

$$g_{i:n}(x) = \frac{n!\, g(x)}{(i-1)!(n-i)!}[G(x)]^{i-1}[1-G(x)]^{n-i}. \tag{5}$$

Using an alternative representation,

$$g_{i:n}(x) = \frac{n!\, g(x)}{(i-1)!\,(n-i)!}\sum_{p=0}^{n-i}(-1)^p\binom{n-i}{p}\frac{\left[1-(1+cx)^{-2\beta}\right]^{\delta(p+i-1)}}{\left\{2-\left[1-(1+cx)^{-2\beta}\right]^{\delta}\right\}^{p+i-1}}$$

which simplifies to

$$g_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}\sum_{p=0}^{n-i}(-1)^p\binom{n-i}{p}\frac{4\delta\beta c(1+cx)^{-(2\beta+1)}\left[1-(1+cx)^{-2\beta}\right]^{\delta p+\delta i-1}}{\left\{2-\left[1-(1+cx)^{-2\beta}\right]^{\delta}\right\}^{p+i+1}}$$

Let $A = \frac{n!}{(i-1)!(n-i)!}$, then

$$\begin{aligned} g_{i:n}(x) &= A\sum_{p=0}^{n-i}\sum_{q=0}^{\infty}(-1)^{p+q}\binom{n-i}{p}\binom{-(p+i+1)}{q} \\ &\times \frac{2^{-(p+q+i)}2\delta\beta c(1+cx)^{-(2\beta+1)}\left[1-(1+cx)^{-2\beta}\right]^{\delta q+\delta p+\delta i-1}}{\left\{1-\frac{1}{2}\left[1-(1+cx)^{-2\beta}\right]^{\delta}\right\}^{p+i+1}}. \end{aligned}$$

Finally,

$$\begin{aligned} g_{i:n}(x) &= \frac{n!}{(i-1)!(n-i)!}\sum_{p=0}^{n-i}\sum_{q=0}^{\infty}(-1)^{p+q}\binom{n-i}{p}\binom{-(p+i+1)}{q}\frac{2^{-(p+q+i)}}{(p+q+i)} \\ &\times g_*(x;\beta,c,\delta(p+q+i)), \end{aligned} \tag{6}$$

where $g_*(x; \beta, c, \delta(p+q+i))$ is the TLLx distribution with parameters c, β, and $\delta(p+q+1) > 0$.

Thus, the distribution of the i^{th} order statistic from the HL-TLLx distribution is a linear combination of the TLLx distribution with parameters c, β, and $\delta(p + q + 1)$, where

$$\frac{n!}{(i-1)!(n-i)!}\binom{n-i}{p}\binom{-(p+i+1)}{q}\frac{(-1)^{p+q}2^{-(p+q+i)}}{(p+q+i)}$$

are the coefficients.

The r^{th} moment of the i^{th} order statistic is given by

$$E[X_{i:n}^r] = \frac{n!}{(i-1)!(n-i)!}\sum_{p=0}^{n-i}\sum_{q=0}^{\infty}\binom{n-i}{p}\binom{-(p+i+1)}{q}\frac{(-1)^{p+q}2^{-(p+q+i)}}{(p+q+i)}E_{G_{TLLx}}(X^r),$$

where $E_{G_{TLLx}}(X^r)$ is the r^{th} moment of the TLLx distribution with parameters c, β, and $\delta(p + q + 1) > 0$.

3.4 R´enyi Entropy

R´enyi entropy proposed (R´enyi, 1960) is defined as

$$I_R(v) = \frac{1}{1-v}\log\left\{\int_0^{\infty} f^v(x)\,dx\right\}.$$

The function $f^v(x)$ is given by

$$f^v(x) = \frac{(4\delta\beta c)^v(1+cx)^{-(2\beta+1)v}[1-(1+cx)^{-2\beta}]^{v(\delta-1)}}{\{2-[1-(1+cx)^{-2\beta}]^{\delta}\}^{2v}},$$

or alternatively

$$f^v(x) = \frac{(4\delta\beta c)^v(1+cx)^{-(2\beta+1)v}[1-(1+cx)^{-2\beta}]^{v(\delta-1)}}{2^{-2v}\left\{1-\frac{1}{2}[1-(1+cx)^{-2\beta}]^{\delta}\right\}^{2v}}.$$

Expanding $f^v(x)$ as a summation

$$f^v(x) = \sum_{m=0}^{\infty}(-1)^m\binom{-2v}{m}(\delta\beta c)^v(1+cx)^{-(2\beta+1)v}[1-(1+cx)^{-2\beta}]^{\delta m+\delta v-v}.$$

Let $y = (1 + cx)^{-2\beta}$, so that

$$dy = -2\beta c(1+cx)^{-2\beta-1}dx,$$

and

$$dx = \frac{-dy}{2\beta c y^{\frac{2\beta+1}{2\beta}}}.$$

Also, $1 + cx = y^{-\frac{1}{2\beta}}$.

