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A Fuzzy Minimum Cost Fuzzy Flow Problem with Fuzzy Time-Windows and Fuzzy Interval Bounds

Nasser A. El-Sherbeny[1]

Abstract

In this paper, we present and describe a new version of the Minimum Cost Flow Problem (MCFP). This version is a Fuzzy Minimum Cost Fuzzy Flow Problem with Fuzzy Time-Windows and Fuzzy Interval Bounds (FMCFFPFTWFIB). The FMCFFPFTWFIB is a combinatorial optimization and an NP-hard problem. The FMCFFPFTWFIB of fuzzy interval data can be using two fuzzy minimum cost fuzzy flow of fuzzy time-windows problems with fuzzy crisp data. In this paper, the idea of Ghiyasvand was extended a fuzzy minimum cost fuzzy flow of fuzzy time-windows problem with fuzzy interval-valued lower, upper bounds and fuzzy flows. Also, this work is extended to the network with fuzzy lower, upper bounds and fuzzy flows. An application example network is given.

Mathematics Subject Classification: 90C27; 90C35; 68R10

Keywords: Optimization network, Minimum flow problem, Interval data, Time-windows

1 Introduction

Consider a fuzzy directed network $\tilde{G} = \left(V, A, \tilde{u}_{v_i v_j}, \tilde{l}_{v_i v_j}, \tilde{t}_{v_i v_j}\right)$ where V is a set of n vertices and A is a set of n arcs. We associate with each arc $\left(v_i, v_j\right) \in V, i \neq j; i, j =$

[1] Mathematics Department, Faculty of Science, Al-Azhar University, Egypt

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$1, \ldots, n$ the fuzzy upper bound $\tilde{u}_{v_i v_j}$ that denotes the maximum amount that can the fuzzy flow on the arc and fuzzy lower bound $\tilde{l}_{v_i v_j}$ that denotes the minimum amount that must fuzzy flow on the arc. Each arc has a non-negative fuzzy transit time $\tilde{t}_{v_i v_j}, \forall v_i, v_j \in V$. Each vertex $v_i \in V$, has a fuzzy time-windows $[\tilde{a}_{v_i}, \tilde{b}_{v_i}]$, within which the vertex may be served, i.e., $\tilde{t}_{v_i} \in [\tilde{a}_{v_i}, \tilde{b}_{v_i}], \tilde{t}_{v_i v_j} \in \tilde{T}$, is a non-negative fuzzy service and leaving for that vertex. To define the Minimum Cost Flow Time-Windows Problem (MCFTWP), we distinguish two special vertices in the network, namely a source vertex s and a sink vertex τ with fuzzy time-windows $[a_s, b_s]$ and $[a_\tau, b_\tau]$ respectively, see, ([14], [16], [17], [18], and [24]). The problem is to find a minimum cost flow with time-windows from the source vertex s to the sink vertex τ that satisfy the lower, upper bounds and balance constraints at all vertices. The decision variables in the minimum cost flow time-windows problem are the arc flows, $f_{v_i v_j}$ on an arc $(v_i, v_j) \in V$, see, ([20], [21], [22], and [23]).

There are several approaches to solve the Minimum Flow Problem (MFP). For decreasing path algorithms by ([4], [5] and [7]), pre-flow algorithms by ([3], [10], [6], and [8]). For minimax which consists of finding a maximum flow from the sink vertex to the source vertex in the residual network by [1] and [5], using dynamic tree implementations by [9]. Also, [13] solved the minimum flow problem for bipartite networks. In [11], and [12], solved the inverse minimum flow problem.

In [19], a new method to solve the minimum cost flow problem with interval data is presented. First, it solves a minimum cost flow problem with lower bounds, flows, and costs, second it, shows a minimum cost flow problem with upper bounds, flows, and costs. Then, the method combines these two solutions to form an interval solution. In [19], also proved that is the interval solution is optimal for the minimum cost flow problem with interval bounds, flows, and costs. Here, we extend their idea to present and describe the minimum cost flow time-windows problem with interval bounds and flows. We show that the minimum cost flow time-windows problem can be using two minimum flow time-windows problems with crisp data.

The reminder of this paper consists of five sections including Introduction. Section 2 presents the basic concepts of the time-windows and a fuzzy time-windows. In section 3, we presented, described the mathematical model of FMCFFPFTWFIB and presented the relationship between the minimum fuzzy cost fuzzy flow of fuzzy time-windows problems with fuzzy interval data and crisp data. In section 4, we presented a fuzzy minimum cost fuzzy flow of fuzzy time-windows problem with fuzzy data according to Zadeh's extension principle and given an application network instance. Finally, the conclusion is given in Section 5.

2 Basic Concepts and Definitions

Consider a fuzzy directed network $\tilde{G} = \left(V, A, \tilde{u}_{v_i v_j}, \tilde{l}_{v_i v_j}, \tilde{t}_{v_i v_j}\right)$, where V is a set of n vertices, A is a set of n arcs with non-negative fuzzy transit time $\tilde{t}_{v_i v_j}$,$\forall v_i, v_j \in V$, $i \neq j; i, j = 1, \dots, n$. For each vertex $v_i \in V$ the fuzzy time-windows $\left[\tilde{a}_{v_i}, \tilde{b}_{v_i}\right]$ within which the vertex may be served with $\tilde{t}_{v_i} \in \left[\tilde{a}_{v_i}, \tilde{b}_{v_i}\right], \tilde{t}_{v_i v_j} \in \tilde{T}$, is a non-negative fuzzy service and fuzzy leaving time of $v_i \in V$. A source vertex s, a sink vertex τ has a fuzzy time-windows $\left[\tilde{a}_s, \tilde{b}_s\right]$ and $\left[\tilde{a}_\tau, \tilde{b}_\tau\right]$ respectively. We also associate, each arc $\left(v_i, v_j\right) \in V, i \neq j; i, j = 1, \dots, n$ has a fuzzy upper bound $\tilde{u}_{v_i v_j}$ that denotes the fuzzy maximum amount. A fuzzy flow on the arc and a fuzzy lower bound $\tilde{l}_{v_i v_j}$ that denotes the minimum fuzzy amount that must a fuzzy flow on the arc. The decision variables in the minimum fuzzy cost fuzzy flow of the fuzzy time-windows problem are the arc fuzzy flows and we represent the fuzzy flow on the arc $\left(v_i, v_j\right) \in V$ by $\tilde{f}_{v_i v_j}$. A Fuzzy Minimum Cost Fuzzy Flow Problem with Fuzzy Time-Windows and Fuzzy Interval Bounds (FMCFFPFTWIB) can be stated formally as follows:

