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Common Fixed Point for Weakly Compatible Mappings for Type (A) in Metric Space

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Abstract

In this paper, we prove common fixed point theorems for weak compatible mappings of type (A) in metric space under ϕ-contractive conditions. Our results extend some of the known results due to Prasad, Jungck-Murthy and Cho and Rani, Kumar and Chugh.

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1 Introduction

In 1922, Banach proved a common fixed-point theorem, which ensures under

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appropriate conditions, the existence and uniqueness of a fixed point. This result of Banach is known as Banach's fixed point theorem or Banach contraction principle. This theorem provides a technique for solving a variety of applied problems in mathematical sciences and engineering. Many authors have extended generalized and improved Banach's fixed point theorem in different ways.

Jungck [2] proved a common fixed point theorem for commuting maps generalizing the Banach's fixed point theorem, which states the that "Let (X, d) be a complete metric space. If T satisfies $d(Tx, Ty) \leqq kd(x, y)$ for each $x, y \in X$ where $0 \leqq k < 1$, then T has a unique fixed point in X. This result was further generalized and extended in various ways by many authors. On the other hand, S. Sessa [8] coined the notion of weak commutativity and proved common fixed point theorems for these mappings. Further, Jungck [2] introduced a more generalized commutativity, the so called compatibility, which is more general than that of weak commutativity. This concept has been useful for obtaining fixed point theorems for compatible mappings satisfying contractive type conditions and assuming continuity of at least one of mappings. It has been known from the paper of Kannan [5] that there exist maps that have a discontinuity in the domain but which have fixed points, moreover, the maps involved in every case were continuous at the fixed point. This paper was a genesis for a multitude of fixed point papers over the next two decades.

Jungck-Murthy and Cho [4] introduced the concept of compatible mappings of type (A) in metric space and improved the results of Pathak and Parsad. We use the idea of weak compatible mappings of type (A) in metric space as used by Pathak-Kang-Baek [11] in menger and 2-metric spaces respectively which is equivalent to concept of compatible and compatible mappings of type (A) under some conditions. The intent of this paper is to generalize the results of Jungck Murthy and Cho [4], Prasad [12] and Rani, Kumar and Chugh [13].

2 Preliminary Notes

Definition 2.1. The pair A,S is said to be compatible if $\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some t in X.

Definition 2.2. The mappings A and S are said to be compatible of type (A) if $\lim_{n\to\infty} d(ASx_n, SSx_n) = 0$ and $\lim_{n\to\infty} d(SAx_n, AAx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ax_n = t$ for some t in X.

Definition 2.3. The pair A, S is said to be weak compatible of type (A) if

$$\lim_{n\to\infty} d(ASx_n, SSx_n) \le \lim_{n\to\infty} d(SAx_n, SSx_n)$$

and

$$\lim_{n\to\infty} d(SAx_n, AAx_n) \le \lim_{n\to\infty} d(ASx_n, AAx_n)$$

whenever $\{x_n\}$ is a sequence in X that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ax_n = t$ for some $t \in X$.

Proposition 2.1. [11] Let S and T be weak compatible mappings of type (A) from a metric space (X, d) into itself. If one of S and T is continuous, then S and T are compatible.

Proposition 2.2. [11] Let S and T be continuous mappings from a metric space (X, d) into itself. Then

i) S and T are compatible of type (A) if and only if they are weak compatible of type (A).

ii) S and T are compatible if and only if they are weak compatible of type (A).

3 Main Results

In 1998 Chugh and Rani *et al.* [12] prove the following fixed point theorem

Let A, B, P and Q be four self-maps of complete metric space (X, d) such that

(3.1) $A(X) \subset Q(X)$ and $B(X) \subset P(X)$.