When $x = 0$, $y = 1$, and when $x \to \infty$, $y = 0$.
Substituting into the integral

$$\int_0^\infty f^v(x)dx = \sum_{m=0}^{\infty}(-1)^m \left(\frac{1}{2}\right)^m \binom{-2v}{m}(\delta\beta c)^v \int_0^1 y^{\frac{-1}{2\beta}[-(2\beta+1)v]}(1-y)^{\delta m+\delta v-1}\frac{dy}{2\beta c y^{\frac{2\beta+1}{2\beta}}}$$

Simplify the expression

$$= \sum_{m=0}^{\infty}(-1)^m \left(\frac{1}{2}\right)^{m+1} \delta^v(\beta c)^{v-1} \int_0^1 y^{\frac{2\beta v+v-1}{2\beta}-1}(1-y)^{\delta(m+v)-1}\,dy.$$

Using the beta function

$$\int_0^1 y^{a-1}(1-y)^{b-1}\,dy = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$

we have

$$\int_0^1 y^{\frac{2\beta v+v-1}{2\beta}-1}(1-y)^{\delta(m+v)-1}\,dy = \frac{\Gamma\left(\frac{2\beta v+v-1}{2\beta}\right)\Gamma\left(\delta(m+v)\right)}{\Gamma\left(\frac{v(2\beta+1)-1}{2\beta}+\delta(m+v)\right)}$$

Thus,

$$\int_0^\infty f^v(x)dx = \sum_{m=0}^{\infty}(-1)^m \left(\frac{1}{2}\right)^{m+1} \delta^v(\beta c)^{v-1}\frac{\Gamma\left(\frac{2\beta v+v-1}{2\beta}\right)\Gamma\left(\delta(m+v)\right)}{\Gamma\left(\frac{v(2\beta+1)-1}{2\beta}+\delta(m+v)\right)}$$

Finally, the R´enyi entropy is

$$I_R(v) = \frac{1}{1-v}\log\left\{\sum_{m=0}^{\infty}(-1)^m \left(\frac{1}{2}\right)^{m+1} \delta^v(\beta c)^{v-1}\frac{\Gamma\left(\frac{2\beta v+v-1}{2\beta}\right)\Gamma\left(\delta(m+v)\right)}{\Gamma\left(\frac{v(2\beta+1)-1}{2\beta}+\delta(m+v)\right)}\right\}$$

3.5 Shannon Entropy

Shannon entropy (Shannon, 1951) is defined as

$$H[g(x)] = E[-\log(g(x))].$$

Using an expanded representation

$$H[g(x)] = -\log(4) - \log(\delta\beta c) + (2\beta+1)E[\log(1+cx)] - (\delta-1)E[\log\{1-(1+cx)^{-2\beta}\}] +$$
$$2\log(2) - E\left[\log\left\{1 - \frac{1}{2}[1-(1+cx)^{-2\beta}]^{\delta}\right\}\right].$$

Note:

For $|x| < 1$, using the series expansion

$$\log(1+x) = \sum_{q=1}^{\infty} \frac{(-1)^{q+1}x^q}{q},$$

we have:

$$E[\log(1+cx)] = \sum_{q=1}^{\infty} \frac{(-1)^{q+1}c^q E[x^q]}{q}$$

Substituting q for r in $E[X^r]$, we obtain,

$$E[\log(1+cx)] = \sum_{i=1}^{\infty}\sum_{j=0}^{q}\sum_{q=1}^{\infty} \left(\frac{1}{2}\right)^{i} \frac{\delta}{q} \binom{q}{j} \frac{\Gamma\left(\frac{2\beta-j}{2\beta}\right)\Gamma(\delta i)}{\Gamma\left(\frac{2\beta-j}{2\beta}+\delta i\right)}.$$

Let

$$B = \frac{\Gamma\left(\frac{2\beta-j}{2\beta}\right)\Gamma(\delta i)}{\Gamma\left(\frac{2\beta-j}{2\beta}+\delta i\right)}.$$

For $E[\log\{1-(1+cx)^{-2\beta}\}]$, we have

$$E[\log\{1-(1+cx)^{-2\beta}\}] = \sum_{s=1}^{\infty} \frac{(-1)^{s+1}}{s} E\{-(1+cx)^{-2\beta s}\}$$

Using series expansion

$$= \sum_{s=1}^{\infty} \frac{(-1)^{s-2\beta s+1}}{s} \sum_{t=0}^{\infty} \binom{-2\beta s}{t} c^t E[x^t]$$

Substituting t for r in $E[X^r]$, we get

$$= \sum_{i=1}^{\infty} \sum_{j=0}^{t} \sum_{s=1}^{\infty} \sum_{t=0}^{\infty} \left(\frac{1}{2}\right)^i \frac{(-1)^{t+j+s-\beta s+1}}{s} \binom{t}{j} \binom{-2\beta s}{t} \delta B.$$

For $E\left[\log\left\{1-\frac{1}{2}[1-(1+cx)^{-2\beta}]^{\delta}\right\}\right]$ we have

$$E\left[\log\left\{1-\frac{1}{2}[1-(1+cx)^{-2\beta}]^{\delta}\right\}\right] = -\sum_{u=1}^{\infty} \left(\frac{1}{2}\right)^u \frac{1}{u} \sum_{v=0}^{\infty} (-1)^v \binom{\delta u}{v} E\{(1+cx)^{-2\beta v}\}$$

