

# Evaluating the Goodness of Fit of Copulas on Equity Returns

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## Abstract

This paper examines copulas that best fits the equity returns. Using nine years data of daily returns of 30 representative stocks, this study finds that t copula unanimously dominates the goodness of fit criteria. The conclusion reveals the inappropriateness of using high-dimensional multivariate Gaussian distribution to model the dependence of asset returns, because the nested distribution underestimates the volatility and anomaly of asset performance. Furthermore, Gumbel, Clayton, and Frank copula do not capture the extreme value dependence among assets. The results suggest that the optimal procedure for Monte Carlo simulation of asset return is to fit the individual asset return marginal and model the dependence of asset trends through the copula.

**Mathematics Subject Classification:** 26D20

**Keywords:** copula; asset return; distribution; information criterion; Schwarz's inequality; Triangle inequality

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## 1 Introduction

This paper examines copulas that best fits the equity returns and suggest copula classes that should be adopted in modeling equity return dependence. Copulas as better solutions of modeling asset return dependence receive more attention in the recent uses in both academia and industry (Fernandez, 2008, Meucci, 2011). This paper thus attempts to provide empirical solution in terms of copula selection and examine the robustness of the suggestion of copula using different groups of equities at various dimensions.

Linear correlation carries major disadvantages that prevent its application to the Monte Carlo simulation. One of the vital disadvantages is that a market shock that drives returns of different assets move simultaneously to the other side of the expected return can inflate the correlation and disregard the different correlation nature of the assets at the tranquil period.

A recently prevalent approach that overcomes the disadvantages of linear correlation is to model the dependence of returns by copulas. A copula separates the randomness of one variable from the dependencies between it and other variables. Certain copulas can model the asymmetric distribution of tails by different level of dependencies at various market environments. This is the major advantage of using copulas to model and simulate interdependent variables compared to Cholesky decomposition, which is the current standard procedure in Monte Carlo simulations with asset return dependence incorporated.

A copula method models each variable separately and then measures the relations between variables. One of the benefits of this two-stage model is that the univariate probability distribution of the return of each different asset can be modelled differently by a specific distribution type of choice without affecting the dependencies among the assets. Such dependencies among the returns of assets can be described by a multivariate probability distribution function. Copula is thus the aforementioned multivariate distribution function that generates joint outcomes of the variables.

Strictly defined and consistent with Nelsen, (1999),  $\mathbf{C}: [0,1]^d \rightarrow [0,1]$  is a d-dimensional copula if  $\mathbf{C}$  is a joint distribution function of a d-dimensional random vector on the unit cube  $[0,1]^d$  with marginal following uniform distribution. More specifically,  $\mathbf{C}: [0,1]^d \rightarrow [0,1]$  is a d-dimensional copula if

$$\mathbf{C}(u_1, \dots, u_{i-1}, 0, u_{i+1}, \dots, u_d) = 0 \quad (1)$$

$$\mathbf{C}(1, \dots, u, 1, \dots, 1) = u \quad (2)$$

$\mathbf{C}$  is d-increasing, i.e., for each hyper-rectangle

$$\mathbf{B} = \prod_{i=1}^d [x_i, y_i] \subseteq [0,1]^d \quad (3)$$

the C-volume of  $\mathbf{B}$  is non-negative:

$$\int_{\mathbf{B}} d\mathbf{C}(u) = \sum_{\mathbf{Z} \in \times_{i=1}^d \{x_i, y_i\}} (-1)^{N(\mathbf{Z})} \mathbf{C}(\mathbf{Z}) \geq 0 \quad (4)$$

where the  $N(\mathbf{Z}) = \#\{k: z_k = x_k\}$

Two most frequently used copula families are elliptical copulas and Archimedean copulas. An elliptical copula is the copula corresponding to an elliptical distribution by the Sklar's theorem. Specifically, it is defined in the following Equation (5).

$$\mathcal{C}(u_1, \dots, u_p) = F[F_1^{-1}(u_1), \dots, F_p^{-1}(u_p)] \quad (5)$$

where  $F$  is the multivariate Cumulative Density Function of an elliptical distribution,  $F_i$  is the Cumulative Density Function of the  $i$ th margin and  $F_i^{-1}$  is its inverse function.

An Archimedean copula is set up through a generator  $\varphi$  as described in Equation (6).

$$\mathcal{C}(u_1, \dots, u_p) = \varphi^{-1}[\varphi(u_1) + \dots + \varphi(u_p)] \quad (6)$$

where  $\varphi^{-1}$  is the inverse of the generator  $\varphi$ . The generator, which determines a specific copula, must be a p-order monotonic function to make  $\mathcal{C}(u_1, \dots, u_p)$  qualify for the aforementioned three features of copula. The most common Archimedean copula classes are the one-parameter families, such as Clayton

copula (Clayton, 1978), Frank copula (Frank, 1979), and Gumbel copula (Gumbel, 1960), which are summarized in Table 1.