$$min \tilde{v}$$

$$\text{subject to: } \sum_{\{v_j:(v_i,v_j)\in A} \tilde{f}_{v_i v_j} - \sum_{\{v_j:(v_j,v_i)\}\in A} \tilde{f}_{v_j v_i} = \begin{Bmatrix} \tilde{v}, & v_i = s \\ -\tilde{v}, & v_i = \tau \\ 0, \forall v_i \in V - \{s, \tau\} & \end{Bmatrix} \quad (1)$$

$$\tilde{l}_{v_i v_j} \leq \tilde{f}_{v_i v_j} \leq \tilde{u}_{v_i v_j}, \forall \left(v_i, v_j\right) \in A \quad (2)$$

$$\tilde{t}_{v_i} + \tilde{t}_{v_i v_j} \leq \tilde{t}_{v_j}, \forall \left(v_i, v_j\right) \in A; \tilde{t}_{v_i}, \tilde{t}_{v_i v_j} \in \tilde{T}; \tilde{t}_{v_i} \in \left[\tilde{a}_{v_i}, \tilde{b}_{v_i}\right]; \tilde{t}_{v_j} \in \left[\tilde{a}_{v_j}, \tilde{b}_{v_j}\right] \quad (3)$$

The FMCFFPFTWIB is one of the fuzzy networks with the fuzzy flow that computes the fuzzy minimum cost fuzzy flow with fuzzy time-windows and fuzzy interval bounds between two given vertices called a source and a sink vertex.

Definition 2.1 *A time-windows constraint is defined by, each vertex,* $(v_i, v_j) \in A$ *has a time-windows* $[a_{v_i}, b_{v_i}]$ *and* $[a_{v_j}, b_{v_j}]$ *respectively. Each arc* $(v_i, v_j) \in A$ *has a non-negative transit time* $t_{v_iv_j}$*;* $i \neq j; i, j = 1, \ldots, n$ *where* $t_{v_i} \in [a_{v_i}, b_{v_i}]; t_{v_j} \in [a_{v_j}, b_{v_j}]$ $t_{v_i}, t_{v_j} \in T \in R^+$, *see Figure 1*

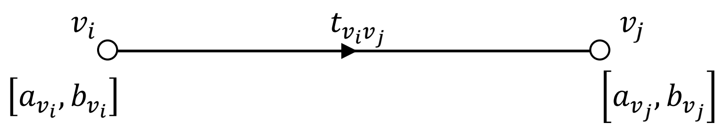


Figure 1: A vertex service time-windows constraints of each arc.

Let I denote the class of non-empty compact fuzzy intervals $[\underline{\tilde{x}}, \bar{\tilde{x}}]$ on $[0, \infty)$. If $\bar{\tilde{x}} = \underline{\tilde{x}} = \tilde{a}$, identify the fuzzy interval with a fuzzy real number $\tilde{a}$.

Definition 2.2 *Let* $[\underline{\tilde{x}}_1, \bar{\tilde{x}}_1]$ *and* $[\underline{\tilde{x}}_2, \bar{\tilde{x}}_2]$ *be two compact fuzzy intervals, then*

$$[\underline{\tilde{x}}_1, \bar{\tilde{x}}_1] + [\underline{\tilde{x}}_2, \bar{\tilde{x}}_2] = [\underline{\tilde{x}}_1 + \underline{\tilde{x}}_2, \bar{\tilde{x}}_1 + \bar{\tilde{x}}_2] \quad (4)$$

$$[\underline{\tilde{x}}_1, \bar{\tilde{x}}_1][\underline{\tilde{x}}_2, \bar{\tilde{x}}_2] = [min(\underline{\tilde{x}}_1\underline{\tilde{x}}_2, \underline{\tilde{x}}_1\bar{\tilde{x}}_2, \bar{\tilde{x}}_1\underline{\tilde{x}}_2, \bar{\tilde{x}}_1\bar{\tilde{x}}_2), max(\underline{\tilde{x}}_1\underline{\tilde{x}}_2, \underline{\tilde{x}}_1\bar{\tilde{x}}_2, \bar{\tilde{x}}_1\underline{\tilde{x}}_2, \bar{\tilde{x}}_1\bar{\tilde{x}}_2)] \quad (5)$$

$$[\underline{\tilde{x}}_1, \bar{\tilde{x}}_1] \leq [\underline{\tilde{x}}_2, \bar{\tilde{x}}_2] \text{ if } \underline{\tilde{x}}_1 \leq \underline{\tilde{x}}_2; \bar{\tilde{x}}_1 \leq \bar{\tilde{x}}_2. \quad (6)$$

The fuzzy infimum and supremum of $[\underline{\tilde{x}}_1, \bar{\tilde{x}}_1]$ *and* $[\underline{\tilde{x}}_2, \bar{\tilde{x}}_2]$, *respectively, are defined by:*

$$[\underline{\tilde{x}}_1, \bar{\tilde{x}}_1] \wedge [\underline{\tilde{x}}_2, \bar{\tilde{x}}_2] = [min\{\underline{\tilde{x}}_1, \underline{\tilde{x}}_2\}, min\{\bar{\tilde{x}}_1, \bar{\tilde{x}}_2\}] \quad (7)$$

$$[\underline{\tilde{x}}_1, \bar{\tilde{x}}_1] \vee [\underline{\tilde{x}}_2, \bar{\tilde{x}}_2] = [max\{\underline{\tilde{x}}_1, \underline{\tilde{x}}_2\}, max\{\bar{\tilde{x}}_1, \bar{\tilde{x}}_2\}] \quad (8)$$