Expanding further,

$$= \sum_{u=1}^{\infty} \sum_{v=1}^{\infty} \sum_{w=1}^{\infty} \left(\frac{1}{2}\right)^u \frac{1}{u} \binom{\delta u}{v} \binom{-2\beta v}{w} (-1)^{v+1} c^w E[x^w].$$

Substituting w for r in $E[X^r]$, we have

$$= \sum_{i=1}^{\infty} \sum_{j=0}^{w} \sum_{u=1}^{\infty} \sum_{v=0}^{\infty} \sum_{w=0}^{\infty} \frac{1}{u} \left(\frac{1}{2}\right)^{u+i} \binom{\delta u}{v} \binom{-2\beta v}{w} (-1)^{w+v+j+1} \delta\beta.$$

Thus, the Shannon entropy is given by

$$H[g(x)] = -\log(\delta\beta c) + (2\beta+1) \sum_{i=1}^{\infty} \sum_{j=0}^{q} \sum_{q=1}^{\infty} \frac{(-1)^{j+1}}{q} \binom{q}{j} \delta \left(\frac{1}{2}\right)^i \delta c$$

$$-(\delta-1) \sum_{i=1}^{\infty} \sum_{j=0}^{t} \sum_{s=i}^{\infty} \sum_{t=0}^{\infty} \frac{(-1)^{t+j+s-\beta s+1}}{s} \binom{t}{j} \binom{-2\beta s}{t} \left(\frac{1}{2}\right)^i \delta c$$

$$+ \sum_{i=1}^{\infty} \sum_{j=0}^{w} \sum_{u=1}^{\infty} \sum_{v=0}^{\infty} \sum_{w=0}^{\infty} \frac{(-1)^{w+v+j+1}}{u} \binom{\delta u}{v} \binom{-2\beta v}{w} \left(\frac{1}{2}\right)^{u+i} \delta c,$$

where

$$c = \frac{\Gamma\left(\frac{2\beta-j}{2\beta}\right) \Gamma(\delta i)}{\Gamma\left(\frac{2\beta-j}{2\beta} + \delta i\right)}$$

4. Maximum Likelihood Estimation

The log-likelihood function, *L*, for a single observation of the HL-TLLx is given by

$$L = \log(4) + \log(\delta) + \log(\beta) + \log(c) - (2\beta+1)\log(1+cx) \\ + (\delta-1)\log\left(1-(1+cx)^{-2\beta}\right) - 2\log\left\{2-\left[1-(1+cx)^{-2\beta}\right]^{\delta}\right\}$$

The partial derivatives of the parameters are

$$\frac{\partial L}{\partial \delta} = \frac{1}{\delta} + \log\left[1-(1+cx)^{-2\beta}\right] - 2\cdot\frac{\left[1-(1+cx)^{-2\beta}\right]^{\delta}\log\left[1-(1+cx)^{-2\beta}\right]}{2-\left[1-(1+cx)^{-2\beta}\right]^{\delta}},$$

$$\frac{\partial L}{\partial \beta} = \frac{1}{\beta} - 2\log(1+cx) + 2(\delta-1)(1+cx)^{-2\beta-1} + \frac{4\delta(1+cx)^{-2\beta-1}[1-(1+cx)^{-2\beta}]^{\delta-1}}{2-[1-(1+cx)^{-2\beta}]^{\delta}},$$

and

$$\frac{\partial L}{\partial c} = \frac{1}{c} - \frac{(2\beta+1)x}{1+cx} - \frac{2\beta(\delta-1)x(1+cx)^{-2\beta-1}}{1-(1+cx)^{-2\beta}} - \frac{4\delta\beta x(1+cx)^{-2\beta-1}[1-(1+cx)^{-2\beta}]^{\delta-1}}{\{2-[1-(1+cx)^{-2\beta}]^{\delta}\}}$$

Solutions to these equations yields the maximum likelihood estimates of the parameters.

5. Simulations

In this section, we present results of Monte Carlo simulation studies. Sample sizes n = 25, 50, 100, 200, 400, 800, and 1600 and N = 3000 were considered.
Different parameter combinations were considered Set *I:* $\beta = 1.0$, $\delta = 1.0$, $c = 1.0$, Set *II:* $\beta = 1.0$, $\delta = 0.5$, $c = 1.5$, Set *III:* $\beta = 1.0$, $\delta = 1.5$, $c = 1.1$, and Set *IV*: $\beta = 0.9$, $\delta = 0.5$, $c = 0.9$. The results are presented in Table 1 and performance of the mle method was assessed using average bias (ABias) and root mean square error (RMSE). Abias and RMSE are calculated for a parameter, say, β, using the formulae:

$$ABias(\hat{\beta}) = \frac{\sum_{i=1}^{N}\hat{\beta}_i}{N} - \beta, \quad \text{and} \quad RMSE(\hat{\beta}) = \sqrt{\frac{\sum_{i=1}^{N}(\hat{\beta}_i-\beta)^2}{N}},$$

respectively. The Abias and RMSE decreases towards zero for all parameter values indicating consistency of the mle estimates.