Table 1: Common Archimedean Copulas

Family	Parameter Space	Generator $\varphi$	Generator Inverse $\varphi^{-1}$
Clayton	$\alpha \geq 0$	$t^{-\alpha} - 1$	$(1 + s)^{-1/\alpha}$
Frank	$\alpha \geq 0$	$-\ln \frac{e^{-\alpha t} - 1}{e^{-\alpha} - 1}$	$-\alpha^{-1} \ln(1 + e^{-s}(e^{-\alpha} - 1))$
Gumbel	$\alpha \geq 1$	$(-\ln t)^\alpha$	$e^{-s^{1/\alpha}}$

According to Rachev, Stoyanov, and Fabozzi (2007) the most widely used copula is the Gaussian copula. Following Bouye et al. (2000), let  $\rho$  be a symmetric, positive definite matrix with  $\text{diag } \rho = 1$  and  $\Phi_\rho$  the standardized multivariate normal distribution with correlation matrix  $\rho$ . The multivariate Gaussian copula is

$$\mathbf{C}(u_1, \dots, u_n, \dots, u_N; \rho) = \Phi_\rho(\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_n), \dots, \Phi^{-1}(u_N)) \quad (7)$$

The corresponding density is

$$\mathbf{c}(u_1, \dots, u_n, \dots, u_N; \rho) = \frac{1}{|\rho|^{1/2}} \exp\left(-\frac{1}{2} \boldsymbol{\zeta}^T (\rho^{-1} - \mathbf{I}) \boldsymbol{\zeta}\right) \quad (8)$$

with  $\zeta_n = \Phi^{-1}(u_n)$ . The bivariate form, consistent with Schmidt (2006), is

$$\mathcal{C}_\delta^G(u_1, u_2) = \Phi_\delta\{\Phi^{-1}(u_1), \Phi^{-1}(u_2)\} \quad (9)$$

$$= \int_{-\infty}^{\Phi^{-1}(u_1)} \int_{-\infty}^{\Phi^{-1}(u_2)} \frac{1}{2\pi\sqrt{1-\delta^2}} \exp\left\{-\frac{(s^2 - 2\delta st + t^2)}{2(1-\delta^2)}\right\} ds dt$$

Gaussian copula allows generating joint symmetric dependence but it does not incorporate tail dependence. Specifically, Gaussian copula does not exhibit either lower or upper tail dependence unless the value of  $\delta$  is 1 (Fernandez, 2008).

However, assets returns may present extreme-value dependency in both tails. The wide use of the Gaussian copula model causes the dramatic increase in market participants and volumes, and may cause significant losses when the asset return is non-normal. Therefore, recent studies have focused on the Student's t-copula, for example, Mashal, Naldi, and Zeevi (2003).

Following Bouye et al. (2000), let  $\rho$  be a symmetric, positive definite matrix with  $\text{diag } \rho=1$  and  $T_{\rho,v}$  the standardized multivariate Student's distribution with  $v$  degrees of freedom and correlation matrix  $\rho$ . The multivariate Student's copula is

$$\mathbf{C}(u_1, \dots, u_n, \dots, u_N; \rho, v) = T_{\rho,v}(t_v^{-1}(u_1), \dots, t_v^{-1}(u_n), \dots, t_v^{-1}(u_N)) \quad (10)$$

with  $t_v^{-1}$  the inverse of the univariate Student's distribution. The corresponding density is

$$\mathbf{c}(u_1, \dots, u_n, \dots, u_N; \rho) = \frac{1}{|\rho|^{\frac{1}{2}}} \frac{\Gamma(\frac{v+N}{2}) [\Gamma(\frac{v}{2})]^N (1 + \frac{1}{v} \zeta^T \rho^{-1} \zeta)^{-\frac{v+N}{2}}}{[\Gamma(\frac{v+1}{2})]^N \Gamma(\frac{v}{2}) \prod_{n=1}^N (1 + \frac{\zeta_n^2}{v})^{-\frac{v+1}{2}}} \quad (11)$$

with  $\zeta_n = t_v^{-1}(u_n)$ .

Guo and Zhong (2015) document that due to the inherent instability, it is inappropriate to employ Pearson correlation coefficient to measure the volatility of a portfolio, such as

$$\sigma_p^2 = \sum_i \omega_i^2 \sigma_i^2 + \sum_i \sum_j w_i w_j \sigma_i \sigma_j \rho_{ij} \quad (12)$$

Guo and Zhong (2015) also document that for the similar reason, using Cholesky decomposition<sup>2</sup> in Monte Carlo simulations presents biased forecast