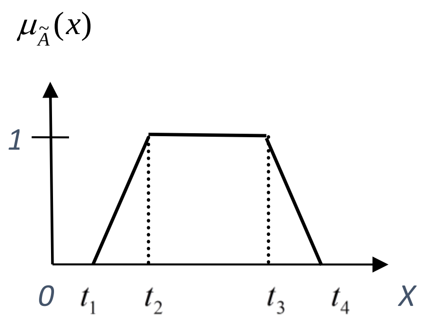
If $[\underline{\tilde{x}}_1, \bar{\tilde{x}}_1], \ldots, [\underline{\tilde{x}}_n, \bar{\tilde{x}}_n] \in I$, *then the fuzzy infimum* $\wedge_i[\underline{\tilde{x}}_i, \bar{\tilde{x}}_i]$, *fuzzy supremum* $\vee_i[\underline{\tilde{x}}_i, \bar{\tilde{x}}_i]$ *are well-defined and*

$$\sum_{\{i:i=1,\ldots,n\}}[\underline{\tilde{x}}_i, \bar{\tilde{x}}_i] = [\sum_{\{i:i=1,\ldots,n\}}\underline{\tilde{x}}_i, \sum_{\{i:i=1,\ldots,n\}}\bar{\tilde{x}}_i] \quad (9)$$

- **Fuzzy Time-Windows** ([15])

Let $X = \mathcal{R}^n$ be a non-empty set, $\tilde{A} \subseteq X$. The fuzzy set $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$ is the set of ordered pairs where $\mu_{\tilde{A}} : X \to [0,1]$ is the membership function of the fuzzy set $\tilde{A}$. The fuzzy constraint is a fuzzy set $\tilde{A} = (t_1, t_2, t_3, t_4)$ with flexible time-windows where (t_1, t_4) is the interval of non-zero satisfaction level and (t_2, t_3) is the interval of non-zero satisfaction level equal to 1 see, Figure 2.

The first step is to ask the expert to give a range for travel time between two places along with the most likely time; For example, the time $\tilde{T}$ to travel from point A to point B is between t_1 and t_3, but must possibly it is t_2. This sort of knowledge lets us construct 3-point fuzzy travel times see Figure 3.



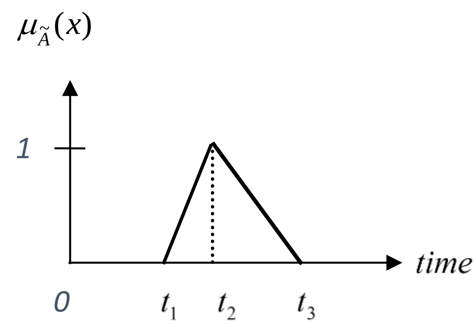
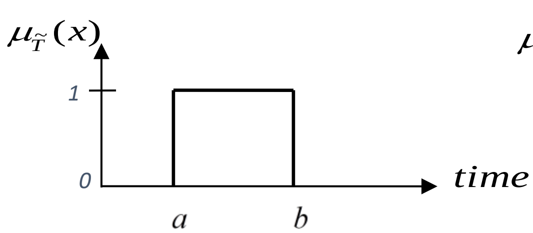


Figure 2: 4-Points representation of fuzzy interval Figure 3: Fuzzy travel time

Similarly, obtain a fuzzy time-windows. Every vertex $v_i \in V$ is assigned by the expert to one of two predetermined groups; a classical fuzzy time-windows and fuzzy time-windows of a normal vertex. In an extreme case, fuzzy time-windows are tighter than the classical counterpart see, Figure 4 and 5. The shown characteristics of fuzzy time-windows are suggested to the shipper who can modify them.



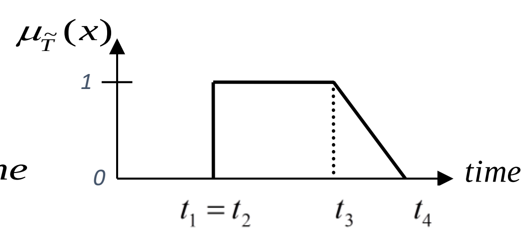


Figure4: Classical fuzzy time-windows Figure5: Fuzzy time-windows of a normal vertex

3 A Fuzzy Minimum Cost Fuzzy Flow Problem with Fuzzy Time-Windows and Fuzzy Interval Bounds (FMCFFPFTWFIB)

We describe the mathematical model of FMCFFPFTWFIB and presented the relationship between the fuzzy minimum cost fuzzy flow of fuzzy time-windows problems with fuzzy interval data and crisp data. Consider a directed fuzzy network $\tilde{G} = \left(V, A, \tilde{u}_{v_i v_j}, \tilde{l}_{v_i v_j}, \tilde{t}_{v_i v_j}\right)$, where V is a set of n vertices, A is a set of n arcs such that the fuzzy time-windows, fuzzy lower bound, fuzzy upper bound, and fuzzy flow of each arc are known to fall within specific ranges expressed as compact fuzzy intervals $\tilde{\bar{w}}, \tilde{\bar{l}}, \tilde{\bar{u}}$ and $\tilde{\bar{f}}$ respectively. Thus, for each arc $(v_i, v_j) \in A$, we have

$$\tilde{w}_{v_i v_j} \in \tilde{\bar{w}}_{v_i v_j} = [\tilde{l}_{\tilde{w}}(v_i, v_j), \tilde{r}_{\tilde{w}}(v_i, v_j)], \tag{10}$$

$$\tilde{l}_{v_i v_j} \in \tilde{\bar{l}}_{v_i v_j} = [\tilde{l}_{\tilde{l}}(v_i, v_j), \tilde{r}_{\tilde{l}}(v_i, v_j)], \tag{11}$$

$$\tilde{u}_{v_i v_j} \in \tilde{\bar{u}}_{v_i v_j} = [\tilde{l}_{\tilde{u}}(v_i, v_j), \tilde{r}_{\tilde{u}}(v_i, v_j)], \tag{12}$$

$$\tilde{f}_{v_i v_j} \in \tilde{\bar{f}}_{v_i v_j} = [\tilde{l}_{\tilde{f}}(v_i, v_j), \tilde{r}_{\tilde{f}}(v_i, v_j)] \tag{13}$$