Table 1: Results of Monte Carlo simulations for the HL-TLLx distribution

			β = 1.0, δ = 1.0, c = 1.0			**β = 1.0, δ = 0.5, c = 1.5**	
	n	**Mean**	**RMSE**	**AvBIAS**	**Mean**	**RMSE**	**AvBIAS**
	25	3.7988	8.9077	2.7988	8.4818	20.3840	7.4818
	50	1.7385	3.1600	0.7385	3.2314	8.9492	2.2314
	100	1.2255	1.2268	0.2255	1.5338	3.3850	0.5338
β	200	1.0647	0.2968	0.0647	1.0495	0.3305	0.0495
	400	1.0328	0.1673	0.0328	1.0028	0.1902	0.0028
	800	1.0135	0.1058	0.0135	0.9675	0.1092	-0.0325
	1600	1.0084	0.0725	0.0084	0.9742	0.0471	-0.0258
	25	2.4744	1.2650	1.4744	6.9815	4.9099	5.4815
	50	1.9391	0.8057	0.6391	5.1494	2.5060	4.6494
	100	1.1010	0.3809	0.1010	0.5638	0.1358	0.0638
δ	200	1.0537	0.2067	0.0537	0.5470	0.0868	0.0470
	400	1.0208	0.1353	0.0208	0.5396	0.0656	0.0396
	800	1.0081	0.0879	0.0081	0.5323	0.0450	0.0323
	1600	1.0046	0.0600	0.0046	0.5313	0.0356	0.0313
	25	5.4269	2.4960	1.4269	3.0905	2.6580	2.5905
	50	3.6596	1.5144	0.4596	2.4742	1.9090	1.9742
	100	1.2279	0.9973	0.2279	1.9653	1.4991	0.4653
c	200	1.0888	0.4944	0.0888	1.7760	0.8317	0.2760
	400	1.0374	0.3343	0.0374	1.7381	0.6374	0.2381
	800	1.0136	0.2159	0.0136	1.6978	0.3916	0.1978
	1600	1.0066	0.1454	0.0066	1.6146	0.1146	0.1146
			β = 1.0, δ = 1.5, c = 1.1			**β = 0.9, δ = 0.5, c = 0.9**	
	n	**Mean**	**RMSE**	**AvBIAS**	**Mean**	**RMSE**	**AvBIAS**
	25	3.1564	7.3044	2.1564	6.2263	14.1189	5.2263
	50	1.4689	2.1183	0.4689	2.9407	6.9743	1.9407
	100	1.1474	0.6961	0.1474	1.4697	2.6448	0.4697
β	200	1.0473	0.2366	0.0473	1.0800	0.4741	0.0800
	400	1.0259	0.1459	0.0259	1.0172	0.1955	0.0172
	800	1.0116	0.0966	0.0116	0.9948	0.1318	0.0052
	1600	1.0075	0.0658	0.0075	0.9958	0.1008	0.0042
	25	4.7094	2.4590	1.6094	3.2838	3.9165	2.7838
	50	2.0402	1.7781	1.3402	2.2469	48.2416	1.7469
	100	2.0150	0.9978	0.5150	0.5507	0.1300	0.0507
δ	200	1.6183	0.4278	0.1183	0.5355	0.0791	0.0355
	400	1.5436	0.2602	0.0436	0.5293	0.0575	0.0293
	800	1.5150	0.1600	0.0150	0.5257	0.0417	0.0257
	1600	1.5076	0.1086	0.0076	0.5250	0.0336	0.0250
	25	3.7212	2.2270	1.6212	2.4273	6.8590	1.5273
	50	2.4256	1.6100	0.9326	1.6120	3.0512	0.7120
	100	1.8296	1.0655	0.7296	1.1272	0.8517	0.2272
c	200	1.2304	0.6133	0.1304	1.0200	0.4579	0.1200
	400	1.1516	0.3868	0.0516	1.0032	0.3493	0.1032
	800	1.1171	0.2425	0.0171	0.9816	0.2320	0.0816
	1600	1.1073	0.1625	0.0073	0.9544	0.1352	0.0544

6. Applications

We present in these section two demonstrations to illustrate the utility of the HL-TLLx distribution compared to the other selected generalizations of the Lomax distribution. The HL-TLLx is also compared to the TLLx distribution to prove the significant effect of the half logistic transformation on the TLLx distribution. The half logistic transformation improves the flexibility of the TLLx distribution as shown by the results presented in this section. The other selected generalizations of the Lomax distribution considered in this paper are the exponentiated Lomax- (Exp-Lx) distribution (Abdul-Moniem and Abdel-Hameed, 2012), Weibull-Lomax (WLx) (Tahir et al., 2015), and power Lomax (PLX) (El-Houssainy et al., 2016). Model performance was evaluated using the following goodness-of-fit metrics: -2loglikelihood (-2log(L)), Bayesian information criterion (BIC), Consistent Akaike Information Criterion (CAIC), Cram´er-von Mises (W*), Anderson-Darling (AD), and Kolmogorov-Smirnov (K-S) and the corresponding p-value. The model with the least values for the goodness-of-fit statistics and bigger p-value for the K-S statistic is deemed as the best fitting model.

