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# Inverse Eigenvalue Problem for a class of Singular Hermitian Matrices

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**Abstract**

In this paper, we discuss inverse eigenvalue problems for singular Hermitian matrices. In particular, we investigate how to construct $n \times n$ singular Hermitian matrices of rank 2 and 3 from a given prescribed spectral data. It is found that given the spectrum and the multipliers $k_i$ where $i = 1, 2, 3, \cdots, n-r$, the inverse eigenvalue problem for $n \times n$ singular Hermitian matrices of rank $r$ is solvable. Numerical examples are presented in each case.

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# 1 Introduction

An *Inverse Eigenvalue Problem*($IEP$) is to reconstruct a matrix which possesses both a prescribed eigenvalue and desired structure. Inverse eigenvalue problems arise in broad application areas such as control design, system identification, principle component analysis, structure analysis etc. There are many different types of inverse eigenvalue problems and despite of a great deal of research effort being put into this topic many of them are still open and are hard to be solved.

In [1] Gyamfi studied the solution to the Inverse Eigenvalue Problems (IEP) for a class of singular symmetric and singular Hermitian matrices. On the case of singular Hermitian matrices, he presented results up to rank 1. In this paper, we extend earlier results found by Gyamfi [1] on the solution to the IEP for a class of singular Hermitian matrices of rank 1 to rank 2 and 3.

The paper is organised as follows: In section 2 we review basics on how to reconstruct singular Hermitian matrices of rank 1 from prescribed spectrum. Our main work on the solution to inverse eigenvalue problems for singular Hermitian matrices of rank 2 and 3 is presented in section 3. we give conclusion and recommendation in the fourth section. Readers are referred to Gyamfi [1] for the details on inverse eigenvalue problems for singular symmetric matrices.

# 2 Preliminary Notes

In this section we review previous results obtained by Gyamfi [1] in respect of the inverse eigenvalue problems for singular Hermitian matrices of rank 1. We begin with with $2 \times 2$ singular symmetric matrices and extend to $n \times n$ singular hermitian matrices of rank 1.

**Lemma 2.1.** *Let $A$ be a non-traceless, symmetric matrix of rank $r$ with non-vanishing elements. Then there exits an isomorphism between the elements of $A$ and its distinct non-zero eigenvalues if and only if $r = 1$.*

**Corollary 1:** The inverse eigenvalue problem has a unique solution for singular symmetric matrices of rank 1 with prescribed linear dependence relation.

Specific case 1:

Given that $n = 2, r = 1$, we begin by considering $A_{(2,1)}$. By definition, $A_{(2,1)}$ has the form:

$$A_{(2,1)} = \begin{pmatrix} a_{11} & ka_{11} \\ ka_{11} & k^2 a_{11} \end{pmatrix} = a_{11} \begin{pmatrix} 1 & k \\ k & k^2 \end{pmatrix}.$$

Let $\Lambda_2 = \{\lambda_1, \lambda_2\}$. Since $A_{(2,1)}$ is singular of rank 1, it means that $\lambda_2 = 0$. We have $\mathrm{tr}(A_{(2,1)}) = \lambda = a_{11} + k^2 a_{11} = a_{11}(1+k^2)$. Therefore $a_{11} = \dfrac{\lambda}{1+k^2}$. Hence

$$A_{(2,1)} = \frac{\lambda}{1+k^2} \begin{pmatrix} 1 & k \\ k & k^2 \end{pmatrix}.$$

Thus $A_{(2,1)}$ has been reconstructed for a given $\lambda$ and prescribed scalar $k$.

We see from this formula that for any given $\lambda$ and parameter $k$, we can generate any $2 \times 2$ singular matrix of rank 1. For example if $k = 3, \lambda = 10$ we have

$$A_{(2,1)} = \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix}.$$

## 2.1 Extension to Hermitian matrices

We now extend the above to Hermitian matrices of dimension $2 \times 2$. $A$ is Hermitian implies that $a_{21} = \overline{a}_{12}$. Linear dependence of rows is given by $a_{21} = ka_{11}$ and $a_{22} = ka_{12}$, so that $a_{21} = \overline{a}_{12} = \overline{k}a_{11}$. Then $a_{22} = k(\overline{k}a_{11}) = |k|^2 a_{11}$. We now write the matrix as;

$$A_{(2,1)} = \begin{pmatrix} a_{11} & \overline{k}a_{11} \\ ka_{11} & |k|^2 a_{11} \end{pmatrix} = a_{11} \begin{pmatrix} 1 & \overline{k} \\ k & |k|^2 \end{pmatrix}.$$