Where, $\tilde{l}_{\tilde{w}}(v_i, v_j), \tilde{r}_{\tilde{w}}(v_i, v_j), \tilde{l}_{\tilde{l}}(v_i, v_j), \tilde{r}_{\tilde{l}}(v_i, v_j), \tilde{l}_{\tilde{u}}(v_i, v_j), \tilde{r}_{\tilde{u}}(v_i, v_j), \tilde{l}_{\tilde{f}}(v_i, v_j)$ and $\tilde{r}_{\tilde{f}}(v_i, v_j)$ are non-negative fuzzy values. A minimum fuzzy cost fuzzy flow problem with fuzzy time-windows with fuzzy compact interval-valued lower and upper bounds and a fuzzy flow can be stated as follows:

$$\min[\tilde{v}_{\tilde{l}}, \tilde{v}_{\tilde{r}}]$$

subject to:

$$\sum_{\{v_j:(v_i,v_j)\in A\}} \tilde{\bar{f}}_{v_i v_j} - \sum_{\{v_j:(v_j,v_i)\in A\}} \tilde{\bar{f}}_{v_j v_i} = \begin{Bmatrix} [\tilde{v}_{\tilde{l}}, \tilde{v}_{\tilde{r}}], & v_i = s \\ -[\tilde{v}_{\tilde{l}}, \tilde{v}_{\tilde{r}}], & v_i = \tau \\ [0,0], \forall v_i \in V - \{s, \tau\} & \end{Bmatrix} \tag{14}$$

$$\tilde{\bar{l}}_{v_i v_j} \le \tilde{\bar{f}}_{v_i v_j} \le \tilde{\bar{u}}_{v_i v_j}, \forall (v_i, v_j) \in A \tag{15}$$

$$\tilde{\bar{l}}_{\tilde{w}}(v_i, v_j) \le \tilde{\bar{w}}_{v_i v_j} \le \tilde{\bar{u}}_{\tilde{w}}(v_i, v_j), \forall (v_i, v_j) \in A \tag{16}$$

$$\tilde{\bar{t}}_{v_i} + \tilde{\bar{t}}_{v_i v_j} \le \tilde{\bar{t}}_{v_j}; \tilde{\bar{a}}_{v_i} \le \tilde{\bar{t}}_{v_i} \le \tilde{\bar{b}}_{v_i}; \tilde{\bar{t}}_{v_i}, \tilde{\bar{t}}_{v_i v_j} \in \tilde{\bar{T}}; v_i \ne v_j; \forall v_i, v_j \in V \tag{17}$$

We call this problem the fuzzy interval-minimum fuzzy cost fuzzy flow of fuzzy time-windows problem. Let $\bar{f}^*$ be an answer of this problem. For each arc $(v_i, v_j) \in A$.

we define any element of the interval $\bar{\tilde{f}}^*_{v_i v_j}$ as an answer for the fuzzy interval-minimum fuzzy cost fuzzy flow of a fuzzy time-windows problem. From definition 2.2 conditions (14), (15), (16) and (17) can be written by the following:

$$[\sum_{\{v_j:(v_i,v_j)\in A\}} \tilde{l}_{\tilde{f}}(v_i,v_j), \sum_{\{v_j:(v_i,v_j)\in A\}} \tilde{r}_{\tilde{f}}(v_i,v_j)] - \left[\sum_{\{v_j:(v_j,v_i)\in A\}} \tilde{l}_{\tilde{f}}(v_j,v_i), \sum_{\{v_j:(v_j,v_i)\in A\}} \tilde{r}_{\tilde{f}}(v_j,v_i)\right] = \begin{Bmatrix} [\tilde{v}_{\tilde{l}}, \tilde{v}_{\tilde{r}}], & v_i = s \\ -[\tilde{v}_{\tilde{l}}, \tilde{v}_{\tilde{r}}], & v_i = \tau \\ [0,0], \forall v_i \in V - \{s,\tau\} \end{Bmatrix} \quad (18)$$

$$\tilde{l}_{\tilde{l}}(v_i,v_j) \le \tilde{l}_{\tilde{f}}(v_i,v_j) \le \tilde{l}_{\tilde{u}}(v_i,v_j), \forall (v_i,v_j) \in A \quad (19)$$

$$\tilde{r}_{\tilde{l}}(v_i,v_j) \le \tilde{r}_{\tilde{f}}(v_i,v_j) \le \tilde{r}_{\tilde{u}}(v_i,v_j), \forall (v_i,v_j) \in A \quad (20)$$

$$\tilde{l}_{\tilde{w}}(v_i,v_j) \le \tilde{w}(v_i,v_j) \le \tilde{u}_{\tilde{w}}(v_i,v_j), \forall (v_i,v_j) \in A \quad (21)$$

$$\tilde{t}_{v_i} + \tilde{t}_{v_i v_j} \le \tilde{t}_{v_j}; \tilde{a}_{v_i} \le \tilde{t}_{v_i} \le \tilde{b}_{v_i}; \tilde{t}_{v_i}, \tilde{t}_{v_i v_j} \in \tilde{T}; v_i \ne v_j; \forall v_i, v_j \in V \quad (22)$$

There for, a fuzzy flow $\tilde{\bar{f}}$ is feasible for the fuzzy interval-minimum fuzzy cost fuzzy flow of the fuzzy time-windows problem if it satisfies the conditions (19), (20), (21), and (22). Thus, the fuzzy interval-minimum fuzzy cost fuzzy flow of the fuzzy time-windows problem can be written by the following:

$\wedge[\tilde{v}_{\tilde{l}}, \tilde{v}_{\tilde{r}}]: \tilde{f}$ satisfies the conditions (18), (19), (20), (21) and (1) (*)

We define the fuzzy $\tilde{l}$–minimum fuzzy cost fuzzy flow of the fuzzy time-windows problem by the following:

$min \tilde{v}_{\tilde{l}}$

subject to:

$$\sum_{\{v_j:(v_i,v_j)\in A\}} \tilde{l}_{\tilde{f}}(v_j,v_i) - \sum_{\{v_j:(v_j,v_i)\in A\}} \tilde{l}_{\tilde{f}}(v_j,v_i) = \begin{Bmatrix} \tilde{v}_{\tilde{l}}, & v_i = s \\ -\tilde{v}_{\tilde{l}}, & v_i = \tau \\ 0, \forall v_i \in V - \{s.\tau\} \end{Bmatrix} \quad (23)$$

$$\tilde{l}_{\tilde{l}}(v_i,v_j) \le \tilde{l}_{\tilde{f}}(v_i,v_j) \le \tilde{l}_{\tilde{u}}(v_i,v_j), \forall (v_i,v_j) \in A \quad (24)$$

$$\tilde{t}_{v_i} + \tilde{t}_{v_i v_j} \le \tilde{t}_{v_j}; \tilde{a}_{v_i} \le \tilde{t}_{v_i} \le \tilde{b}_{v_i}; \tilde{t}_{v_i}, \tilde{t}_{v_i v_j} \in \tilde{T}; v_i \ne v_j; \forall v_i, v_j \in V \quad (25)$$