Graphical techniques were also employed to demonstrate how flexible our model is in data fitting. Fitted pdfs for all the fitted models are considered as well as the probability plots showing the sum of squares values (SS). The model with least value for the SS statistic is deemed the best in fitting the given data set. Other plots considered for the HL-TLLx distribution for each data set are empirical cdf, Kaplan Meier (K-M), scaled total time of test (TTT), and hrf plots. To each example, we plot the profile log-likelihood plots to demonstrate that the mles represent global maxima.

6.1 Carbon Fibers Data

The first data are measurements for single carbon fibers impregnated at gauge lengths of 1, 10, 20 and 50 mm which were tested at gauge lengths of 20, 50, 150 and 300 mm. A sample of 63 from single fibers of 20 mm is considered with the following observations: 1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020. The data set was first published by (Badar and Priest, 1982).

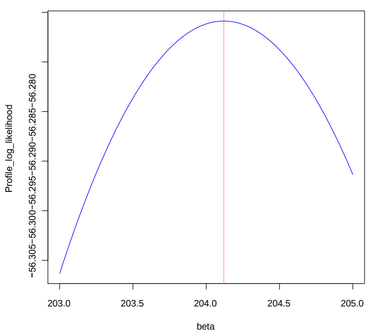
Table 2: Parameter estimates and GoF statistics for carbon fibers data

Statistics											
Distribution	**Estimates**			**-2log(L)**	**AIC**	**CAIC**	**BIC**	**W***	**AD**	**K−S**	**p−value**
HL-TLLx	$\boldsymbol{\beta}$	$\boldsymbol{\delta}$	$\boldsymbol{c}$								
	1.4132	1.1541×10^{5}	1.9048	155.05	161.05	161.83	165.72	0.0356	0.2493	0.0831	0.9690
	(0.1918)	(9.9956×10^{-4})	(11.0180)								
TLLx	1.4296	193.4700	1.4128	155.38	161.38	162.15	166.04	0.0382	0.2644	0.0861	0.9576
	(0.2762)	(0.0028)	(0.5874)								
Exp-Lx	$\boldsymbol{\alpha}$	$\boldsymbol{\delta}$	$\boldsymbol{\theta}$								
	193.4500	1.4127	2.8592	155.38	161.38	162.15	166.04	0.0382	0.2644	0.0861	0.9577
	(0.0033)	(0.5874)	(0.5524)								
WLx	$\boldsymbol{\alpha}$	$\boldsymbol{\alpha}$	$\boldsymbol{b}$								
	4.2176	0.2548	2.4039	162.19	168.19	168.96	172.85	0.1265	0.8062	0.1329	0.5663
	(2.4859)	(0.1825)	(5.0122)								
PLx	$\boldsymbol{\alpha}$	$\boldsymbol{b}$	$\boldsymbol{\delta}$								
	0.4554	5.0969	504.1200	157.51	163.51	164.29	168.18	0.0451	0.3308	0.0866	0.9577
	(0.2101)	(0.7883)	(0.0019)								

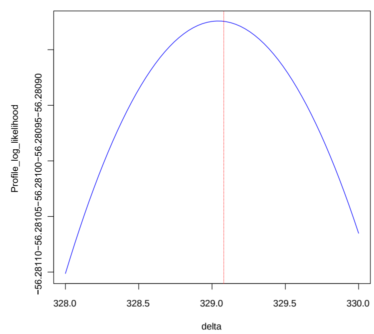
The parameter estimates and standard errors, together with the goodness-of-fit statistics for the various fitted models are presented in Table 2. The estimated variance-covariance (var-cov) matrix of the HL-TLLx model on carbon fibers dataset is defined as follows:

$$\begin{bmatrix} 1.4031 \times 10^{-17} & -1.3808 \times 10^{-18} & 5.1188 \times 10^{-13} \\ -1.3807 \times 10^{-18} & 1.3587 \times 10^{-19} & -5.0373 \times 10^{-14} \\ 5.1188 \times 10^{-13} & -5.0373 \times 10^{-14} & 1.8674 \times 10^{-8} \end{bmatrix}$$

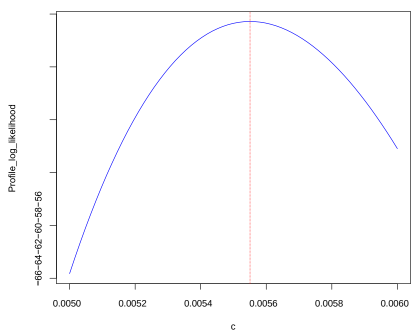
The var-cov matrix is useful in estimating the 95% asymptotic confidence intervals for the model parameters. Results shown in Table 2 confirm that the HL-TLLx distribution provides a better fit to the carbon fibers dataset compared to the TLLx and the other selected distributions. This is because the HL-TLLx distribution has the lowest values for the goodness-of-fit statistics and a bigger p-value for the K-S statistic.



