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Numerical Tau Method for Solving DAEs in Banach Spaces with Schauder Bases

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Abstract

Differential algebraic equations (DAEs) appear in many fields of physics and have a wide range of applications in various branches of science and engineering. Finding reliable methods to solve DAEs has been the subject of many investigations in recent years. In this paper, we present a numerical method for approximating the solution of a DAE. We formulate a general problem in a Banach space making use of a Schauder basis and the Tau method to approximate the load function and the solution of the differential problem. Finally, we offer a numerical example. This technique provides converges to the exact solution of the problem. The scheme is tested for some high-index DAEs and the results demonstrate that the method is very straightforward and can be considered as a

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powerful mathematical tool.

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1 Introduction

Many physical problems are naturally described by a system of differential algebraic equations (DAEs). These type of systems occur in the modelling of electrical networks, flow of incompressible fluids, optimal control, mechanical systems subject to constraints, power systems, chemical process simulation, computer-aided design and in many other applications. Finding new methods for solving DAEs has become an interesting task for mathematicians. The numerical approaches include the backward differentiation formulae (BDF) [20, 21], RungeKutta method [17], specialized RungeKutta method, which is a modification of the classic RungeKutta method to solve index-2 DAEs [14] and Krylov deffered correction (KDC) method [15]. Recently, Adomian decomposition method [18, 19] and the variational iteration method (VIM) [16] have been used to solve the linear and nonlinear DAEs. For some applications of the VIM and Adomian decomposition method (ADM) in science and engineering the interested readers can see [6-13].

In this paper, we present a different approach. The main aim of this paper is to extend the Banach Spaces with Schauder Bases, proposed by the Chinese mathematician J. Houn He [4, 5], to find the solution of linear and nonlinear DAEs. The solution of many problems consists in finding the inverse of a given function by means of an operator. This technique, we construct approximations of the load function starting from some functions for which we can easily solve the DAEs. Such functions are the elements of certain Schauder basis. In practice, the numerical solution usually requires calculating some particular solutions using the Tau method (see [1, 2, 3]).

We will consider the Banach spaces C[0, 1] and $C^2[0, 1]$ endowed with their usual norms

$$\|y\|_\infty = \max_{t\in[0,1]} |y(t)| \ , \qquad (y \in C[0,1])$$

and

$$\|x\|_{C^2} = \|x''\|_\infty + \|x'\|_\infty + \|x\|_\infty \ , \quad (x \in C^2[0,1])$$

2 DAEs and reduction of index

A system of DAEs is one that consists of ordinary differential equations (ODEs) coupled with purely algebraic equations, on the other hand, DAEs are everywhere singular implicit ODEs. The general form of DAEs are

$$F(x(t),\ x'(t),\ t) = 0\ , \quad F \in C^1(R^{2m+1},\ R^m)\ , \quad t \in [0,T]$$

where $\partial F / \partial x$ is singular on R^{2m+1} [22]. Most DAEs arising in applications are in semi-explicit form and many are in the further restricted Hessenberg form [21]. The index-1 semi explicit DAEs is given by:

$$\begin{cases} x'(t) = f(x(t),\ y(t),\ t) & f \in C(R^{m+k+1}, R^m) \\ 0 = g(x(t),\ y(t),\ t) & g \in C^1(R^{m+k+1}, R^k) \end{cases} \quad t \in [0,T]$$

where $\partial g / \partial y$ is non-singular.

The index-2 Hessenberg DAEs is given by:

$$\begin{cases} x'(t) = f(x(t),\ y(t),\ t) & f \in C^1(R^{m+k+1}, R^m) \\ 0 = g(x(t),\ t) & g \in C^2(R^{m+1}, R^k) \end{cases} \quad t \in [0,T]$$

where $(\partial f/\partial y)$ $(\partial g/\partial x)$ is non-singular [22].