(3.2) $(d(Ax, By))^2 \leq \phi(d(Px, Ax)\, d(Qy, By), d(Px, By)d(Qy, Ax),$
$d(Px, Ax)\, d(Px, By), d(Qy, Ax)\, d(Qy, By)\, [d(Px, Qy)]^2,$
$d(Px, Ax)\, d(Qy, Ax), d(Qy, by)\, d(Px, By),$
$d(Px, Qy)\, d(Px, Ax), d(Px, Qy)\, d(Qy, Ax),$
$d(Px, Qy)\, d(Qy, By)\}$ for all $x, y \in X$

where the function $\phi : [0, \infty)^{10} \to [0, \infty)$ satisfies the conditions:

(a) ϕ is upper semi-continuous and non-decreasing in each coordinate variable.

(b) The function $\psi : R^+ \to R^+$ is non-decreasing and satisfies

$$\psi(t) = \max\{\phi(t, 0, 2t, 0, t, 0, 2t, t, 0, t), \phi(t, 0, 0, 2t, t, 2t, 0, t, 2t, t), \phi(0, t, 0, 0, t, 0, 0, 0, t, 0\} < t \text{ for some } t > 0.$$

Inductively, we can define a sequence $\{y_n\}$ in X such that

(3.3) $y_{2n-1} = Qx_{2n-1} = Ax_{2n-2}$ and $y_{2n} = Bx_{2n-1} = Px_{2n}$ for $n = 1, 2, 3,\ldots$

(3.4) pairs (A, P) and (B, Q) are compatible of type A. Then A, B, S and T have a unique common fixed point.

Before proving our main result we need the following results.

Lemma 3.1[13] Let A, B, P and Q be four mappings from a metric space (X, d) into itself satisfying the conditions (3.1), (3.2) and (3.3). Then $\{y_n\}$ is a Cauchy sequence in X defined by (3.3).

Lemma 3.2. Let A, B, P and Q be four mappings from a metric space (X,d) into itself satisfying the conditions (3.1), (3.2) and the following

(3.5) $P(X) \cap Q(X)$ is a complete subspace of X.

Then A and P have a coincidence point in X, and B and Q have also a coincidence point in X.

Proof. By Lemma (3.1) the sequence $\{y_n\}$ defined by (3.3) is a Cauchy sequence is $P(X) \cap Q(X)$. Since $P(X) \cap Q(X)$ is a complete subspace of X, so $\{y_n\}$ converges to a point α (say) in $P(X) \cap Q(X)$. On the other hand, since the subsequences $\{y_{2n}\}$ and $\{y_{2n-1}\}$ of $\{y_n\}$ are also Cauchy sequences in $P(X) \cap Q(X)$. So, they also converge to the limit α. Hence there exist points α_1 and α_2 in X such that $P\alpha_1 = \alpha$ and $Q\alpha_2 = \alpha$, respectively.

By (3.2), we have

$$[d(A\alpha, y_{2n+2})]^2 = [d(A\alpha_1, Bx_{2n+1})]^2$$

$$\leq \phi\{d(P\alpha_1, Q\alpha_1)\, d(Qx_{2n+1},Bx_{2n+1}),\, d(P\alpha_1, Bx_{2n+1})\, d(Qx_{2n+1}, A\alpha_1),$$

$$\alpha(P\alpha_1, A\alpha_1)d(P\alpha_1, Bx_{2n+1}),\, d(Qx_{2n+1}, A\alpha_1)\, d(Qx_{2n+1}, Bx_{2n+1}),$$

$$[d(P\alpha_1, Qx_{2n+1})]^2,\, d(P\alpha, A\alpha_1)\, d(Qx_{2n+1}, A\alpha_1),$$

$$d(Tx_{2n+1}, Bx_{2n+1})\, d(P\alpha_1, Bx_{2n+1}),\, d(P\alpha_1, Qx_{2n+1})\, d(P\alpha_1, A\alpha_1),$$

$$d(P\alpha_1, Qx_{2n+1})\, d(Qx_{2n+1}, A\alpha_1),\, d(P\alpha_1, Qx_{2n+1})\, d(Qx_{2n+1}, Bx_{2n+1}),$$