Hence $\mathrm{tr}(A_{(2,1)}) = \lambda = a_{11} + |k|^2 a_{11} = a_{11}(1+|k|^2) \implies a_{11} = \dfrac{\lambda}{1+|k|^2}$. ¿From this we see that any $2 \times 2$ Hermitian matrix which has a parameter with the same value as the modulus of $k$ satisfies the above formula.

Example: Let $k = 1 + 2i, \lambda = 3$ and $\overline{k} = 1 - 2i$. We have $a_{11} = \dfrac{1}{2}$ and

$$A_{(2,1)} = \frac{1}{2}\begin{pmatrix} 1 & 1-2i \\ 1+2i & 5 \end{pmatrix}.$$

We use numerical examples to illustrate small singular Hermitian matrices os size $3 \leq n \leq 4$ of rank 1.

Example: For $n = 3, r = 1, A_{(3,1)}$ is the form:

$$A_{(3,1)} = a_{11}\begin{pmatrix} 1 & \overline{k}_1 & \overline{k}_1\overline{k}_2 \\ k_1 & |k_1|^2 & |k_1|^2\overline{k}_2 \\ k_1k_2 & |k_1|^2k_2 & |k_1|^2|k_2|^2 \end{pmatrix}$$

where $a_{11} = \dfrac{\lambda}{1 + |k_1|^2 + |k_1|^2|k_2|^2}$. Any parameter which has the same value as the modulus of $k_1$ and $k_2$ generates the $3 \times 3$ singular Hermitian matrix. Suppose $\lambda = 3, k_1 = 2i, \overline{k}_1 = -2i, k_2 = 1 + i, \overline{k}_2 = 1 - i$, we have $a_{11} = \dfrac{3}{13}$ and

$$A_{(3,1)} = \frac{3}{13}\begin{pmatrix} 1 & -2i & -2-2i \\ 2i & 4 & 4-4i \\ -2+2i & 4+4i & 8 \end{pmatrix}.$$

Example: For $n = 4, r = 1$. Given $k_1, k_2$ and $k_3$ we obtain the following singular Hermitian matrix:

$$A_{(4,1)} = a_{11}\begin{pmatrix} 1 & \overline{k}_1 & \overline{k}_1\overline{k}_2 & \overline{k}_1\overline{k}_2\overline{k}_3 \\ k_1 & |k_1|^2 & |k_1|^2\overline{k}_2 & |k_1|^2\overline{k}_2\overline{k}_3 \\ k_1k_2 & |k_1|^2k_2 & |k_1|^2|k_2|^2 & |k_1|^2|k_2|^2\overline{k}_3 \\ k_1k_2k_3 & |k_1|^2k_2k_3 & |k_1|^2|k_2|^2k_3 & |k_1|^2|k_2|^2|k_3|^2 \end{pmatrix}.$$

In this case $a_{11} = \dfrac{\lambda}{1 + |k_1|^2 + |k_1|^2|k_2|^2 + |k_1|^2|k_2|^2|k_3|^2}$. When $\lambda = 2, k_1 = 2i, k_2 = 2 + i, k_3 = i$, we have $a_{11} = \dfrac{2}{45}$ and hence

$$A_{(4,1)} = \frac{2}{45}\begin{pmatrix} 1 & -2i & 2-4i & -4+2i \\ 2i & 4 & 8-4i & -4-8i \\ -2+4i & 8+4i & 40 & -20i \\ -4-2i & -4+8i & 20i & 20 \end{pmatrix}.$$