Now, we briefly review the reducing index method for semi-explicit DAEs, which is mentioned in [23, 24]. Consider a linear (or linearized) semi-explicit DAEs:

$$X^{(m)} = \sum_{j=1}^{m} A_j X^{(j-1)} + BY + q \tag{1}$$

$$0 = CX + r \tag{2}$$

where A_j, B and C are smooth functions of t,

$t_0 \le t \le t_f$, $A_j(t) \in R^{n\times n}$, $j = 1,...,m$, $B(t) \in R^{n\times k}$, $C(t) \in R^{k\times n}$, $n \ge 2$, $1 \le k \le n$

and CB is non-singular (DAE has index m+1) except possibly at a finite number of isolated points of t, which in this case, the DAEs (Equations (1,2)) have constraint singularity. The in homogeneities are $q(t) \in R^n$ and $r(t) \in R^k$. Now suppose that CB is non-singular, so we can rewrite Equation (1) as follows:

$$Y = (CB)^{-1} C\left[X^{(m)} - \sum_{j=1}^{m} A_j X^{(j-1)} - q\right],\ t \in [t_0, t_f] \tag{3}$$

Substituting Equation (3) into Equation (1), we obtain:

$$[I - (CB)^{-1} C]\left[X^{(m)} - \sum_{j=1}^{m} A_j X^{(j-1)} - q\right] = 0$$

The problem (1) transforms to the overdetermined system:

$$[I - (CB)^{-1} C]\left[X^{(m)} - \sum_{j=1}^{m} A_j X^{(j-1)} - q\right] = 0$$
$$CX + r = 0 \tag{4}$$

Now the system (4) can be transformed to a full-rank DAE system with *n* equations and *n* unknowns with index m [24, 25]. Here for simplicity, we consider problem (1) when $m=1$ (this problem has index 2), $n=2,3$ and $k=1,2$.

Also, if we suppose that DAE is non-singular, i.e. $CB(t)\neq 0$, $t\in[t_0,t_f]$ then by the following theorems, the given index-2 problem will be transformed to the index-1 DAE and the Schauder Bases will be applied to the obtained index-1 problem. This discussion can be extended to the general form (1).

Theorem 1 *Consider DAEs* (Equation (1)), *when it has index*-2, $n=2$ *and* $k=1$. *This problem is equivalent to the following index*-1 *DAE system:*

$$E_1X+E_0X=\tilde{q}$$

such that

$$E_0=\begin{pmatrix} b_1\alpha_{21}-b_2\alpha_{11} & b_1\alpha_{22}-b_2\alpha_{12} \\ c_1 & c_2 \end{pmatrix},\quad E_1=\begin{pmatrix} b_2 & -b_1 \\ 0 & 0 \end{pmatrix}$$

$$\tilde{q}=\begin{pmatrix} b_2q_1-b_1q_2 \\ 0 \end{pmatrix}$$

and

$$y=(CB)^{-1}C[X-AX-q].$$

Proof. The proof of Theorem 1 is presented in [25]. □

Theorem 2 *Consider DAEs* (Equation(1)) *with index*-2, $n=3$ *and* $k=2$. *This problem is equivalent to the following index*-1 *DAE system:*

$$\begin{pmatrix} M \\ 0 \end{pmatrix}X'+\begin{pmatrix} -MA \\ C \end{pmatrix}X=\begin{pmatrix} Mq \\ -r \end{pmatrix}$$

such that

$$M=(b_{21}b_{32}-b_{22}b_{31}\quad b_{12}b_{31}-b_{11}b_{32}\quad b_{11}b_{22}-b_{12}b_{21})_{1\times 3}$$

and $$y=(CB)^{-1}C[X-AX-q]$$

Proof. The proof is presented in [25]. □

3 Formulating the problem

Let X and Y be Banach spaces (over K=R or C), let $D : X \to Y$ be a one-to-one bounded and linear operator and let $y_0 \in Y$. We consider the following problem:

$$\text{Find } x_0 \in X : Dx_0 = Y_0 \qquad (5)$$

When D is a linear differential operator, this problem constitutes a linear differential equation. Thus, a method for solving this problem is, in particular, a method for solving differential equations.

4 Schauder basis

As mentioned in the introduction, we will make use of the concept of Schauder basis. Let us recall that a sequence $\{y_n\}_{n\geq 1}$ in a Banach space Y is a Schauder basis provided that for all $y \in Y$ there exists a unique sequence $\{\alpha_n\}_{n\geq 1} \subset K$ such that $y = \sum_{i=1}^{\infty} \alpha_n y_n$. The scalars $\alpha_n \in K$ are called the coefficients of y in the basis $\{y_n\}_{n\geq 1}$. If, for each $n \geq 1$, $y_n^*(y)$ is the unique α_n such that $y = \sum_{i=1}^{\infty} \alpha_n y_n$, then Y_n^* is a bounded and linear functional on Y. They are called the functionals associated with the basis $\{Y_n\}_{n\geq 1}$ and the sequence of bounded linear operators $P_n : Y \to Y$, given by $P_n y = \sum_{i=1}^{\infty} y_i^*(y) y_i$, $y \in Y$ are known as the sequence of projections of $\{y_n\}_{n\geq 1}$.