$$[d(A\alpha_1, y_{2n+2})]^2 \leq \phi\{d(P\alpha_1, A\alpha_1)\, d(y_{2n+1}, y_{2n+2}),\, d(P\alpha_1, y_{2n+2})\, d(y_{2n+1}, A\alpha_1),$$

$$d(P\alpha_1, A\alpha_1)\, d(P\alpha_1, y_{2n+2}),\, d(y_{2n+1}, Au)\, d(y_{2n+1}, y_{2n+2}),$$

$$[d(P\alpha_1, y_{2n+1})]^2,\, d(P\alpha_1, A\alpha_1)\, d(y_{2n+1}, A\alpha_1),$$

$$d(y_{2n+1}, y_{2n+2})\, d(P\alpha_1, y_{2n+2}),\, d(P\alpha_1, y_{2n+1})\, d(P\alpha_1, A\alpha_1),$$

$$d(P\alpha_1, y_{2n+1})\, d(y_{2n+1}, A\alpha_1),\, d(P\alpha_1, y_{2n+1})\, d(y_{2n+1}, y_{2n+2})\}.$$

Since ϕ is upper semi-continuous and proceeding lim as $n\to\infty$,

$$d[(A\alpha_1,w)]^2 \leq \phi[(0,d(P\alpha_1,\alpha)d(\alpha,A\alpha_1),d(P\alpha_1,A\alpha_1)d(P\alpha_1,\alpha),d(\alpha,A\alpha_1)\, d(\alpha, \alpha)$$

$$[d(P\alpha_1,\alpha)]^2,d(P\alpha_1,A\alpha_1)d(\alpha,A\alpha_1),\, d(\alpha, \alpha)\, d(P\alpha_1, \alpha),\, d(P\alpha_1, \alpha)\, d(P\alpha_1, A\alpha_1),$$

$$d(P\alpha_1, \alpha)\, d(\alpha, A\alpha_1),\, d(P\alpha_1, \alpha)\, d(\alpha, \alpha)\}]$$

$$[d(A\alpha_1, \alpha)]^2 \leq \phi(0, 0, 0, 0, 0, [d(\alpha, A\alpha_1)]^2, 0, 0, 0, 0)$$

$$\leq \phi[d(A\alpha_1, \alpha)]^2 < [d(A\alpha_1, \alpha)]^2, \text{ a contradiction.}$$

Therefore, $A\alpha_1 = P\alpha = \alpha$, i.e., α_1 is a coincidence point of A and P. Similarly, we can show that α_2 is also coincidence point of B and T. □

Lemma 3.3[11] Let A and P be weak compatible mappings of type (A) from a metric space (X, d) into itself. If $A\alpha_1 = P\alpha_1$ for some $\alpha_1 \in X$, then

$$AP\alpha_1 = AA\alpha_1 = PP\alpha_1 = PA\alpha_1$$

Now we prove the following Theorem.

Theorem 3.1. Let A, B, P and Q be mappings from a metric space (X, d) into itself satisfying the conditions (3.1), (3.2), (3.4) and the following

(3.6) the pairs (A, P) and (B, Q) are weak compatible of type (A). Then A, B, P and Q have a unique common fixed point in X.

Proof. By Lemma (3.2), there exist points α_1, α_2 in X such that $A\alpha_1 = P\alpha_1 = \alpha$ and $B\alpha_2 = Q\alpha_2 = \alpha$, respectively. Since A and P are weak compatible of type (A), then by Lemma (3.3) we have $AP\alpha_1 = AA\alpha_1 = PP\alpha_1 = PA\alpha_1$, which implies that $A\alpha = P\alpha$. Similarly, $B\alpha = Q\alpha$. Now, we prove that $A\alpha = \alpha$. If $A\alpha \neq \alpha$, then by (3.2), we have