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<sup>2</sup> Specifically, the Cholesky decomposition of a positive-definite matrix  $\mathbb{A}$  is a decomposition of the form  $\mathbb{A} = \mathbb{L}\mathbb{L}^*$ , where  $\mathbb{L}$  is a lower triangular matrix with real and positive diagonal entries, and  $\mathbb{L}^*$  is the conjugate transpose of  $\mathbb{L}$ . Cholesky decomposition allows for separating dependent variables into independent variables with designated correlation pattern. The first step is to compute the correlation matrix and then decompose the matrix to obtain the lower-triangular  $\mathbb{L}$ . Applying this to a vector of uncorrelated samples,  $\vec{\mathbf{r}}$ , produces a sample vector  $\mathbb{L} \vec{\mathbf{r}}$  with the correlation of the historical values being simulated. For example, for two correlated normally distributed variables with correlation coefficient equals  $\rho$ , one can first simulate two uncorrelated normal variables  $\omega_1$  and  $\omega_2$ ; then  $\omega_1$  and  $\rho\omega_1 + \sqrt{1 - \rho^2}\omega_2$  are correlated at the level of  $\rho$ .

with asymptotic volatility. Somewhat sadly, using the variance of portfolio presented in Equation (12) with *ex post* data and using Cholesky decomposition in Monte Carlo procedure is still the common practice in academia and industry. Underlying Cholesky decomposition is the implicit assumption is the stability of the correlation coefficient in the  $\mathbb{P}$  matrix.

$$\mathbb{P} = \begin{pmatrix} 1 & \cdots & \rho_{1j} \\ \vdots & \ddots & \vdots \\ \rho_{i1} & \cdots & 1 \end{pmatrix} \quad (13)$$

In fact, any model that uses deterministic coefficients, rather than time varying variables suffers from similar problems as correlation coefficient. Examples of such deterministic coefficient include VAR, GARCH, cointegration, Granger causality, VECM, Kendall's tau, Spearman's rank correlation coefficient, and Goodman and Kruskal's gamma. To generate time varying variables to describe the dependence among financial assets, distribution-based simulation dominates constant-based simulation.

## 2 Methods and Models

The further question is then which copula fits the asset return patterns best. Using the Gaussian copula as the benchmark, Student's t copula has fatter head and tails. This contrasts to the leptokurtic distributions which has thinner head but fatter tails. Both Gaussian and Student's t copula have both sides of the extreme observations being dependent. Clayton copula has only one side of the extreme observations being dependent to the same degree, compared to Gumbel copula that has both sides of the extreme observations being dependent to asymmetric degree. Frank copula, however, do not present significant degree to dependence at either side of the observations. Figure 1, 2, and 3 provides intuitive demonstrations of the five copulas in terms of their distribution features. The R program for copula demonstration is available by request.

Well-known goodness of fit statistics are the  $\chi^2$ , Kolmogorov-Smirnoff, and Anderson-Darling goodness statistics. However, this study does not consider these statistics as they are limited to the requirement of precise observations and cannot incorporate truncated data (Vose, Koupeev, et al., 2007). This paper instead considers three information criteria in terms of the goodness of fit of the five copulas to the returns of financial assets. The criteria are Akaike information criterion (AIC); Schwarz information criterion (SIC), also known as Bayesian information criterion (BIC); and Hannan-Quinn information criterion (HQIC).

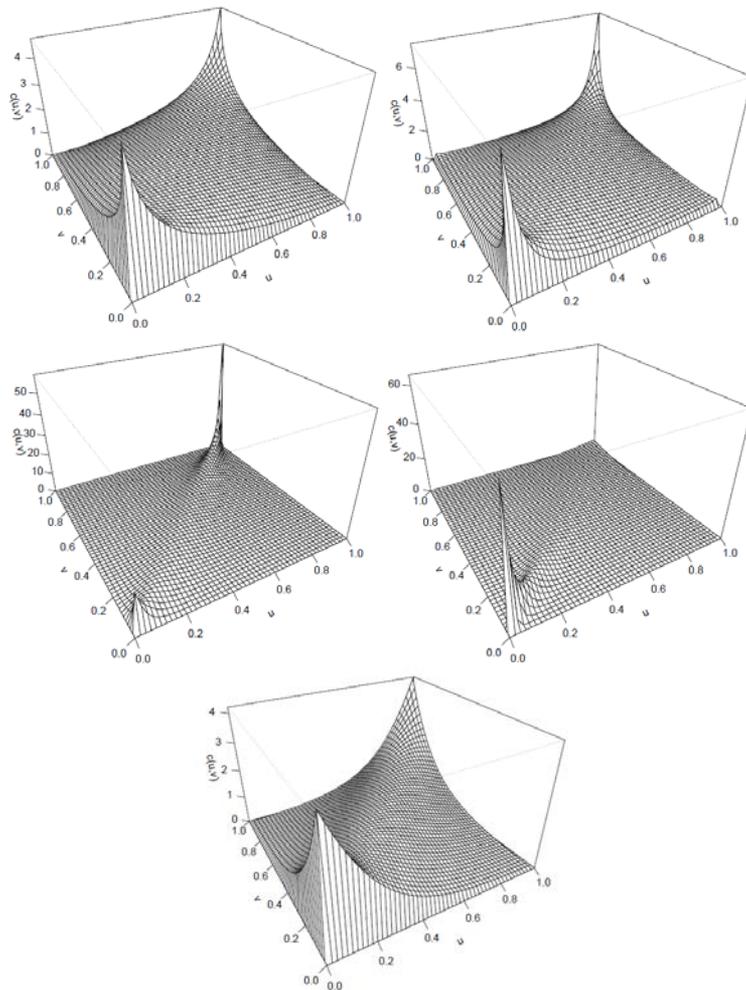


Figure 1: Probability Density of the Gaussian, Student's t, Gumbel, Clayton, and Frank Copulas