If $\{Y_n\}_{n\geq 1}$ is an orthogonal base in a separable Banach space, then it plainly is a Schauder basis. Schauder bases are explicitly known for Banach spaces of sequences as $l_p\ (1 < p < \infty)$ or c_0, and for Banach spaces of functions as $L_p([\alpha, b])(1 \leq p < \infty)$, $C^k([0,\ 1]^d)$ or $W_p^m([0,1]^d)$ (see [26-28]).

Given a partition $T_n = \{t_0 = 0 < t_1 < ... < t_n = 1\}$ of the interval [0, 1], let us denote by $S_1(T_n)$ the (n+1)-dimensional linear space of all continuous spline functions of degree 1 with knots T_n and, by $\{S_{t_i}^{T_n} : i = 0,1,...,n\}$ the Lagrangian basis, i.e., $S_{t_i}^{T_n}$ is the unique function in $S^1(T_n)$ such that $S_{t_i}^{T_n}(t_j) = \delta_{ij}$, $j = 0,1,...,n$. We can set up the usual Schauder basis in C[0, 1] as follows.

Definition 1 Let $T = \{t_n\}_{n\geq 0}$ denote a dense sequence in [0,1], with $t_0 = 0$, $t_n = 1$ and $t_i \neq t_j$, $i \neq j$. For all $n \geq 1$, let $T_n = \{t_j \mid j = 0,1,\ldots,n\}$. The Schauder system designates the functions $\{\varphi_n^T\}_{n\geq 0}$ given by $\varphi^T = S_{t_0}^{T_1}$, $\varphi_n^T = S_{t_n}^{T_n}$, for $n \geq 1$. For a proof of the following result, you can see [26].

Theorem 3 [29] *The Schauder system* $\{\varphi_n\}_{n\geq 0}$ *is a Schauder basis in* $C[0, 1]$.

The following easy property on Schander basis provides the solution of problem (5) as the limit of a sequence, and constitutes a numerical method for solving some ordinary differential equations.

Theorem 4 *Let* X *and* Y *be Banach spaces, let* $y_0 \in Y$ *and let* $D : X \to Y$ *a one-to-one bounded linear operator. We assume that* $\{y_n\}_{n\geq 1} \subset Y$ *is a Schauder basis and* $\{P_n\}_{n\geq 1}$ *is the sequence of the associate projections. Then the unique solution of problem* (5) *is given by*

$$x_0 = \lim_{n\to\infty} D^{-1}(P_n y_0)$$

Moreover, this solution satisfies $\left\| x_0 - D^{-1}(P_n y_0) \right\| \le \left\| D^{-1} \right\| \cdot \left\| y_0 - P_n y_0 \right\|$ (see [2]).

Proof: By virtue of the definition of Schauder basis and the projections P_n, given $y \in Y$, we obtain that

$$\lim_{n\to\infty} \left\| y - P_n y \right\| = 0 \tag{6}$$

Let x_0 be the solution of problem (5). Then, for all n∈N, the boundedness of the operator D^{-1} (by the inverse mapping theorem) and if we suppose that DAE is non-singular, i.e. $CB(t) \neq 0$, $t \in [t_0, t_f]$ then by use of the theorems 1,2, the given index-2 problem will be transformed to the index-1 DAE and will be used to the obtained index-1 problem gives that

$$\left\| x_0 - D^{-1}(P_n y_0) \right\|_\infty = \left\| D^{-1} y_0 - D^{-1}(P_n y_0) \right\|_\infty \le \left\| D^{-1} \right\|_\infty \left\| y_0 - P_n y_0 \right\|_\infty \tag{7}$$

Finally, it follows from (6) and (7) that $\lim_{n\to\infty} \left\| x_0 - D^{-1}(P_n y_0) \right\| = 0$. □