We also define the $\tilde{r}$-minimum fuzzy cost fuzzy flow of the fuzzy time-windows problem by the following:

$min \tilde{v}_{\tilde{r}}$

subject to:

$$\sum_{\{v_j:(v_i,v_j)\in A\}} \tilde{r}_{\tilde{f}}(v_j,v_i) - \sum_{\{v_j:(v_j,v_i)\in A\}} \tilde{r}_{\tilde{f}}(v_j,v_i)] = \begin{Bmatrix} \tilde{v}_{\tilde{r}}, & v_i = s \\ -\tilde{v}_{\tilde{r}}, & v_i = \tau \\ 0, \forall v_i \in V - \{s,\tau\} \end{Bmatrix} \quad (26)$$

$$\tilde{r}_{\tilde{l}}(v_i,v_j) \le \tilde{r}_{\tilde{f}}(v_i,v_j) \le \tilde{r}_{\tilde{u}}(v_i,v_j), \forall (v_i,v_j) \in A \quad (27)$$

$$\tilde{t}_{v_i} + \tilde{t}_{v_iv_j} \le \tilde{t}_{v_j}; \tilde{a}_{v_i} \le \tilde{t}_{v_i} \le \tilde{b}_{v_i}; \tilde{t}_{v_i}, \tilde{t}_{v_iv_j} \in \tilde{T}; v_i \ne v_j; \forall v_i, v_j \in V \quad (28)$$

The relationship among the fuzzy $\tilde{l}$-minimum fuzzy cost fuzzy flow of the fuzzy time-windows problem and the fuzzy interval minimum fuzzy cost fuzzy flow of the fuzzy time-windows problem is shown by the next theorem.

Theorem 3.1 *Let* $\tilde{l}_{\tilde{f}_1^*}$ *(resp,* $\tilde{r}_{\tilde{f}_2^*}$*) is an optimal fuzzy flow for the fuzzy* $\tilde{l}$*-minimum fuzzy cost fuzzy flow of the fuzzy time-windows problem (resp, fuzzy* $\tilde{r}$*-minimum fuzzy cost fuzzy flow of the fuzzy time-windows problem). Then* $\tilde{\tilde{f}}^*=[\tilde{l}_{\tilde{f}_1^*}, \tilde{r}_{\tilde{f}_2^*}]$ *is an optimal fuzzy flow for the fuzzy interval-minimum fuzzy cost fuzzy flow of the fuzzy time-windows problem.*

Proof: We first show that the fuzzy flow $\tilde{l}_{\tilde{f}_1^*}$ is a feasible fuzzy flow for the fuzzy interval-minimum fuzzy cost fuzzy flow of the fuzzy time-windows problem. By the feasibility of $\tilde{l}_{\tilde{f}_1^*}$ in the fuzzy $\tilde{l}$-minimum fuzzy cost fuzzy flow of the fuzzy time-windows problem, we get

$$\tilde{l}_{\tilde{l}}(v_i,v_j) \le \tilde{l}_{\tilde{f}_1^*}(v_i,v_j) \le \tilde{l}_{\tilde{u}}(v_i,v_j), \forall (v_i,v_j) \in A \quad (29)$$

$$\sum_{\{v_j:(v_i,v_j)\in A\}} \tilde{l}_{\tilde{f}_1^*}(v_i,v_j) - \sum_{\{v_j:(v_j,v_i)\in A\}} \tilde{l}_{\tilde{f}_1^*}(v_j,v_i) = 0, \forall v_i \in V - \{s,\tau\} \quad (30)$$

By satisfying a fuzzy time-windows constraint, $\tilde{t}_{v_i} + \tilde{t}_{v_iv_j} \le \tilde{t}_{v_j}; \tilde{a}_{v_i} \le \tilde{t}_{v_i} \le \tilde{b}_{v_i}; \tilde{t}_{v_i}, \tilde{t}_{v_iv_j} \in \tilde{T}; v_i \ne v_j; \forall v_i, v_j \in V$. In the same way, $\tilde{r}_{\tilde{f}_2^*}$ is a feasible fuzzy flow for the fuzzy $\tilde{l}$-minimum fuzzy cost fuzzy flow of the fuzzy time-windows problem, so we have:

$$\tilde{r}_{\tilde{l}}(v_i,v_j) \le \tilde{r}_{\tilde{f}_2^*}(v_i,v_j) \le \tilde{r}_{\tilde{u}}(v_i,v_j), \forall (v_i,v_j) \in A \quad (31)$$

$$\sum_{\{v_j:(v_i,v_j)\in A\}} \tilde{r}_{\tilde{f}_1^*}(v_i,v_j) - \sum_{\{v_j:(v_j,v_i)\in A\}} \tilde{r}_{\tilde{f}_1^*}(v_j,v_i) = 0, \forall v_i \in V - \{s,\tau\} \quad (32)$$

also, by satisfying a fuzzy time-windows constraint. By (18), (30) and (32), $\tilde{\tilde{f}}^*$ satisfies in (14) and by (19), (20), (21), (22) and (31), it satisfies in (17). Thus, $\tilde{\tilde{f}}^*$ is a feasible fuzzy flow for the fuzzy interval-minimum fuzzy cost fuzzy flow of the fuzzy time-windows problem. The fuzzy flow $\tilde{v}_1^*$ (resp. $\tilde{v}_2^*$) is optimal for the fuzzy $\tilde{l}$-minimum fuzzy cost fuzzy flow of the fuzzy time-windows problem (resp. the fuzzy $\tilde{r}$-minimum fuzzy cost fuzzy flow of the fuzzy time-windows problem), so by (*) and definition 2.1, we yield that $[\tilde{v}^*_{\tilde{l}}, \tilde{v}^*_{\tilde{r}}]$ is an optimal fuzzy flow for the fuzzy interval-minimum fuzzy cost fuzzy flow of the fuzzy time-windows problem.