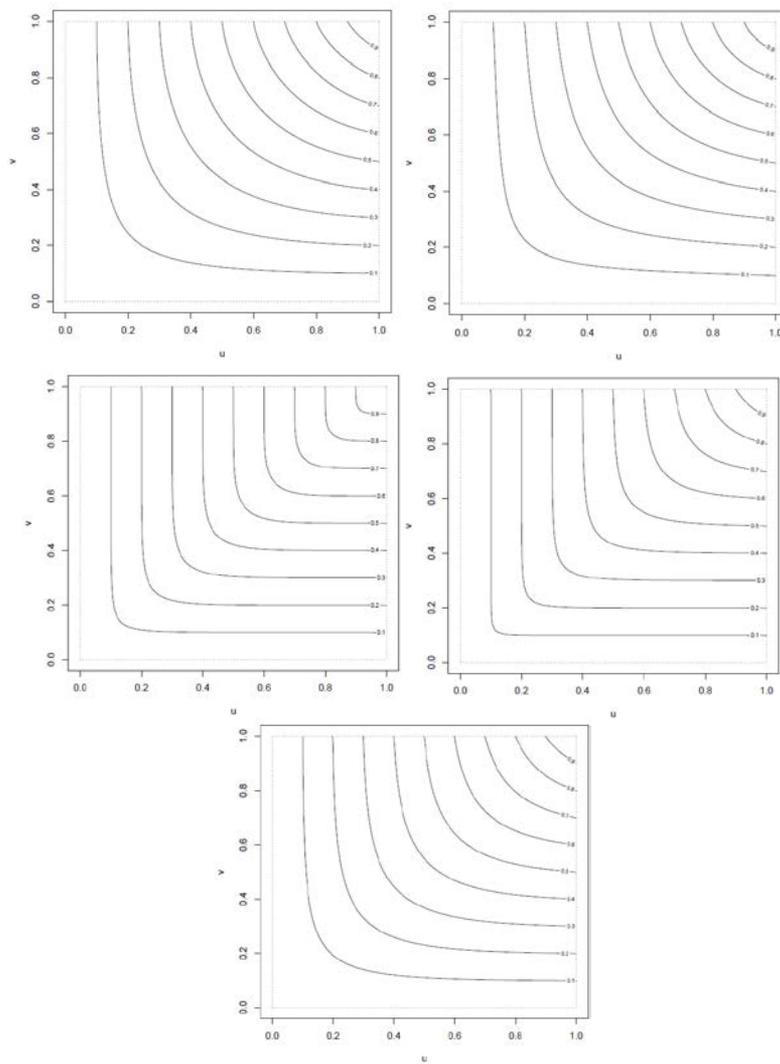


Figure 2: Cumulative Distribution Cross Sections of the Gaussian, Student's  $t$ , Gumbel, Clayton, and Frank Copulas