5 Test problems

The above result allows us to calculate some numerical solutions of differential equations. First of all, we should fix the Banach spaces X and Y and the operator $D : X \to Y$ so that the differential equation be unisolvent, i.e., D is a one-to-one and linear operator. Then the continuity of the operator D (or equivalently D^{-1}) must be proven. The boundary conditions will obviously influence the choice of X. Afterwards, we will consider a Schander basis $\{y_n\}_{n\ge1}$ in the Banach space Y. Since the sequence $\{D^{-1} y_n\}_{n\ge1}$ is a basis in X and the coefficients of y_0 with respect to $\{Y_n\}_{n\ge1}$ are the same as the ones of $x_0 = D^{-1} y_0$ in $\{D^{-1} y_n\}_{n\ge1}$, it suffices to obtain, for all $n \in N$, $D^{-1} y_n$. Such a function can then be calculated by means of the Tan method (see [2, 3]).

Example 1 Consider the linear index-2 semi-explicit DAEs problem:

$$\begin{aligned} X' &= AX + By + q \\ CX + r &= 0 \end{aligned} \tag{8}$$

where

$$A = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1+2t \end{pmatrix}, \quad q = \begin{pmatrix} -\sin(t) \\ 0 \end{pmatrix}, \quad C^T = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and

$$r(t) = -(e^{-t} + \sin(t))$$

with

$$x_1(0) = 1 \text{ and } x_2(0) = 0 .$$

The exact solutions of this problem are:

$$x_1(t) = e^{-t}, \qquad x_2(t) = \sin(t), \qquad y(t) = \frac{\cos(t)}{1+2t}$$

From Theorem 1, the index-2 DAEs (Equation (8)) can be converted to the index-1 DAEs:

$$\begin{cases} x_2 = -x_1 + e^{-t} + \sin(t) \\ x'_1 = -x_2 - x_1 - \sin(t) \end{cases}$$

with $x_1(0) = 1$ and $x_2(0) = 0$. We now apply the preceding method. We shall obtain a numerical solution of the problem. In this case, we take as X, Y the Banach Spaces ; $D: X \to Y$ the operator $Dx_1 = x_1 + 2x_1$ and $y_0 \in Y$ as the load function.

This problem has a unique solution $x_0 \in X$ for a given $y_0 \in Y$. Thus, D is a continuous operator and, according to the inverse mapping theorem, D^{-1} it is also continuous. We proceed to consider the Schander system $\{\varphi_n^T\}_{n\geq 0}$.

Given $n \in N$, $P_n y_0 = \sum_{i=0}^{n} (\varphi_i^T)^* y_0 \varphi_i^T$ and since $S_1(T_n) = \langle \varphi_0^T, \varphi_1^T, \ldots, \varphi_n^T \rangle$ then $P_n y_0 = \sum_{i=0}^{n} y_0(t_i) S_{t_i}^{T_n}$.

Therefore, from Theorem 4, we obtain the following approximation of x_0 :

$$D^{-1}(P_n y_0) = D^{-1}\left(\sum_{i=0}^{n} y_0(t_i) S_{t_i}^{T_n}\right) = \sum_{i=0}^{n} y_0(t_i) D^{-1}(S_{t_i}^{T_n})$$

To explicitly obtain a numerical solution, we calculate $D^{-1}(S_{t_i}^{T_n}),\ i = 0,\ldots,n$ that is, we solve the differential equation for those load functions which are polygonal functions. In the following table, we show some approximation error of the solution $\left\|x_0 - D^{-1}(P_n y_0)\right\|_{C^2}$, for some values of n.

Table 1

n	E
5	0.319617215342398
10	0.158984243890628
20	0.099496522297961
40	0.042540501653029
80	0.005105257392897

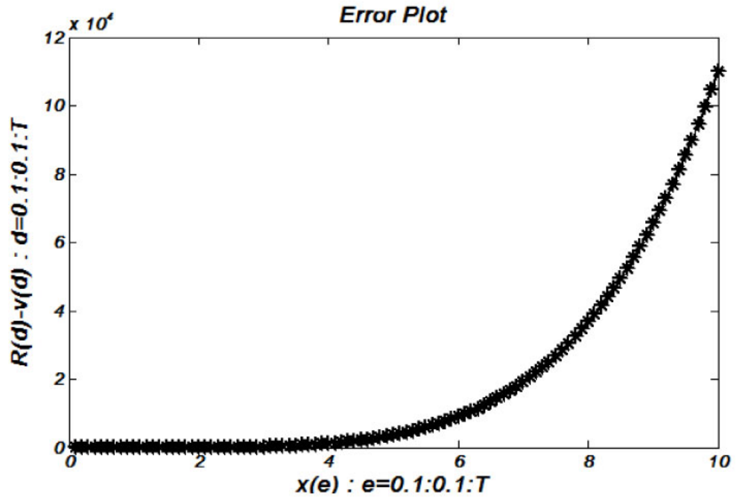

Figure 1.