**Proposition 2** : If the row dependence relations for a Hermitian or anti-Hermitian matrix of rank 1 are specified as follows $R_i = k_{i-1}R_1, i = 2, 3, 4, ..., n-1$ where $R_i$ is the $ith$ row and each $k_i$ is a non-zero scalar. The matrix can be generated from its non-zero eigenvalue $\lambda$:

$$A_{(n,1)} = a_{11} \begin{pmatrix} 1 & \overline{k}_1 & \overline{k}_1\overline{k}_2 & \cdots & \overline{k}_1 \cdots \overline{k}_{n-1} \\ k_1 & |k_1|^2 & |k_1|^2\overline{k}_2 & \cdots & |k_1|^2\overline{k}_2 \cdots \overline{k}_{n-1} \\ k_1k_2 & |k_1|^2k_2 & |k_1|^2|k_2|^2 & \cdots & \cdots \\ \vdots & \vdots & \vdots & & \vdots \\ k_1 \cdots k_{n-1} & |k_1|^2 \cdots k_{n-1} & |k_1|^2|k_2|^2k_3 \cdots k_{n-1} & \cdots & |k_1|^2 \cdots |k_{n-1}|^2 \end{pmatrix} \tag{1}$$

where

$$a_{11} = \frac{\lambda}{1 + |k_1|^2 + |k_1|^2|k_2|^2 + \cdots + |k_1|^2 \times \cdots \times |k_{n-1}|^2}.$$

# 3 Main Results

We now consider the IEP for $n \times n$ singular Hermitian matrices of rank 2. $A_{(3,2)}$ is the form:

$$A_{(3,2)} = \begin{pmatrix} a_{11} & \overline{k}a_{11} & \overline{a}_{13} \\ ka_{11} & |k|^2a_{11} & k\overline{a}_{13} \\ a_{13} & \overline{k}a_{13} & a_{33} \end{pmatrix}.$$

Here, $\text{tr}(A_{(3,2)}) = \lambda_1 + \lambda_2 = a_{11} + |k|^2a_{11} + a_{33} = a_{11}(1 + |k|^2) + a_{33}$. But $\lambda_1\lambda_2 = a_{11}(1 + |k|^2)a_{33} \implies a_{33} = \dfrac{\lambda_1\lambda_2}{a_{11}(1 + |k|)^2}$. Thus

$$\lambda_1 + \lambda_2 = a_{11}(1 + |k|)^2 + \frac{\lambda_1\lambda_2}{a_{11}(1 + |k|)^2}$$

which implies

$$a_{11}^2(1 + |k|^2)^2 - a_{11}(1 + |k|^2)(\lambda_1 + \lambda_2) + \lambda_1\lambda_2 = 0$$

which yields $a_{11} = \dfrac{\lambda_1}{1+|k|^2}$ and $\lambda_2 = a_{33}$. Therefore $a_{13}$ becomes a free variable. When $\lambda_1 = 1, \lambda_2 = 2, k = 3i$ and $a_{13} = 2+i$ for example, we obtain the following singular Hermitian matrix:

$$A_{(3,2)} = \begin{pmatrix} \frac{1}{10} & \frac{-3i}{10} & 2-i \\ \frac{3i}{10} & \frac{9}{10} & 3+6i \\ 2+i & 3-6i & 2 \end{pmatrix}.$$

We illustrate the results for $4 \times 4$ singular Hermitian matrices of rank 2. $A_{(4,2)}$ is of the form:

$$A_{(4,2)} = \begin{pmatrix} a_{11} & \overline{k}_1 a_{11} & \overline{k}_1\overline{k}_2 a_{11} & \overline{a}_{14} \\ k_1 a_{11} & |k_1|^2 a_{11} & |k_1|^2\overline{k}_2 a_{11} & k_1\overline{a}_{14} \\ k_1k_2a_{11} & |k_1|^2k_2a_{11} & |k_1|^2|k_2|^2a_{11} & k_1k_2\overline{a}_{14} \\ a_{14} & \overline{k}_1 a_{14} & \overline{k}_1\overline{k}_3 a_{14} & a_{44} \end{pmatrix}.$$

Then $\operatorname{tr}(A_{(4,2)}) = \lambda_1 + \lambda_2 = a_{11}(1+|k_1|^2+|k_1|^2|k_2|^2) + a_{44}$. This implies $a_{11} = \dfrac{\lambda_1}{1+|k_1|^2+|k_1|^2|k_2|^2}, \lambda_2 = a_{44}$ and $a_{14}$ becomes a free variable.