There for, by theorem 3.1, for solving the fuzzy interval-minimum fuzzy cost fuzzy flow of the fuzzy time-windows problem, it is enough that we solve the fuzzy $\tilde{l}$-minimum fuzzy cost fuzzy flow of the fuzzy time-windows problems, which yields the following theorem.

Theorem 3.2 *The minimum fuzzy cost fuzzy flow of the fuzzy time-windows problem with the fuzzy interval-valued fuzzy lower bound, fuzzy upper bounds and fuzzy flows is solved using two fuzzy minimum cost fuzzy flow of the fuzzy time-windows problem with the crisp data.*

4 The Fuzzy Minimum Cost Fuzzy Flow of the Fuzzy Time-Windows Problem According to Zadeh's Extension Principle

In this section, the minimum fuzzy cost fuzzy flow of the fuzzy time-windows problem with fuzzy lower, upper bounds and fuzzy flows is solved using Theorem 3.2. Consider a fuzzy directed network $\tilde{G} = \left(V, A, \tilde{u}_{v_i v_j}, \tilde{l}_{v_i v_j}, \tilde{t}_{v_i v_j}\right)$, where V is a set of n vertices, A is a set of n arcs such that the fuzzy time-windows, fuzzy lower bound, fuzzy upper bound, and fuzzy flow of each arc are known to fall within specific ranges expressed as compact fuzzy intervals $\tilde{w}, \tilde{l}, \tilde{\tilde{l}}$ and $\tilde{f}$, respectively. We call the minimum fuzzy cost fuzzy flow of the fuzzy time-windows problem with fuzzy data as the minimum fuzzy cost fuzzy flow of the fuzzy time-windows problem. As it was mentioned in the above of the application of the minimum fuzzy flow problem, the fuzzy interval representation of the fuzzy $\tilde{\beta}$-level allows extending classical interval arithmetic to the case of fuzzy numbers. Interval arithmetic can be directly applied to every $\tilde{\beta}$-level to obtain the resulting fuzzy set. For each $\tilde{\beta} \in [0,1]$ and each arc $(v_i, v_j) \in A$, we define the $\tilde{\beta}$-level sets corresponding to $\tilde{w}, \tilde{l}, \tilde{u}$ and $\tilde{f}$ as follows:

$$[\tilde{f}_{v_i v_j}]^{\tilde{\beta}} = \tilde{f}_{v_i v_j}(\tilde{\beta}) = [\tilde{f}_{\tilde{l}}\left((v_i, v_j), \tilde{\beta}\right), \tilde{f}_{\tilde{r}}\left((v_i, v_j), \tilde{\beta}\right)] \quad (33)$$

$$[\tilde{u}_{v_i v_j}]^{\tilde{\beta}} = \tilde{u}_{v_i v_j}(\tilde{\beta}) = [\tilde{u}_{\tilde{l}}\left((v_i, v_j), \tilde{\beta}\right), \tilde{u}_{\tilde{r}}\left((v_i, v_j), \tilde{\beta}\right)] \quad (34)$$

$$[\tilde{l}_{v_iv_j}]^{\tilde{\beta}} = \tilde{l}_{v_iv_j}(\tilde{\beta}) = [\tilde{l}_{\tilde{l}}\left(\left(v_i, v_j\right), \tilde{\beta}\right), \tilde{l}_{\tilde{r}}\left(\left(v_i, v_j\right), \tilde{\beta}\right)] \quad (35)$$

$$[\widetilde{w}_{v_iv_j}]^{\tilde{\beta}} = \widetilde{w}_{v_iv_j}(\tilde{\beta}) = [\widetilde{w}_{\tilde{l}}\left(\left(v_i, v_j\right), \tilde{\beta}\right), \widetilde{w}_{\tilde{r}}\left(\left(v_i, v_j\right), \tilde{\beta}\right)] \quad (36)$$

The minimum fuzzy cost fuzzy flow of the fuzzy time-windows problem with compact fuzzy interval-valued lower, upper bounds and fuzzy flows are given by the following:

$$min[\tilde{v}_{\tilde{l}}(\tilde{\beta}), \tilde{v}_{\tilde{r}}(\tilde{\beta})]$$

subject to:

$$\left[\sum_{\{v_j:(v_i,v_j)\in A\}} \tilde{l}_{\tilde{f}}((v_i, v_j), \tilde{\beta})), \sum_{\{v_j:(v_i,v_j)\in A\}} \tilde{r}_{\tilde{f}}((v_i, v_j), \tilde{\beta}))\right] -$$

$$\left[\sum_{\{v_j:(v_j,v_i)\in A\}} \tilde{l}_{\tilde{f}}((v_j, v_i), \tilde{\beta})), \sum_{\{v_j:(v_j,v_i)\in A\}} \tilde{r}_{\tilde{f}}((v_j, v_i), \tilde{\beta}))\right] =$$

$$\left\{\begin{array}{ll} [\tilde{v}_{\tilde{l}}(\tilde{\beta}), \tilde{v}_{\tilde{r}}(\tilde{\beta})], & v_i = s \\ -[\tilde{v}_{\tilde{l}}(\tilde{\beta}), \tilde{v}_{\tilde{r}}(\tilde{\beta})], & v_i = \tau \\ [0,0], & \forall v_i \in V - \{s.\tau\} \end{array}\right\} \quad (37)$$

$$\tilde{r}_{\tilde{l}}\left(\left(v_i, v_j\right), \tilde{\beta}\right) \le \tilde{r}_{\tilde{f}}\left(\left(v_i, v_j\right), \tilde{\beta}\right) \le \tilde{r}_{\tilde{u}}\left(\left(v_i, v_j\right), \tilde{\beta}\right), \forall\left(v_i, v_j\right) \in A \quad (38)$$

$$\tilde{l}_{\tilde{l}}\left(\left(v_i, v_j\right), \tilde{\beta}\right) \le \tilde{l}_{\tilde{f}}\left(\left(v_i, v_j\right), \tilde{\beta}\right) \le \tilde{l}_{\tilde{u}}\left(\left(v_i, v_j\right), \tilde{\beta}\right), \forall\left(v_i, v_j\right) \in A \quad (39)$$

$$\widetilde{w}_{\tilde{l}}\left(\left(v_i, v_j\right), \tilde{\beta}\right) \le \widetilde{w}_{\tilde{f}}\left(\left(v_i, v_j\right), \tilde{\beta}\right) \le \widetilde{w}_{\tilde{u}}\left(\left(v_i, v_j\right), \tilde{\beta}\right), \forall\left(v_i, v_j\right) \in A \quad (40)$$

