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Some common fixed point theorems of sequentially compact and complete cone -2- metric spaces of a contractive mapping

L Shambhu Singh[1], Th. Chhatrajit Singh[2] and K Anthony Singh[1]

Abstract

In this paper, we prove some common fixed point theorems of sequentially compact and complete cone 2-metric spaces of a contractive mappings. We have proved four lemmas for the main theorems. Our results improve and generalize some of the results of B Singh etal(15) etc.

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1 Introduction

The notion of 2-metric space was introduced by S. Gahler in series of papers [1], [2],[3] etc. published in the 60's whose abstract properties were suggested by the area function for a triangle determined by a triplet in the Euclidean Space.

L.G. Huang and X. Zhang [10] introduced the concept of cone metric space as a generalization of a metric space on replacing real numbers by an ordered Banach Space and obtained some fixed points of contractive mappings. Using both the two concepts B.Singh etal [15] introduced cone 2-metric space and

[1] Department of Mathematics, D.M. College of Science. Imphal Manipur India
[2] Department of Mathematics, Manipur Technical University, Imphal Manipur India

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proved some common fixed point theorems. We now extend and generalize the results of [15] for a pair of self-mappings using a contractive mapping.

2 Preliminary Notes

We need the following definitions for our main theorems.

Definition 1.1: A non-empty set X with at least three points together with a mapping $d : X \times X \times X \to R^+$, is called a 2-metric space if d satisfies.

i) To each pair of points x and $y, x \neq y$ in X there exists a point $z \in X$ such that $d(x, y, z) \geq 0$.

ii) $d(x, y, z) = 0$ if at least two of $x, y\, and\, z$ are equal.

iii) $d(x, y, z) = d(y, z, x) = d(z, x, y) \forall x, y, z \in X$

iv) $d(x, y, z) \leq d(x, y, w) + d(x, w, z) + d(w, y, z)$ for all $x, y, z, w \in X$

It is denoted by the ordered pair (X, d). The mapping d is called a 2-metric defined on X

Definition 1.2 [15]: A cone 2-metric space is defined as follows. Let E be a real Banach Space and P, a subset of E. P is called a cone of E if

i) P is closed, non-empty and $P \neq \{0\}$

ii) $a, b \in IR^+, a, b \geq 0\, and\, x, y \in P \Rightarrow ax + by \in P$

iii) $x \in P\, and - x \in P \Rightarrow x = 0$

Given a cone $P \subset E$, we define a partial ordering $\leq$ in E with respect to P by $x \leq y$ if and only if $y - x \in P$. A cone P is normal if for every $x, y \in P$ there exist a number $K > 0$ such that $x \leq y \Rightarrow \|x\| \leq K\|y\|$.

The least of K is the normal constant of the cone P. The symbol $x << y$ is used to denote that $y - x \in Int(P)$, interior of P in E.

A cone P is regular if for every non-decreasing sequence which is bounded above is convergent i.e. if $\{x_n\}$ be a sequence such that $x_1 \leq x_2 \leq .... \leq x_n \leq ..... \leq y \leq ....$ for some $y \in P$ then there exists some $x \in P$ such that $x_n \to x\, or \|x_n - x\| \to 0\, as\, n \to \infty$

Equivalently $x_n \to x\, as\, n \to \infty$ if $\{x_n\}$ is a non-increasing sequence which is bounded from below..

Definition 1.3 : Let X be a non-empty set with at least three points. Let $d : X \times X \times X \to P$ be a mapping such that

i) $d(x,y,z)\geq 0,\forall x,y,z\in X$ and $d(x,y,z)=0$ if at least two of $x,y\,or\,z$ are equal.

ii) $d(x,y,z)=d(y,z,x)=d(z,x,y),\forall x,y,z\in X$.

iii) $d(x,y,z)\leq d(x,y,w)+d(x,w,z)+d(w,x,z)$ for all $x,y,z,w\in X$.

Such a mapping '*d*' is called a cone 2-metric and the pair *(X,d)* is called a cone 2-metric space.

Definition 1.4 : Let (X,d) be a cone 2-metric space with respect to a cone P in a real Banach Space E, Let $\{x_n\}$ be a sequence in X and let $x\in X$. If for every $c\in Int(P)$ with 0<<c there exist a number N(c) such that $d(x_n,x,a)<<c\forall a\in X,n\geq N(c)$.