Numerical example, for $\lambda_1 = 3, \lambda_2 = 5, k_1 = 2i, k_2 = 1+2i$ and $a_{14} = 1+i$, we obtain the singular Hermitian matrix below:

$$A_{(4,2)} = \begin{pmatrix} \frac{3}{25} & \frac{-6i}{25} & \frac{-12-6i}{25} & 1-i \\ \frac{6i}{25} & \frac{12}{25} & \frac{12-24i}{25} & 2+2i \\ \frac{-12+6i}{25} & \frac{12+24i}{25} & \frac{60}{25} & -2+6i \\ 1+i & 2-2i & -2-6i & 5 \end{pmatrix}.$$

In general, the solution of the IEP for $A_{(n,r)}$ leads to the solution of an $r$th degree polynomial equation in $a_{11}$ of the form:

$$0 = a_{11}^r(1+|k_1|^2+\cdots+|k_{n-r}|^2)^r - (\sum_{i=1}^{r}\lambda_i)(1+|k_1|^2+\cdots+|k_{n-r}|^2)a_{11}^{r-1}$$
$$\sum_{k=1}^{r}(\prod_{i=k}^{k+1}\lambda_i)(1+|k_1|^2+\cdots+|k_{n-r}|^2)a_{11}^{r-2}$$
$$-\sum_{k=1}^{r}(\prod_{i=k}^{k+2}\lambda_i)(1+|k_1|^2+\cdots+|k_{n-r}|^2)a_{11}^{r-3}+\cdots-(\prod_{i=1}^{r}\lambda_i).$$

We generalise the method above in the following two theorems, first an $n\times n$ singular Hermitian matrix of rank 2 and the of rank r, where $2\le r<n$.

**Theorem 3.1.** *Given the spectrum and the row multipliers $k_i, i=1,...,n-2$, the inverse eigenvalue problem for an $n\times n$ singular Hermitian matrix of rank 2 is solvable.*

*Proof.* Given the spectrum $\Lambda_n=\{\lambda_1,\lambda_2,...,\lambda_n\}$, since the rank of $\Lambda_2=2$, it follows from our notation above that $\lambda_1\neq 0\neq\lambda_2$ and $\lambda_i=0$, for $i=3,4,...,n$. Let $k_i, i=1,2,...,k_{n-2}$ be row multiples. Letting

$$A_{(n,2)} = \begin{pmatrix} a_{11} & \overline{k}_1a_{11} & \overline{k}_1\overline{k}_2a_{11} & \cdots & \overline{k}_1\overline{k}_2\cdots\overline{k}_{n-2}a_{11} & \overline{a}_{1n} \\ k_1a_{11} & |k_1|^2a_{11} & |k_1|^2\overline{k}_2a_{11} & \cdots & |k_1|^2\overline{k}_2\cdots\overline{k}_{n-2}a_{11} & k_1\overline{a}_{1n} \\ k_1k_2a_{11} & |k_1|^2k_2a_{11} & |k_1|^2|k_2|^2a_{11} & \cdots & |k_1|^2|k_2|^2k_3\cdots\overline{k}_{n-2}a_{11} & k_1k_2\overline{a}_{1n} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{1n} & \overline{k}_1a_{1n} & \overline{k}_1\overline{k}_2a_{1n} & \cdots & \overline{k}_1\overline{k}_2\cdots\overline{k}_{n-2}a_{1n} & a_{nn} \end{pmatrix} \quad (2)$$

Then

$$\mathrm{tr}(A_{(n,2)}) = \lambda_1+\lambda_2 = a_{11}(1+|k_1|^2+|k_1|^2|k_2|^2+\cdots+|k_1|^2\times\cdots\times|k_{n-2}|^2)+a_{nn}.$$

Hence

$$a_{11} = \frac{\lambda_1}{1+|k_1|^2+|k_1|^2|k_2|^2+\cdots+|k_1|^2\times\cdots\times|k_{n-2}|^2},$$

$\lambda_2=a_{nn}$ and $a_{1n}$ becomes a free variable. The result follows by induction on $n$.