(a)



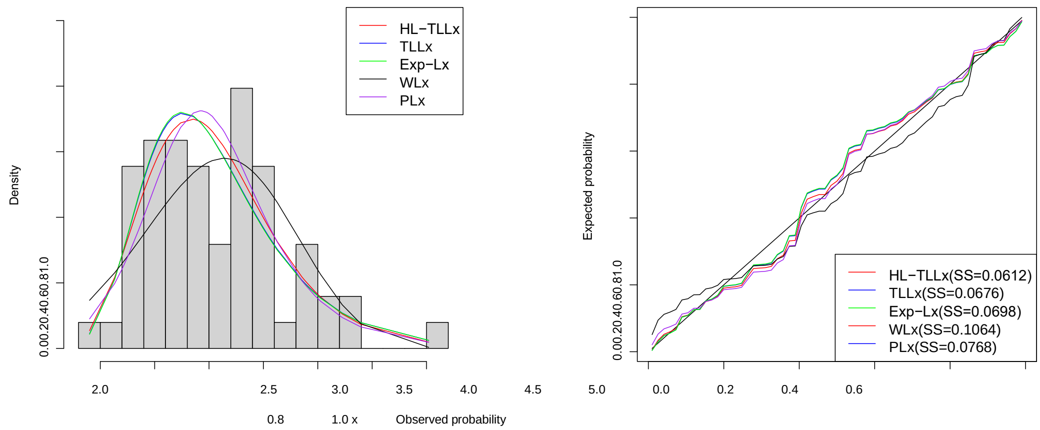
(b)



(c)

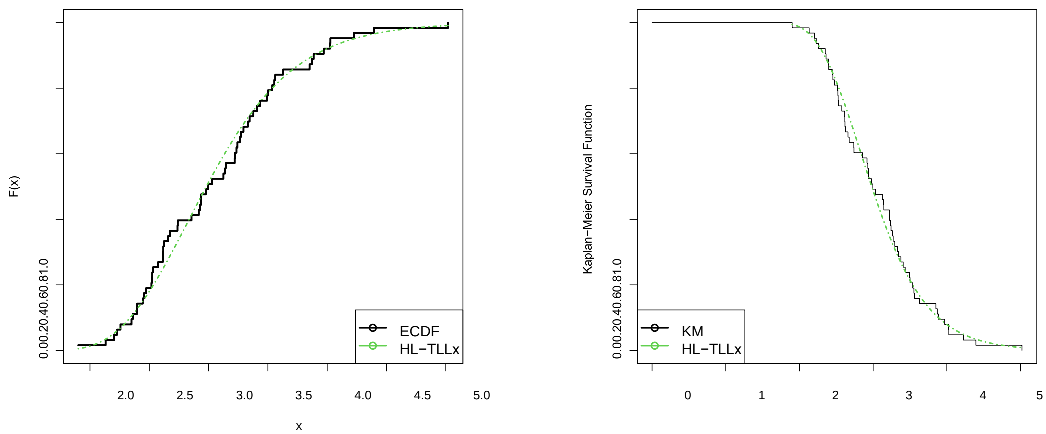
Figure 2: Profiles plots for first dataset

Figure [2] represents profile plots of the model parameters. The results show that we have accurately estimated the model parameters since the loglikelihood reaches its maximum value at the estimated parameter value for all the model parameters



(a) **(b)**

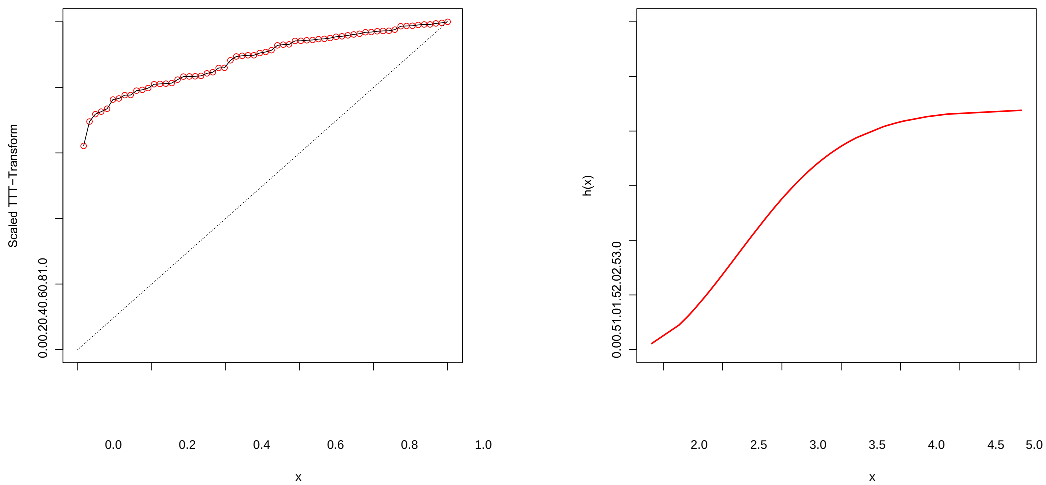
Figure 3: Density and probability plots for first dataset



(a) **(b)**

Figure 4: Fitted ECDF curve and K-M plots for first dataset

Figure [3] represents the fitted densities and probability plots for all the models in Table 2. The HL-TLLx distribution offers a better fit to carbon fibers data compared to the other selected models. Furthermore, Figure [4] shows the empirical versus theoretical cdf as well as empirical versus theoretical K-M survival curves for carbon fibers dataset. The HL-TLLx distribution correspondence with the observed data, as the empirical lines run closer to the theoretical lines.