We call the fuzzy interval-valued fuzzy time-windows network with data (38), (39), (40) and (41) as the $\tilde{\beta}$-fuzzy interval minimum fuzzy flow of fuzzy time-windows network. The interval fuzzy flow $[\tilde{f}]^{\tilde{\beta}}$ is feasible in the $(\tilde{G}, \tilde{\beta})$ network if it satisfies in (37), (38), (39) and (40). There for $\tilde{f}$ is a feasible fuzzy flow for the fuzzy-minimum flow of the fuzzy time-windows problem if, at each $\tilde{\beta}$-level, $[\tilde{f}]^{\tilde{\beta}}$ is a feasible fuzzy flow in the $\tilde{\beta}$-interval minimum fuzzy flow of the fuzzy time-windows problem. At each $\tilde{\beta}$-level, we define the $\tilde{\beta}$-interval minimum fuzzy flow of the fuzzy time-windows problem

$$min[\tilde{v}_{\tilde{l}}(\tilde{\beta}), \tilde{v}_{\tilde{r}}(\tilde{\beta})]$$

subject to: $\tilde{\tilde{f}}(\cdot, \tilde{\beta})$ satisfies in (44), (45), (46) and (47).

Hence, for each $\tilde{\beta} \in [0,1]$, a fuzzy interval-valued of minimum fuzzy flow $\tilde{\tilde{f}}^*\left(\left(v_i, v_j\right), \tilde{\beta}\right) = [\tilde{l}_{\tilde{f}^*}\left(\left(v_i, v_j\right), \tilde{\beta}\right), \tilde{r}_{\tilde{f}^*}\left(\left(v_i, v_j\right), \tilde{\beta}\right)]$, for each arc $\left(v_i, v_j\right) \in A$, is found by solving the $\tilde{\beta}$-interval minimum fuzzy flow of the fuzzy time-windows problem. By

Theorem 3.2, $\tilde{l}_{\tilde{f}^*}\left(\left(v_i,v_j\right),\tilde{\beta}\right)$'s and $\tilde{r}_{\tilde{f}^*}\left(\left(v_i,v_j\right),\tilde{\beta}\right)$'s are computed using $\tilde{l}-\tilde{\beta}$ −interval and $\tilde{r}-\tilde{\beta}$ −interval minimum fuzzy cost fuzzy flow of the fuzzy time-windows problems defined by the following:

- The $\tilde{l}-\tilde{\beta}$ −interval minimum fuzzy cost fuzzy flow of the fuzzy time-windows problem:

$$min\tilde{v}_{\tilde{l}}(\tilde{\beta})$$

subject to:

$$\sum_{\{v_j:(v_i,v_j)\in A\}}\tilde{l}_{\tilde{f}}((v_i,v_j),\tilde{\beta})-\sum_{\{v_j:(v_j,v_i)\in A\}}\tilde{l}_{\tilde{f}}((v_j,v_i),\tilde{\beta})=\begin{Bmatrix}\tilde{v}_{\tilde{l}}(\tilde{\beta}), & v_i=s\\ -\tilde{v}_{\tilde{l}}(\tilde{\beta}), & v_i=\tau\\ 0,\forall v_i\in V-\{s,\tau\}\end{Bmatrix} \quad (41)$$

$$\tilde{l}_{\tilde{l}}\left(\left(v_i,v_j\right),\tilde{\beta}\right)\le\tilde{l}_{\tilde{f}}\left(\left(v_i,v_j\right),\tilde{\beta}\right)\le\tilde{l}_{\tilde{u}}\left(\left(v_i,v_j\right),\tilde{\beta}\right),\forall\left(v_i,v_j\right)\in A \quad (42)$$

$$\widetilde{w}_{\tilde{l}}\left(\left(v_i,v_j\right),\tilde{\beta}\right)\le\widetilde{w}_{\tilde{f}}\left(\left(v_i,v_j\right),\tilde{\beta}\right)\le\widetilde{w}_{\tilde{u}}\left(\left(v_i,v_j\right),\tilde{\beta}\right),\forall\left(v_i,v_j\right)\in A \quad (43)$$

- The $\tilde{r}-\tilde{\beta}-$ interval minimum fuzzy cost fuzzy flow of the fuzzy time-windows problem:

$$min\tilde{v}_{\tilde{r}}(\tilde{\beta})$$

subject to:

$$\sum_{\{v_j:(v_i,v_j)\in A\}}\tilde{r}_{\tilde{f}}((v_i,v_j),\tilde{\beta})-\sum_{\{v_j:(v_j,v_i)\in A\}}\tilde{r}_{\tilde{f}}((v_j,v_i),\tilde{\beta})=\begin{Bmatrix}\tilde{v}_{\tilde{r}}(\tilde{\beta}), & v_i=s\\ -\tilde{v}_{\tilde{r}}(\tilde{\beta}), & v_i=\tau\\ 0,\forall v_i\in V-\{s,\tau\}\end{Bmatrix} \quad (44)$$

$$\tilde{r}_{\tilde{l}}\left(\left(v_i,v_j\right),\tilde{\beta}\right)\le\tilde{r}_{\tilde{f}}\left(\left(v_i,v_j\right),\tilde{\beta}\right)\le\tilde{r}_{\tilde{u}}\left(\left(v_i,v_j\right),\tilde{\beta}\right),\forall\left(v_i,v_j\right)\in A \quad (45)$$

$$\widetilde{w}_{\tilde{l}}\left(\left(v_i,v_j\right),\tilde{\beta}\right)\le\widetilde{w}_{\tilde{f}}\left(\left(v_i,v_j\right),\tilde{\beta}\right)\le\widetilde{w}_{\tilde{u}}\left(\left(v_i,v_j\right),\tilde{\beta}\right),\forall\left(v_i,v_j\right)\in A \quad (46)$$

Since the representation of the $\tilde{\beta}$ −levels are used instead of the fuzzy numbers, by Zadeh's extension, the result is accorded with Zadeh's extension principle. In general, any function of k fuzzy intervals $\tilde{F}(\tilde{A}_1,\tilde{A}_2,\ldots,\tilde{A}_k)$ of k fuzzy intervals $\tilde{A}_1,\tilde{A}_2,\ldots,\tilde{A}_k$ can be extended to fuzzy by defining $[\tilde{F}(\tilde{A}_1,\ \tilde{A}_2,\ldots,\ \tilde{A}_k)]^{\tilde{\beta}}=\tilde{F}([\tilde{A}_1]^{\tilde{\beta}},[\tilde{A}_2]^{\tilde{\beta}},\ldots,[\tilde{A}_k]^{\tilde{\beta}})$.