Then $\{x_n\}$ is called a convergent sequence converging to *x*. *x* is the limit of the sequence $\{x_n\}$. We denote it by $x_n\to x$ or $\lim_{n\to\infty}x_n=x$.

Definition 1.5 : Let $\{x_n\}$ be a sequence in a cone 2-metric space (X,d). For each $c\in Int(P)$ there is a number N(c) such $d(x_n,x_m,a)<<c$ for all $a\in X\,and\,n,m\geq N(c)$. Such a sequence $\{x_n\}$ is called a Caucly Sequence.

Definition 1.6 : If every sequence $\{x_n\}$ in a cone 2-metric space has a convergent subsequence then X is said to be a sequentially compact cone 2-metric space. A compact space is a sequentially compact space.

3 Main Results

We prove some results in cone 2-metric spaces.

Lamma 2.1 : Every regular cone is a normal cone with a normal constant.

Proof: Let $\{x_n\}$ be a non-decreasing sequence which is bounded. Then exists a real number $M\in N$ such that each $x_i\leq M,\forall i$ if $\{x_n\}\subset X\;i.e. M-x_i\in Int(P)$ i.e $\|M-x_i\|\to 0\;\;as\,i\to\infty$. i.e $\{x_n\}$ is convergent. Also for some $i\,and\,j,i\leq j$ We have, $\|x_i\|\leq K\|x_j\|,\forall i\,and\,j$. Such K has least value and hence P is a normal cone with a normal constant. If $\{x_n\}$ be a non-increasing sequence and $x_{n+1}\leq x_n\forall n\in N$ then $\|x_{n+1}\|\leq K\|x_n\|,\forall n$ and hence P is normal cone with a normal constant K.

Lamma 2.2 : Every convergent sequence $\{x_n\}$ in a cone 2-metric space has a convergent subsequence in X or Every subsequence of a convergent sequence in a cone 2-metric space converges.

Proof : Let $\{x_n\}$ be convergent to some $x \in X$. Then for each $c \in Int(P), P$ is normal cone with normal constant K. there exists a number N(c) such that

$d(x_n, x, a) << c, \quad \forall n \geq N(c)$

Choosing $c, 0 << c$ such that $K\|c\| <\in, \; for \in > 0$.

We have

$\|d(x_n, x, a)\| \leq K\|c\| <\in \qquad \forall n \geq N(c)$

$\Rightarrow \|d(x_{n(\lambda)}, x, a)\| \leq \in \qquad \forall n(\lambda) \geq N \; (c \qquad a \in X$

$\Rightarrow d(x_{n(\lambda)}, x, a) \to 0 \qquad as\, \lambda \to \infty$

$\Rightarrow \lim_{\lambda\to\infty} x_{n(\lambda)} = x$

$\therefore \left\{x_{n(\lambda)}\right\} \subset \left\{x_n\right\}$ converges to $x \in X$.

As $\left\{x_{n(\lambda)}\right\}$ is arbitrary, any subsequence of a convergent sequence $\left\{x_n\right\}$ convereges.

Lemma 2.3 : Every convergent sequence is Cauchy in a cone 2-metric space but the converse is not true.

Proof : Let $\left\{x_n\right\}$ be convergent

There for each $c \in Int(P)$ there exists a number $N(c)$ such that $d(x_n, x, a) << c$ for all $a \in X, n \geq N(c)$, Now for some $m, n \geq N(c)$

$d(x_n, x_m, a)$

$\leq d(x_n, x, a) + d(x, x_m, a) + d(x_n, x_m, x)$

$< d(x_n, x, a) + d(x, x_m, a) + d(x_n, x_m, x)$

$< \frac{c}{3} + \frac{c}{3} + \frac{c}{3} = c$

$i.e. d(x_n, x_m, a) \to 0\, as\, n, m \to \infty$

$\therefore \{x_n\}$ is a Cauchy sequence.

Conversely if $\{x_n\}$ is Cauchy sequence, then for some $c \in Int(P)$.