(a) **(b)**

Figure 5: Scaled TTT and hrf plots for first dataset

Figure [5] presents the scaled TTT for the carbon fiber dataset. The scaled TT plot suggests an increasing hrf which corresponds to the fitted HL-TLLx hrf for carbon fibers data.

6.2 Growth Hormone Data

The second dataset represent estimated time since growth hormone medication until the children reached the target age. The data was analysed by (Alizadeh et al., 2017) and (Oluyede et al., 2020). The measurements are: 2.15, 2.20, 2.55, 2.56, 2.63, 2.74, 2.81, 2.90, 3.05, 3.41, 3.43, 3.43, 3.84, 4.16, 4.18, 4.36, 4.42, 4.51, 4.60, 4.61, 4.75, 5.03, 5.10, 5.44, 5.90, 5.96, 6.77, 7.82, 8.00, 8.16, 8.21, 8.72, 10.40, 13.20, 13.70.

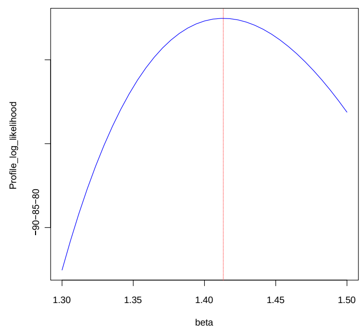
Table 3: Parameter estimates and GoF statistics for growth hormone data

Statistics											
Distribution	**Estimates**			**-2log(L)**	**AIC**	**CAIC**	**BIC**	**W***	**AD**	**K−S**	**p−value**
HL-TLLx	$\boldsymbol{\beta}$	$\boldsymbol{\delta}$	$\boldsymbol{c}$	112.56	118.56	118.97	124.99	0.0603	0.3210	0.0789	0.8272
	204.1200	329.0800	5.5505×10^{-3}								
	(3.7458×10^{-9})	(3.6861×10^{-10})	(1.3666×10^{-4})								
TLLx	168.0200	228.6300	5.8875×10^{-3}	113.07	119.07	119.48	125.50	0.0713	0.3693	0.0885	0.7074
	(5.0964×10^{-9})	(7.0174×10^{-10})	(1.4431×10^{-4})								
Exp-Lx	$\boldsymbol{\alpha}$	$\boldsymbol{\delta}$	$\boldsymbol{\theta}$	113.17	119.17	119.58	125.60	0.0727	0.3767	0.0903	0.6833
	258.8200	0.0199	103.2600								
	(7.1166×10^{-9})	(4.8845×10^{-4})	(9.7143×10^{-8})								
WLx	$\boldsymbol{\alpha}$	$\boldsymbol{\alpha}$	$\boldsymbol{b}$	121.95	127.95	128.36	134.38	0.1121	0.7817	0.0808	0.8055
	9.6542	0.2884	3.0414								
	(1.8239)	(0.0630)	(1.7539)								
PLx	$\boldsymbol{\alpha}$	$\boldsymbol{b}$	$\boldsymbol{\delta}$	115.68	121.68	122.09	128.11	0.0868	0.4953	0.0897	0.6913
	1.0678	8.4244	1.1305×10^{4}								
	(0.4759)	(0.6797)	(9.3435×10^{-5})								

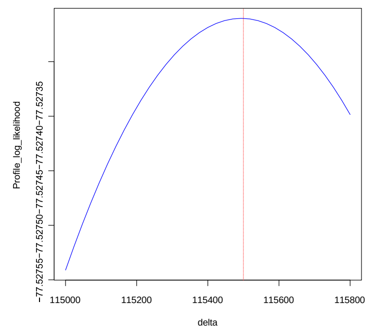
The estimated var-cov matrix for the HL-TLLx model of growth hormone is

$$\begin{bmatrix} 0.0367 & 0.0001 & -2.0970 \\ 0.0001 & 4.8937 \times 10^{-7} & -0.0077 \\ -2.0970 & -0.0077 & 121.3968 \end{bmatrix}$$

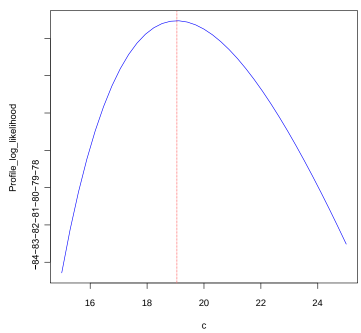
From the results presented in Table 3, we further conclude that the HL-TLLx model is an important alternative to the other generalizations of the Lomax distribution as shown by the performance on this second dataset.