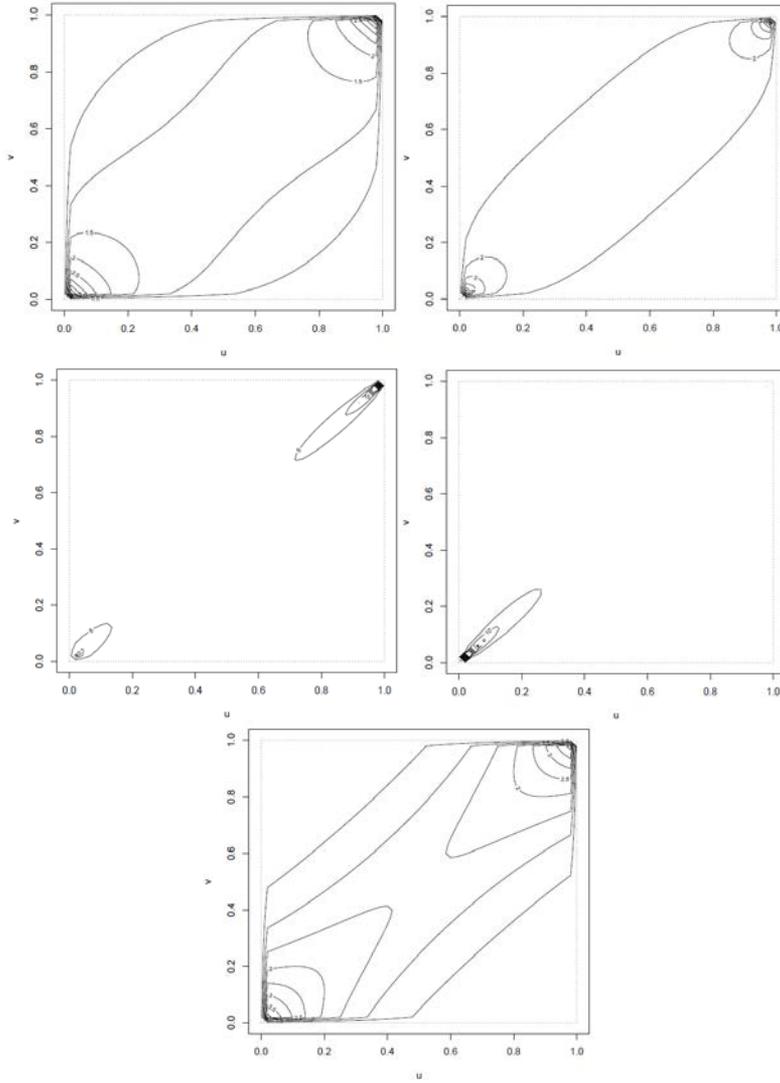


Figure 3: Probability Density Cross Sections of the Gaussian, Student's t, Gumbel, Clayton, and Frank Copulas

The information criteria statistics are computed as:

$$AIC = \left( \frac{2n}{n-k-1} \right) k - \ln L_{max}^2 \quad (14)$$

$$SIC = \ln n^k - \ln L_{max}^2 \quad (15)$$

$$HQIC = \ln(\ln k)^{2k} - \ln L_{max}^2 \quad (16)$$

where  $n$  is the number of observations;  $k$  is the number of parameters; and  $L_{\max}$  is the maximized value of the log-Likelihood for the estimated model. This study ranks the information criterion from the lowest to highest for a designated copula. The first item at the right side of Equations (14), (15), and (16) are the penalize term, as more parameters in a distribution ought to generate more precise description to the quantiles of distribution. This paper ranks the above-mentioned three information criteria for the five bivariate copulas for each pair of asset returns. For the 435 asset returns pairs generated from 30 assets, this study attempts to explore the copula that dominates the others.

## 2 Data

This paper randomly selects 30 stocks from the Russell 3000 index. The index list is according to the latest June 27, 2014 version as of December 18, 2014. Similar to Guo and Zhong (2015), this study first assigns random value between 0 and 1 for all 3000 assets, and then select assets with random values between 0.49 and 0.51. The assets that do not carry full historical data between December 5, 2005 and December 8, 2014 are excluded. The random values assigned using the following procedure are uniformly distributed (Rotz, Falk, Wood, and Mulrow, 2001) and are free of sampling bias or data mining concerns. Specifically, according to Wichman and Hill (1982, 1987), because the fractional part of the sum of three random numbers on  $[0,1]$  is still a random number on  $[0,1]$ ,

For integer  $a$ ,  $b$ , and  $c$  between 1 and 30000, assign the values to  $a$ ,  $b$ , and  $c$ :

$$a \leftarrow \text{MOD}(170 * a, 30323)$$

$$b \leftarrow \text{MOD}(171 * a, 30269)$$

$$c \leftarrow \text{MOD}(172 * a, 30307)$$

The random number is the fractional part of the sum of  $a$ ,  $b$ , and  $c$ .

Between December 5, 2005 and December 8, 2014, for each of the 30 stocks there is 2269 daily adjusted close price information from the Center for Research in Security Prices (CRSP). This generates 2268 daily returns for each asset; in other words, these are the daily returns in nine years assuming 252 trading days of every year. To convert the prices into continuously compounded returns, this paper applies the following Equation (17).

$$r_t^Y = \log_e p_t^Y - \log_e p_{t-1}^Y \quad (17)$$

The assets selected in this paper are summarized in Table 2, and the key features of the assets are summarized in Table 3.

Table 2: The 30 Assets Randomly Selected for Copula Goodness of Fit Tests

Company	Ticker	Company	Ticker
ASPEN INSURANCE HOLDING	AHL	INGRAM MICRO INC	IM
BE AEROSPACE INC	BEAV	MARKEL CORP	MKL
Belmond	BEL	MONARCH CASINO & RESORT	MCRI
COCA COLA BOTTLING	COKE	MOSAIC COMPANY	MOS
DIEBOLD INC	DBD	NU SKIN ENTERPRISES	NUS
MULTI-COLOR	LABL	PEPCO	POM
DOT HILL SYS CORP	HILL	POWELL INDUSTRIES INC	POWL
EZCORP INC	EZPW	PRAXAIR	PX
FIRST FINL BANKSHARES	FFIN	RAMCO-GERSHENSON	RPT
FIRST LONG ISLAND CORP	FLIC	ROCKWOOD HOLDINGS INC	ROC
GENERAL COMMUNICATION	GNCMA	SCHOLASTIC	SCHL
HEARTLAND FINANCIAL USA	HTLF	SANGAMO	SGMO
HOVNANIAN	HOV	SKYWORKS SOLUTIONS INC	SWKS
J & J	JJSF	MOLSON	TAP
INTER PARFUMS	IPAR	UNIFI	UFI