We do the same steps for x_2 and so for y use of the following formulation:

$$Y = (CB)^{-1} C[X' - AX - q]$$

Example 2 Consider the index-2 problem [18, 19]:

$$\begin{aligned} X' &= AX + By + q \\ 0 &= CX + r \end{aligned} \qquad t \in [0,\ 1] \tag{9}$$

where

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t \end{pmatrix}, \qquad C^T = B = \begin{pmatrix} 1 & -1 \\ 0 & t \\ -1 & 0 \end{pmatrix},$$

q and r are compatible to the exact solutions

$$x_1(0) = x_2(0) = 0$$

and the exact solutions are:

$$x_1(t) = x_2(t) = x_3(t) = t^4 + t^5, \qquad y_1(t) = y_2(t) = \frac{t}{1 + t^4 + t^5}.$$

By Theorem 2, the index-2 DAEs (Equation (9)), with $M = \begin{pmatrix} t & 1 & t \end{pmatrix}$, transforms to the following index-1 DAEs:

$$\begin{cases} x_1 = -tx_2 + g_1(t) \\ x_3 = \ \ x_1 - g_2(t) \\ x'_2 = -tx'_1 - tx'_3 + tx_1 + x_2 + t_2 x_3 + g_3(t) \end{cases} \tag{10}$$

with

$$x_1(0) = x_2(0) = x_3 = 0$$

when

$$g_1(t) = t^6 - t^4, \quad g_2(t) = 0 \quad \text{and } g_3(t) = 4t^3 + 12t^4 + 8t^5 - 2t^6 - t^7$$

To solve the index-1 DAEs (Equation (10)), in this case, we take as X, Y the Banach space; $D : X \to Y$ the operator $Dx = x_2 + f(t)x_2$,

where

$$f(t) = \frac{2t - t^2 - t^3 - 1}{1 + 2t^2}$$

with

$$y_0 = \frac{11t^6 - 2t^7 + 9t^5 - t^8 + 4t^3 + 4t^4}{1 + 2t^2}$$

as the load function. This problem has a unique solution $x_0 \in X$ for a given $y_0 \in Y$.

Table 2

n	E
5	0.684645362643197
10	0.280067356217877
20	0 .066766480995862
40	0.004928003046549
80	0.001152165284518
100	8.521410442523754e-005

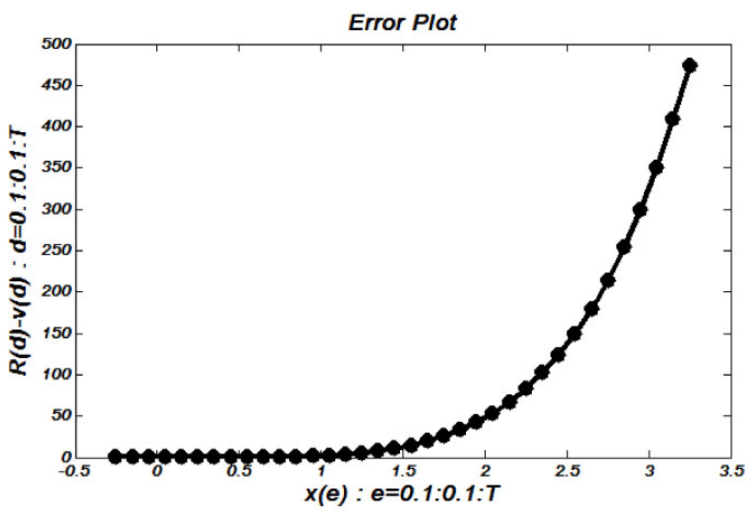

Figure 2.

We do the same steps for x_1 , x_3 and so for y use of the following formulation:

$$Y = (CB)^{-1} C[X' - AX - q]$$

6 Concluding remarks

We have presented a numerical method that allows some DAEs to be solved with a low computational cost. The use of the Schauder basis provided for the convergence to the exact solution, whereas the introduction of a basis of the Lagrangian splines simplifies the calculation of the approximations.

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