(a)



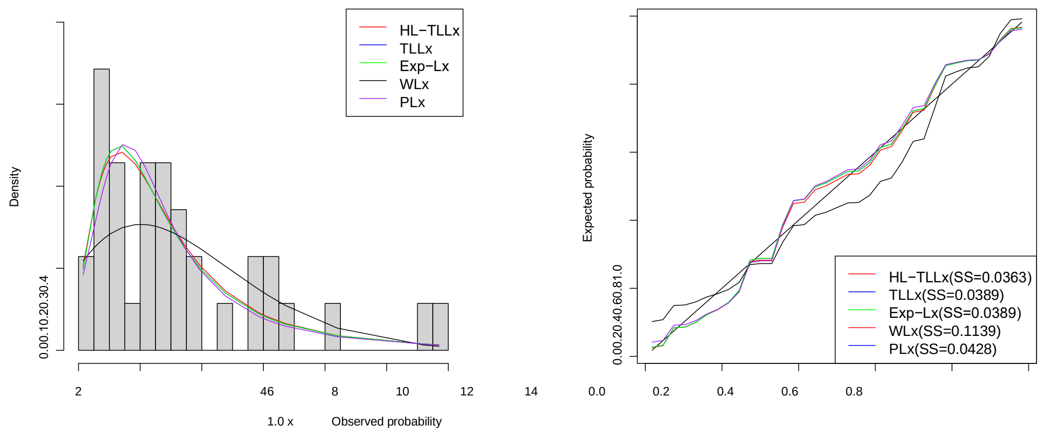
(b)



(c)

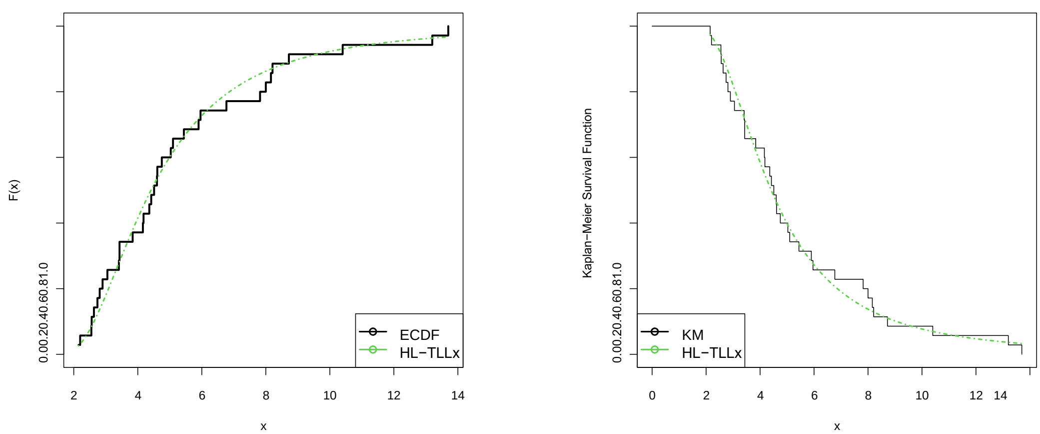
Figure 6: Profiles plots for second dataset

Figure [6] represents the profile plots for the parameters of the HL-TLLx model on growth hormone data. The results confirm that the mles are accurate and the chosen model estimation technique is good.



(a) **(b)**

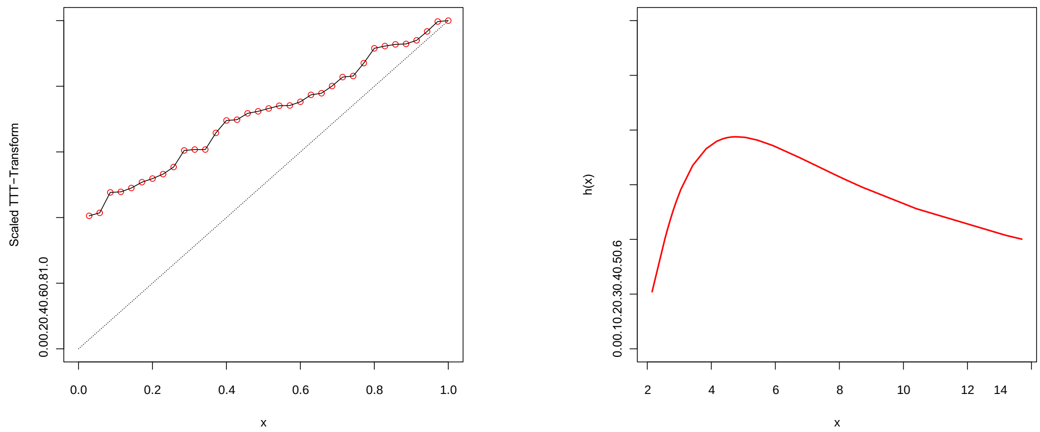
Figure 7: Density and probability plots for second dataset



(a) **(b)**

Figure 8: Fitted ECDF curve and K-M plots for second dataset

Figures [8] and 9 demonstrate that the HL-TLLx distribution fit the growth hormone data effectively.



(a) (b)

Figure 9: Scaled TTT and hrf plots for second dataset

7. Conclusions

A new generalization of the TLLx distribution was developed via the half logistic transformation. The half logistic transformation improved the flexibility of the TLLx as evidence by the results in the application section. Some important statistical properties of the proposed distribution were derived to enhance understanding of this model. Maximum likelihood estimation method was used to estimate the model parameters and the consistency and efficiency of the mle estimates were assessed via Monte Carlo simulation studies. The new model was applied to two real datasets in comparison to some selected generalizations of the Lomax distribution. The proposed distribution outperformed the other selected Lomax generalizations.

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