$$\begin{aligned}
[d(A\alpha, Bx_{2n+1})]^2 = [d(A\alpha, y_{2n+2})]^2 \\
&\leq \phi\{d(P\alpha, A\alpha)\, d(y_{2n+1}, y_{2n+2}), \\
&\quad d(P\alpha, y_{2n+2})\, d(y_{2n+1}, A\alpha), d(P\alpha, A\alpha)\, d(P\alpha, y_{2n+2}), \\
&\quad d(y_{2n+1}, A\alpha)\, d(y_{2n+1}, y_{2n+2}), [d(P\alpha, y_{2n+1})]^2, \\
&\quad d(P\alpha, A\alpha)\, d(y_{2n+1}, A\alpha), d(y_{2n+1}, y_{2n+2})\, (P\alpha, y_{2n+2}), \\
&\quad d(P\alpha, y_{2n+1})\, d(P\alpha, A\alpha), d(P\alpha, y_{2n+1})\, d(y_{2n+1}, A\alpha), \\
&\quad d(P\alpha, y_{2n+1})\, d(y_{2n+1}, y_{2n+2})\}.
\end{aligned}$$

Since ϕ is upper semi-continuous and proceeding limit as $n \to \infty$, it follows

$$[d(A\alpha, \alpha)]^2 \leq \phi\{0, d(A\alpha, \alpha)^2, 0, 0, [d(A\alpha, \alpha)]^2, 0, 0, 0, [d(A\alpha, \alpha)]^2, 0]$$

$< [d(A\alpha, \alpha)]^2$,which is a contradiction; hence $\alpha = \alpha = P\alpha$. Similarly, we have $B\alpha = \alpha = Q\alpha$, i.e., α is the common fixed point of A, B, P and Q.

Uniqueness. Let $z \neq \alpha$ be another common fixed point of A, B, P and Q. We have

$$\begin{aligned}
[d(\alpha, z)]^2 = [d(A\alpha, Bz)]^2 \\
&\leq \phi[d(P\alpha, A\alpha)\, d(Qz, Bz), d(P\alpha, Bz)\, d(Qz, A\alpha), d(P\alpha, A\alpha)\, d(P\alpha, Bz), \\
&\quad d(Qz, A\alpha)\, d(Qz, Bz), [d(P\alpha, Qz)]^2, d(P\alpha, A\alpha)\, d(Qz, A\alpha),
\end{aligned}$$

$d(Qz, Bz)\, d(P\alpha, Bz), d(P\alpha, Qz)\, d(P\alpha, A\alpha), d(P\alpha, Qz)\, d(Qz, A\alpha),$

$d(P\alpha, Qz)\, d(Qz, Bz)\}$

$\leq \phi\, (0, [d(\alpha, z)]^2, 0, 0, [d(\alpha, z)]^2, 0, 0, 0, [d(\alpha, z)]^2\, [d(\alpha, z)]^2\},$

which is a contradiction. Thus $z = \alpha$. □

Theorem 3.2. Let A, B, P and Q be mappings from a complete metric space (X, d) into itself satisfying the conditions

(3.7) $A^a(X) \subset Q^t(X)$, $B^b(X) \subset P^s(X)$ for some positive integer a, b, s and t.

(3.8) $[d(A^a x, B^b y)]^2 \leq \phi\{d(P^s x, A^a x)\, d(Q^t y, B^b y), d(P^s x, B^b y)\, d(Q^t y, A^a x),$

$d(P^s x, A^a x)\, d(P^s x, B^b y), d(Q^t y, A^a x)\, d(Q^t y, B^b y),$

$[d(P^s, Q^t y)]^2, d(P^s x, A^a x)\, d(Q^t y, A^a x),$

$d(Q^t y, B^b y)\, d(P^s x, B^b y), d(P^s x, Q^t y)\, d(P^s x, A^a x),$

$d(P^s x, Q^t y)\, d(Q^t y, A^a x), d(P^s x, Q^t y)\, d(Q^t y, B^b y)\}$

for all $x, y \in X$.