Table 3: The Key Features of the 30 Assets for Copula Goodness of Fit Tests

	Market Cap	EV	Trailing P/E	Forward P/E	PEG	P/S	P/B	EV/Revenue	EV/EBITDA
AHL	2.77B	477.73M	8.72 <sup>[MIN]</sup>	10.73	8.71	1.07	0.83	0.18	0.9
BEAV	8.25B	10.58B	19.93	15.58	0.89	2.01	2.85	2.58	11.78
BEL	1.22B	1.70B	N/A	62 <sup>[MAX]</sup>	23.66 <sup>[MAX]</sup>	2.06	1.51	2.85	16.66
COKE	799.72M	1.28B	33.67	19.42	N/A	0.47	3.7	0.75	10.24
DBD	2.26B	2.52B	52.43	17.25	1.34	0.75	3.63	0.84	10.54
LABL	905.28M	1.34B	24.94	16.09	1.34	1.15	2.93	1.72	9.9
HILL	268.74M <sup>[MIN]</sup>	223.35M <sup>[MIN]</sup>	222 <sup>[MAX]</sup>	15.88	1.54	1.28	5.05	1.08	43.88 <sup>[MAX]</sup>
EZPW	593.16M	882.93M	N/A	7.09 <sup>[MIN]</sup>	0.63	0.58	0.65 <sup>[MIN]</sup>	0.89	6.26
FFIN	1.93B	1.98B	22.11	20.27	2.11	7.56	2.89	7.89	N/A
FLIC	371.45M	680.68M	16.37	14.33	2.15	5.02	1.52	9.71	N/A
GNCMA	507.73M	1.56B	32	24.6	N/A	0.55	2.67	1.73	4.98
HTLF	491.91M	1.09B	13.53	11.05	1.24	1.83	1.19	4.16	N/A
HOV	568.94M	2.38B	32.77	11.14	2.48	0.29	N/A	1.22	21.44
JJSF	2.03B	1.90B	28.45	25.04	2.6	2.16	3.53	2.06	13.11
IPAR	860.81M	587.35M	39.13	27.54	2.32	1.74	2.12	1.23	12.09
IM	4.27B	4.83B	16.65	9.01	0.85	0.1 <sup>[MIN]</sup>	1.04	0.11 <sup>[MIN]</sup>	6.65
MKL	9.62B	8.53B	32.6	30.9	2.92	1.91	1.35	1.68	12.15
MCRI	282.29M	305.66M	23.2	17.87	1.57	1.5	1.62	1.64	7.75
MOS	16.85B	17.55B	59.78	N/A	N/A	2.09	1.53	2.19	12.16
NUS	2.66B	2.74B	10.25	10.82	-2.26 <sup>[MIN]</sup>	0.92	2.94	0.91	5.49
POM	6.85B	12.34B	25.71	20.76	2.85	1.41	1.57	2.55	9.86
POWL	563.23M	441.01M	19.38	14.81	1.59	0.83	1.46	0.68	10.63
PX	37.21B <sup>[MAX]</sup>	46.27B <sup>[MAX]</sup>	20.3	18.78	2.02	3.04	5.7 <sup>[MAX]</sup>	3.76	11.96
RPT	1.41B	2.32B	N/A	13.84	3.24	6.85	1.73	11.4	19.7
ROC	5.68B	6.25B	9.62	28.68	3.58	3.99	2.27	4.4	19.6
SCHL	1.16B	1.32B	28.85	15.4	1.92	0.62	1.28	0.72	10.3
SGMO	948.07M	626.96M	N/A	N/A	N/A	20.89 <sup>[MAX]</sup>	3.75	16.59 <sup>[MAX]</sup>	-21.13 <sup>[MIN]</sup>
SWKS	13.20B	12.38B	29.09	13.74	0.82	5.75	5.17	5.4	17.8
TAP	13.64B	16.42B	24.53	17.32	3.89	3.26	1.63	3.91	19.89
UFI	514.79M	595.61M	20.26	13.55	0.95	0.72	1.77	0.86	11.95

In Table 3, the market capitalizations and enterprise values (EV) are based on the market quotes on December 10, 2014. The trailing Price-to-earnings (P/E), Price-to-Sales (P/S), Enterprise Value-to-Revenue, Enterprise Value-to-EBITDA (EV/EBITDA) are trailing twelve month data, the Price-to-Book (P/B) uses the

most recent quarter data as for December 18, 2014. The forward Price-to-earnings (P/E) in general uses pro forma earnings estimate to the 2015-2016 fiscal year, and the P/E-to-growth (PEG) indicates the 5-year expected growth level.