However, unless $\tilde{F}$ preserves inclusion, to get a fuzzy number as the result, we must modify the definition so that the level set $\tilde{\beta}$ is a subset of $\tilde{\beta} > \tilde{\beta}'$. There for we define by Bondia, Sala and Sainz [2]:

$$[\tilde{F}(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_k)]^{\tilde{\beta}} = \cap_{0 \le \tilde{\beta}' \le \tilde{\beta}} \tilde{F}([\tilde{A}_1]^{\tilde{\beta}'}, [\tilde{A}_2]^{\tilde{\beta}'}, \dots, [\tilde{A}_k]^{\tilde{\beta}'}) \quad (47)$$

For $\tilde{\beta} \in [0,1]$ and its $(\tilde{G}, \tilde{\beta})$ problem, consider the fuzzy interval-valued minimum fuzzy cost fuzzy flow of the fuzzy time-windows $\tilde{\tilde{f}}^*\left((v_i, v_j), \tilde{\beta}\right) = [\tilde{l}_{\tilde{f}^*}\left((v_i, v_j), \tilde{\beta}\right), \tilde{r}_{\tilde{f}^*}\left((v_i, v_j), \tilde{\beta}\right)]$, for each arc $(v_i, v_j) \in A$. Let $\tilde{z}^*(\tilde{\beta}) = \sum_{(v_i, v_j) \in A} \tilde{\tilde{c}}\left((v_i, v_j), \tilde{\beta}\right) \tilde{\tilde{f}}^*\left((v_i, v_j), \tilde{\beta}\right)$.

- **Application Instance network:**

(i) The representation example of a fuzzy network with fuzzy time-widows, fuzzy bounds, and fuzzy flows, for a given $\tilde{\beta}$:

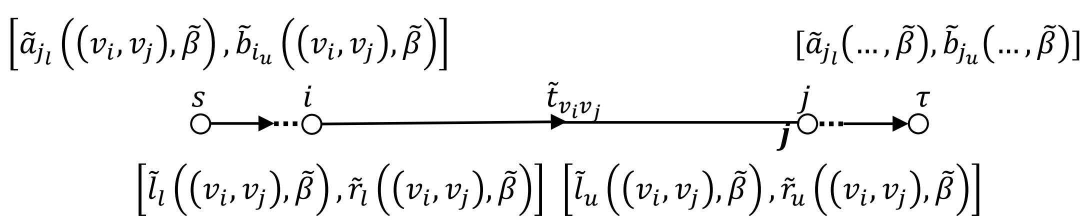


Figure 6: A network with fuzzy time-windows, fuzzy bounds and fuzzy flows

(ii) The representation example of a fuzzy network corresponding to the $\tilde{l} - \tilde{\beta}$ fuzzy interval minimum fuzzy cost fuzzy flow of the fuzzy time-windows problem:

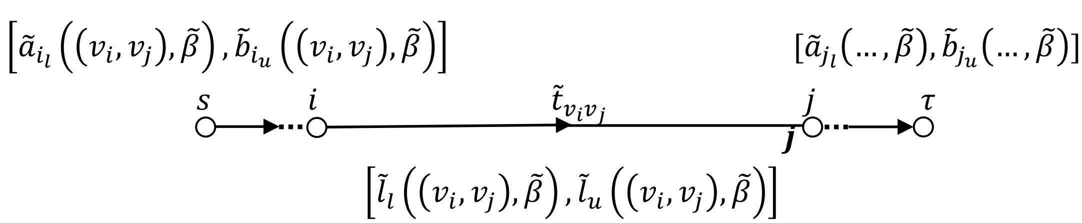


Figure 7: A network of $\tilde{l} - \tilde{\beta}$ fuzzy interval minimum fuzzy cost fuzzy flow of the fuzzy time-windows problem

(iii) The representation example a fuzzy network corresponding $\tilde{r} - \tilde{\beta}$ fuzzy interval minimum fuzzy cost fuzzy flow of the fuzzy time-windows problem:

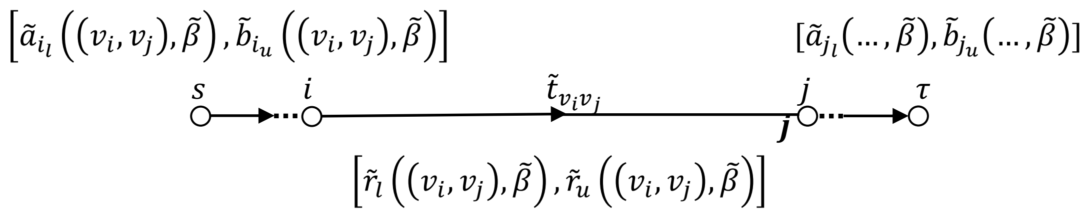


Figure 8: A fuzzy network of the $\tilde{r} - \tilde{\beta}$ fuzzy interval minimum fuzzy cost fuzzy flow of the fuzzy time-windows problem

5 Conclusion

In this paper, we present and described a new version of the Minimum Cost Flow Problem (MCFP), a new version is an FMCFFPFTWIB. In [19], presented the method to solve the minimum cost flow problem with interval date, which solves the problem using the two minimum cost flow problems with crisp data. This paper extended the method of [19], by using the two-minimum fuzzy cost fuzzy flow of the fuzzy time-windows problems with crisp data. Also, this method is extended to the minimum fuzzy cost fuzzy flow problem with fuzzy time-windows, fuzzy lower, fuzzy upper bounds and fuzzy flow. An application example of the fuzzy network is given.

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