(3.9) A, B commute with P and Q respectively. Then A, B, P and Q have a unique common fixed point in X.

Proof. Since A and B commute with P and Q, so A^a and B^b also commute with P^s and Q^t respectively. Thus by Theorem I, there exists a unique z in X such that

$$z = A^a z = B^b z = P^s z = Q^t z$$

Now, $Az = A(A^a z) = A^a(Az)$ and $Az = A(P^s z) = P^s(Az)$.

Therefore, Az is a common fixed point of A^a and P^s.

$Bz = B(B^b z) = B^b(Bz)$ and $Bz = B(Q^t z) = Q^t(Bz)$. Also, Bz is a common fixed point of B^b and Q^t.

Considering $x = Az$ and $y = Bz$ in (3.8), we have

$[d(Az, Bz)]^2$

$= [d(A^a(Az), B^b(Bz))]^2$

$\leq \phi[d(P^s(Az), A^a(Az))\, d(Q^t(Bz), B^b(Bz)),$

$d(P^s(Az), B^b(Bz))\, d(Q^t(Bz), A^a(Az))\, d(P^s(Az), A^a(Az)),$

$d(P^s(Az), B^b(Bz)), d(Q^t(Bz), A^a(Az))\, d(Q^t(Bz), B^b(Bz)),$

$[d(P^s(Az), Q^t(Bz))]^2$, $d(P^s(Az), A^a(Az))\, d(Q^t(Bz), A^a(Az))$,
$d(Q^t(Bz), B^b(Bz))\, d(P^s(Az), B^b(Bz))$, $d(P^s(Az), Q^t(Bz))\, d(Q^s(Az), A^a(Az))$,
$d(S^s(Az), T^t(Bz))\, (T^t(Bz), A^a(Az))$, $d(S^s(Az), T^t(Bz))\, d(T^t(Bz), B^b(Bz))$,

$= \phi[(d(Az, Az)\, d(Bz, Bz), d(Az, Bz)\, d(Bz, Az), d(Az, Az)\, d(Az, Bz)]$,
$d(Bz, Az)\, d(Bz, Bz), [d(Az, Bz)]^2, d(Az, Az)\, d(Bz, Az)$,
$d(Bz, Bz)\, d(Az, Bz), d(Az, Bz)\, d(Az, Az), d(Az, Bz)\, d(Bz, Az)$,

$= \phi(0, [d(Az, Bz)]^2, 0, 0, [d(Az, Bz)]^2, 0,0,0, [d(Az, Bz)]^2, 0)$

$< [d(Az, Bz)]^2$,which is a contradiction. This implies that $Az = Bz$ and it is the common fixed point of A^a and P^s. Also Tz is the common fixed point of B^b and Q^t. Putting $x = Pz$ and $y = Qz$ in (6), we have $Pz = Qz$ and hence it is the common fixed point of A^a, B^b, P^s and Q^t. Uniqueness of z in X shows that $z = Az = Bz = Pz = Qz$. □

Theorem 3.3. Let $\{A_n\}$, $\{B_n\}$, $\{P_n\}$ and $\{Q_n\}$ be sequences of mappings from a complete metric space (X, d) into itself such that $\{A_n\}$, $\{B_n\}$, $\{P_n\}$ and $\{Q_n\}$ converge uniformly to self-mappings A, B, P and Q on X, respectively.

Suppose that for $n = 1, 2,\ldots$, x_n is common fixed point of A_n, B_n, P_n and Q_n. Further let self-mappings A, B, P and Q on X satisfy (3.1), (3.2) and (3.6). If x is a common fixed point of A, B, P and Q, $\sup \{d(x_n, x)\} < \infty$, then $x_n \to x$ as $n \to \infty$.