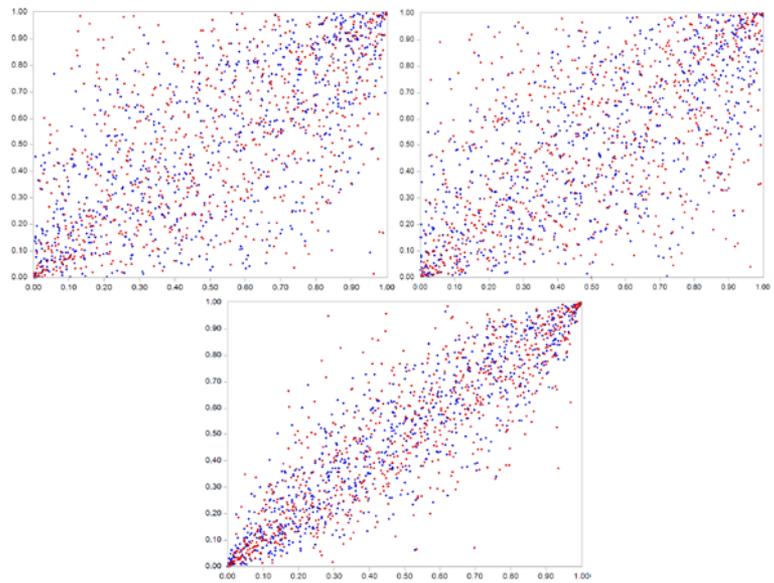
## 4 Results and Discussions

The distributions of returns of financial assets are frequently observed or assumed to be non-normal. Most frequently observed return distributions carry excess kurtosis and non-zero skewness. Harvey and Siddique (2000) discuss symmetrical skewness of asset returns and the premium associated with this skewness. Conrad, Dittmar, and Ghysels (2013) detail the skewness and kurtosis as markers for risk compensation in securities after considering comovements. Dong (2014) and Chen, Chen, and Lee (2013) suggest that the extreme negative investor sentiment leads to high dependence at the lower tail of the return quantile-quantile plot. While the optimistic market sentiment is less contagious, the returns at the upper tail are also dependent due to the massive fund import at market bubbles.

The results of the 6525 information criteria, which are the 3 information criteria of the 5 copulas for 435 pairs of asset returns, nearly unanimously favors the Student's t copula as the joint distribution of asset returns. The sizable outputs are available by request. This paper finds that the goodness of fit copula is not significantly linked to the type of the asset, no matter how the type is defined and categorized. Specifically, the extreme types of assets in various groups are presented in Table 3, and the following examples in Figure 4 are the demonstrations of the fitted t copula using the R copula functions (Hofert, Kojadinovic, Maechler, and Yan, 2014).

The benefit of Student's t copula is that it allows the joint distribution of assets to be fat headed and fat tailed. Such feature fits the nature of asset returns

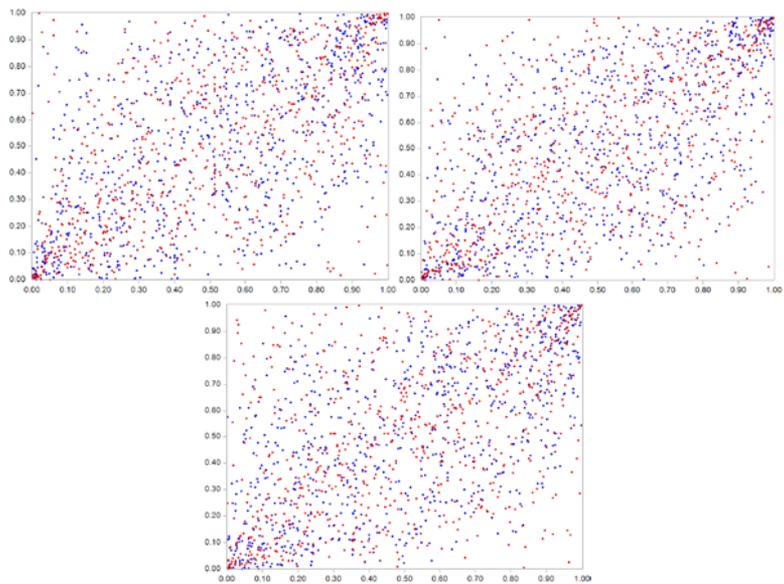
compared to normal distributions with excess kurtosis. Positive excess kurtosis allows for fat tails but limits the head of the distribution to be thinner than Gaussian distribution and is inconsistent with the common observations of volatile assets, which do not concentrate near the mean.