We consider the inverse eigenvalue problem for $n \times n$ singular Hermitian matrices of rank 3. $A_{(4,3)}$ is of the form

$$A_{(4,3)} = \begin{pmatrix} a_{11} & \overline{k}a_{11} & \overline{a}_{13} & \overline{a}_{14} \\ ka_{11} & |k|^2 a_{11} & k\overline{a}_{13} & k\overline{a}_{14} \\ a_{13} & \overline{k}a_{13} & a_{33} & \overline{a}_{34} \\ a_{14} & \overline{k}a_{14} & a_{34} & a_{44} \end{pmatrix} \qquad (3)$$

Here, $\mathrm{tr}(A_{(4,3)}) = \lambda_1 + \lambda_2 + \lambda_3 = a_{11}(1+|k|^2) + a_{33} + a_{44}$. Using equation (1) $A_{(4,3)}$ leads to the following cubic equation in $a_{11}$ where $\lambda_1, \lambda_2$ and $\lambda_3$ are non-zero members of the spectrum.

$$a_{11}^3(1+|k|^2)^3 - a_{11}^2(1+|k|^2)^3(\lambda_1+\lambda_2+\lambda_3) + a_{11}(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3) - \lambda_1\lambda_2\lambda_3 = 0.$$

Solving the above cubic equation we obtain the following roots; $\lambda_1 = a_{11}(1+|k|^2) \implies a_{11} = \dfrac{\lambda_1}{1+|k|^2}, \lambda_2 = a_{33}$ and $\lambda_3 = a_{44}$, where $a_{13}, a_{14}$ and $a_{34}$ are free variables. For instance, when $\lambda_1 = 2, \lambda_2 = -1, \lambda_3 = 5, k = -i, a_{13} = 2+i, a_{14} = 2i$ and $a_{34} = 1-3i$ we have

$$A_{(4,3)} = \begin{pmatrix} 1 & i & 2-i & -2i \\ -i & 1 & -1-2i & -2 \\ 2+i & -1+2i & -1 & 1+3i \\ 2i & -2 & 1-3i & 5 \end{pmatrix}.$$

Finally, we present $5 \times 5$ singular Hermitian matrix of rank 3. Using the same method, we obtain the following equation in $a_{11}$ where $\lambda_1, \lambda_2$ and $\lambda_3$ are non-zero members of the spectrum.

$$\begin{aligned} 0 =& a_{11}^3(1+|k_1|^2+|k_1|^2|k_2|^2)^3 - a_{11}^2(1+|k_1|^2+|k_1|^2|k_2|^2)^2(\lambda_1+\lambda_2+\lambda_3) \\ &+ a_{11}(1+|k_1|^2+|k_1|^2|k_2|^2)(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3) - \lambda_1\lambda_2\lambda_3. \end{aligned}$$

Factoring the above equation gives the following results:

$a_{11} = \dfrac{\lambda_1}{1+|k_1|^2+|k_1|^2|k_2|^2}, \lambda_2 = a_{44}$ and $\lambda_3 = a_{55}$. The free variables are $a_{14}, a_{15}$ and $a_{45}$.

Example: Let $\lambda_1 = 13, \lambda_2 = -3, \lambda_3 = 5, k_1 = 2i, k_2 = 1+i, a_{14} = 4, a_{15} = i$ and $a_{54} = 3 - i$, we get the following $5 \times 5$ singular Hermitian matrix of rank 3:

$$A_{(5,3)} = \begin{pmatrix} 1 & -2i & -2(1+i) & 4 & -i \\ 2i & 4 & 4(1-i) & 8i & 2 \\ -2(1-i) & 4(1+i) & 8 & -8(1-i) & 2(1+i) \\ 4 & -8i & -8(1+i) & -3 & 3+i \\ i & 2 & 2(1-i) & 3-i & 5 \end{pmatrix}.$$

**Theorem 3.2.** *The inverse eigenvalue problem for an $n \times n$ singular Hermitian matrix of rank $r$ is solvable provided that $n$ - $r$ arbitrary parameters are described.*

# 4 Conclusion and recommendation

We found in this study that when the eigenvalues and some parameters are given the inverse eigenvalue problem for $n \times n$ singular Hermitian matrices of rank 2 and 3 are solvable. To illustrate the results, numerical examples were provided.

Finally, we recommend that singular Hermitian matrices of rank $\geq 4$ could be studied for future research work.

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