Theorem 3.4. Let $\{A_n\}$, $\{B_n\}$, $\{P_n\}$ and $\{Q_n\}$ be sequences of mappings from a complete metric space (X, d) into itself such that, for $n = 1, 2, 3,\ldots$,

(3.10) $A_n(X) \subset Q_n(X)$ and $B_n(X) \subset P_n(X)$

(3.11) The pairs A_n, P_n and B_n, Q_n are weak compatible of type (A).

(3.12) There exists a function ψ satisfying (a) and (b) such that

$[d(A_nx, B_ny)]^2 \le \psi(d(P_nx, A_nx)\, d(Q_ny, Y_ny), d(P_nx, B_ny)\, d(Q_ny, A_nx)$
$d(P_nx, A_nx)\, d(P_nx, B_ny), d(Q_ny, A_nx)\, d(Q_ny, B_ny), [d(P_nx, Q_ny)]^2$,
$d(P_nx, A_nx)\, d(Q_ny, A_nx), d(Q_ny, B_ny)\, d(P_nx, B_ny)$,
$d(P_nx, Q_ny)\, d(P_nx, A_nx), d(P_nx, Q_ny)\, d(Q_ny, A_nx)$,

$d(P_nx, Q_ny)\, d(Q_ny, B_ny)\}$

for all x, y $\in$X.

If $\{A_n\}$, $\{B_n\}$, $\{P_n\}$ and $\{Q_n\}$ converge uniformly to self-mappings A, B, P and Q on X, respectively, then A, B, P and Q satisfy the conditions (3.1), (3.2) and (3.5). Further, the sequence $\{x_n\}$ of unique common fixed points x_n of A_n, B_n, P_n and Q_n converges to a unique common fixed point x of A, B, P and Q, if sup $\{d(x_n, x)\} < \infty$.

Example 3.1. Let X = [0, 1) with usual metric on real line; define $d(x, y) = |(x - y)|$. Define P, Q : (X, d) $\to$ (X, d) as follows:

$$P(x) = \begin{cases} x & \text{if} \quad x \in [0,1/2) \\ 1 & \text{if} \quad x \in [1/2,1] \end{cases}$$

$$Q(x) = \begin{cases} 1-x & \text{if} \quad x \in [0,1/2) \\ 1 & \text{if} \quad x \in [1/2, 1] \end{cases}$$

then P and Q are not continuous at t = 1/2. Now we observe that P and Q are not compatible but they are compatible of type (A).

Suppose that $\{x_n\} \subseteq [0, 1]$ and Qx_n, $Px_n \to t$.

By definition of P and Q, $t \in [1/2, 1]$. Since P and Q agree on [1/2, 1] we need only to consider t = 1/2. So we can suppose that $x_n \to 1/2$ and that $x_n < 1/2$ for all n. Then $Q(x_n) = 1 - x_n \to 1/2$ from right and $Px_n = x_n \to 1/2$ from left. Thus, since $1 - x_n > ½$, for all n. $PQx_n = P(1 - x_n) = 1$ and, since $x_n < 1/2$,

and $\quad QPx_n = Qx_n = 1 - x_n \to ½$, $d(PQx_n, QPx_n) \to ½$. But

$$d(PQx_n, QQx_n) = |PQx_n - Qx_n| = |1 - Q(1 - x_n)| = 1 - 1 \to 0$$

$$d(QPx_n, PPx_n) = |QPx_n - PPx_n| = |1 - x_n - x_n| = |1 - 2x_n| \to 0$$

as $x_n \to 1/2$. Therefore P and Q are compatible mappings of type (A). But they are not compatible. Also by Proposition 1.2 every compatible of type (A) is weak compatible of type (A).

Example 3.2. Let X = R, the set of real numbers with usual metric d.