BEL&amp;HILL

BEL&amp;IM

BEL&amp;PX



BEL&amp;SGMO

HILL&amp;IM

HILL&amp;PX

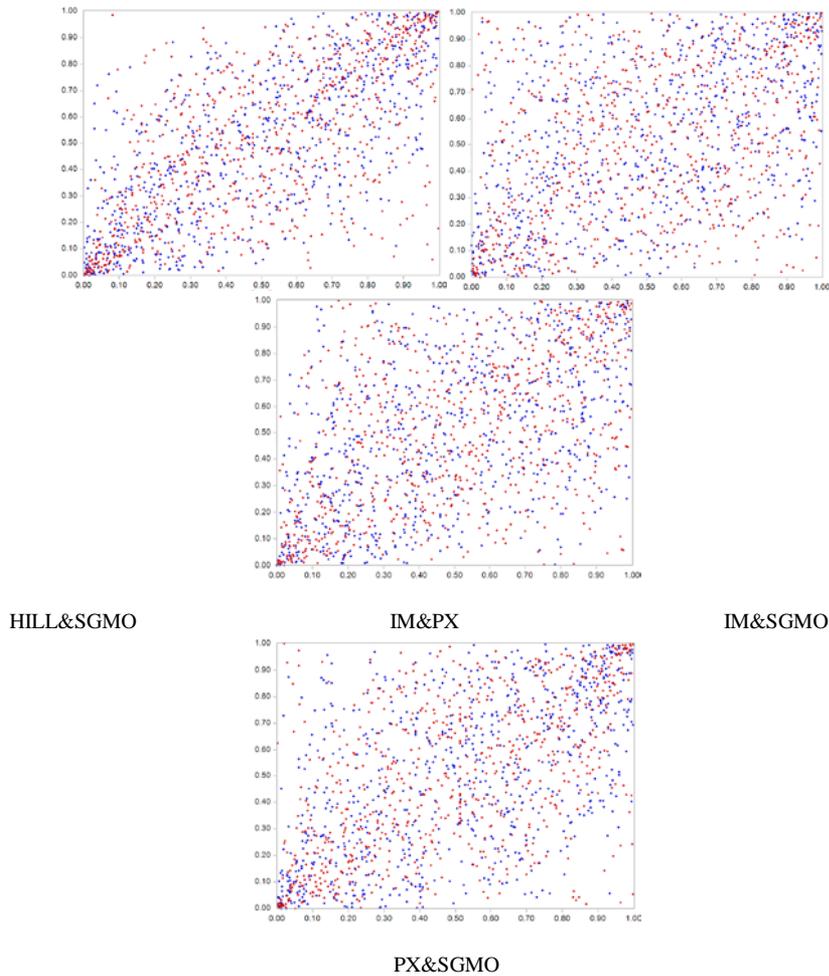


Figure 4: t-Copula Simulation for 5 Assets with Extreme Financial Features

While the Gumbel and Clayton copula allows for the asymmetric joint distribution, such asymmetry is redundant due to the nature of copulas. The different features of extreme value distribution can be captured by the independent marginals, such as the marginal of generalized hyperbolic distribution that Breyman and Lüthi (2013) suggest, of the univariate asset returns. This suggests another favorable feature of using copula to model the asset independence, which is the joint distribution is orthogonal to the statistic moments of individual asset.

In other words, copulas do not repeatedly incorporate the dependence of assets due to the similar variance, skewness, and kurtosis.

## 5 Conclusions

This paper uses asset returns and information criteria to conclude that the Student's  $t$  copula is the most appropriate joint distribution to model the performance of financial assets. For each of the 30 representative stocks, this study fits the bivariate copulas to 9 years of historical returns, or 2268 observations. In the 435 pairs of assets, the SIC, AIC, and HQIC information criteria indicate that  $t$  copula fits asset returns better compared to the Gaussian, Gumbel, Clayton, and Frank copula.

For a single asset, Gaussian distribution is commonly used in academia and industry to describe asset returns in spite of the widely-accepted conclusions that asset returns usually carry non-zero skewness and excess kurtosis. For multiple assets, academia and industry frequently use correlation coefficient to describe the dependence of assets. My study does not support such practice. The conclusions of this paper also suggest that it is inappropriate to use multivariate Gaussian distribution to model the dependence among assets. The reason is that any orthogonal cross section of a multivariate normal distribution must be a univariate normal distribution. However, this does not comprise the initiative to test the Gaussian copula, because joint normality does not dictate the individual distribution.

For the simulation of a single asset, the predicted future returns may be biased if the Monte Carlo procedure is based on Gaussian distribution, as the projected return fails to consider the third and fourth moments of the historical distribution. For the simulation of a portfolio of assets, the predicted future returns may also be biased if the Monte Carlo procedure involves correlation coefficients, as the

dependent relationship among assets is time-varying and is very often a random walk (Guo and Zhong, 2014). Therefore using a univariate time series model, such as ARIMA, to fit the correlation coefficient limits the precision of the forecast. The reason is because there is no evidence to support the stability of coefficients in the time series model, which is highly dependent to the historical sample period selected.

This paper suggests that the currently optimal method of modeling interdependent asset returns in a portfolio is to first fit the individual distributions and then fit the t-copula to the independent distributions. The study at the next step, which is relatively separated, is to explore the generalized return distributions for single assets to complete the chain of a new Monte Carlo procedure.

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