Define P, Q : R $\to$ R by

$$P(x) = \begin{cases} 1/x^2 & \text{if} \quad x \neq 0 \\ 2 & \text{if} \quad x = 0 \end{cases}$$

and
$$Q(x) = \begin{cases} 1/x^3 & \text{if} \quad x \neq 0 \\ 3 & \text{if} \quad x = 0 \end{cases}$$

Then P and Q are not continuous at x = 0. Define a sequence $\{x_n\} \subseteq R$ b $x_n = n^2$, n = 1, 2,…., then as $n\to\infty$, we obtain

$$Px_n = 1/n^4 \to 0,\ Qx_n = 1/n^6 \to 0$$

$$\lim_{n\to\infty} d(PQx_n, QPx_n) = \lim_{n\to\infty} |n^{12} - n^{12}| = 0$$

$$\lim_{n\to\infty} d(PPx_n, QQx_n) = \lim_{n\to\infty} |n^8 - n^{18}| = \infty$$

$$\lim_{n\to\infty} d(PQx_n, QQx_n) = \lim_{n\to\infty} |n^{12} - n^8| = \infty$$

and
$$\lim_{n\to\infty} d(QPx_n, PPx_n) = \lim_{n\to\infty} |n^{12} - n^8| = \infty$$

Hence P and Q are compatible but P and Q are not compatible of type (A).

Example 3.3. Let X = $[1, \infty)$ with usual metric on real line; define A, B, P and Q by $Ax = x^4$, $Bx = x^3$, $Px = 2x^8 - 1$ and $Qx = 2x^6 - 1$ for all $x\in[1, \infty)$. Then, clearly AX = BX = PX = QX = X.

Moreover,

$$|Px-Ax| = |2x^8 - 1 - x^4| = |(x^4-1)(2x^4 + 1)| \to 0 \quad \text{iff } x\to1, \text{ since } x \geq 1$$

$$|PAx - AAx| = |2x^{32} - 1 - x^{16}| = |(x^{16} - 1)(2x^{16} + 1)| \to 0 \text{ iff } x\to1 \text{ and}$$

$$|APx - PPx| = |(2x^8 - 1)^4 - 2(2x^8 - 1)^8 + 1|$$

$$= |[(2x^8 - 1)^4 - 1]\,[2(2x^8 - 1)^4 + 1]| \to 0 \text{ iff } x \to 1.$$

Similarly,

$$|Qx - Bx| = |2x^6 - 1 - x^3| = |(x^3 - 1)(2x^3 + 1)| \to 0 \quad \text{iff } x \to 1$$

$$|BQx - QQx| = |[(2x^6 - 1)^3 - 1]\,[2(2x^6 - 1)^3 + 1]\,| \to 0 \text{ iff } x \to 1$$

$$|QBx - BBx| = |2x^{18} - 1 - x^9| = |(x^9 - 1)(2x^9 + 1)| \to 0 \quad \text{iff } x \to 1$$

Therefore, A, P and B, Q are compatible of type (A) and hence by Proposition(3.2) are weak compatible of type (A).

Let us define the function ψ as

$$\psi(t_1, t_2, t_3, \ldots, t_{10}) = h \max \{t_1, t_2, t_3, \ldots, t_{10}\}$$

where $1/16 \le h < ½$, $t_i \in R^+$ for i = 1, 2,…, 14. Then ψ satisfies (a) and (b).

Furthermore, we obtain

$$\begin{aligned} |Px - Qy| &= |2x^8 - 1 - 2x^6 + 1| = |x^8 - y^6| \\ &= 2(x^4 + y^3)\, |x^4 - y^3| \\ &= 2.2\, |Ax - By| \end{aligned}$$

$$\begin{aligned} |Ax - By|^2 &\le 1/16\, |Px - Qy|^2 \\ &\le \psi(d(Px, Ax)d(Qy, By), d(Px, By)\, d(Qy, Ax), d(Px, Ax)\, d(Px, By) \ldots \\ &\qquad d(Qy, By)\, d(Ax, By)) \end{aligned}$$

so that the condition (3.2) is satisfied and one is the unique common fixed point of A, B, P and Q.

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