We have,

$\therefore d(x_n, x_m, a) \leq d(x_n, x_m, x) + d(x_n, x, a) + d(x, x_m, a), \forall\, \mathrm{n, m} \geq \mathrm{N(c)}$

$\therefore 0 < d(x_n, x, a), \forall n \geq N(c)$

$Hence, \lim d(x_n, x, a) \neq 0$

$or \quad \lim_{n\to\infty} x_n \neq x$

$\therefore \{x_n\}$ does not converge to x.

This completes the proof.

Lemma 2.4 : If every subsequence of a sequence $\{x_n\}$ converges to the same limit x in a cone 2-metric space (X, d) then x is the limit of the sequence $\{x_n\}$ in X.

Proof :

Let $\{x_{n(\lambda)}\} \subset \{x_n\}$ be a subsequence such that $x_{n(\lambda)} \to x \ as\ \lambda \to \infty$

Therefore, given $c \in Int(P)$, there exists a number N(c) and $x \in X$ such that $d(x_{n(\lambda)}, x, a) << c$ or equivalently.

$d(x_{n(\lambda)}, x, a) < \frac{\in}{3}\ if\ K\|c\| < \in \quad and \ \in > 0$ be chosen.

Now,

$d(x_n, x, a)$

$\leq d(x_n, x, x_{n(\lambda)}) + d(x_n, x_{n(\lambda)}, a) + d(x_{n(\lambda)}, x, a)$

$\leq \frac{\in}{3} + \frac{\in}{3} + \frac{\in}{3} = \in$

$i.e, d(x_n, x, a) \to 0\ as\ n \to \infty \forall a \in X\ \&\ n \geq N(\mathrm{c})$

Thus,

$\lim_{n\to\infty} x_n = x, as\ n \to \infty$

Hence the sequence $\{x_n\}$ also converges to x.

Remark 2.5 [15];[12] : If E be a real Banach space with cone P and if $a \leq \lambda a$ where $a \in P\ and\ 0 < \lambda < 1, then\ a = 0.$

Theorem 2.6 : Let $((X, d)$ be a sequentially compact cone 2- metric space and let P be a cone with a normal constant K. Suppose the mappings $S, T : X \to X$ satisfy the following for all $x, y, z \in X$:

(2.6.1) $T(\mathrm{X})$ or $\mathrm{S(X)}$ is closed subset of X.

(2.6.2) $d(Sx, Ty, z) \leq Kd(x, y, z) + \lambda d(Sx, x, z) + \mu d(Ty, y, z) + \beta \max\{d(Sx, y, z), d(Ty, x, z)\}$

Where $0 \leq K, \lambda, \mu, \beta < 1$

and $K + \lambda + \mu + 2\beta < 1, K + \beta < 1, \mu + \beta < 1$

Then S and T have coincident points which are the unique common fixed points of S and T.

Proof : For $x_0 \in X$, define a sequence $\{x_n\}$ as follows

$x_n = Tx_{n-1}$ *if n* is odd

$x_{n+1} = Sx_n$ if n is even.

Now,

$$d(x_{n+1}, x_n, z) = d(Sx_n, Tx_{n-1}, z)$$
$$\leq Kd(x_n, x_{n-1}, z) + \lambda d(Sx_n, x_n, Z) + \mu d(Tx_{n-1}, x_{n-1}, Z)$$
$$+\beta \max\{d(Sx_n, x_{n-1}, Z), d(Tx_{n-1}, x_n, z)\}$$
$$\leq Kd(x_n, x_{n-1}, z) + \lambda d(x_{n+1}, x_n, Z) + \mu d(x_n, x_{n-1}, z)$$
$$+\beta \max\{d(x_{n+1}, x_{n-1}, z), d(x_n, x_n, z)\}$$
$$\leq Kd(x_n, x_{n+1}, z) + \lambda d(x_{n+1}, x_n, Z) + \mu d(x_n, x_{n-1}, z)$$
$$+\beta d(x_{n+1}, x_{n-1}, z)$$
$$\Rightarrow (1-\lambda-\beta) d(x_{n+1}, x_n, z) \leq (K+\mu+\beta) d(x_n, x_{n-1}, z)$$
$$\Rightarrow d(x_{n+1}, x_n, z) \leq \left(\frac{K+\mu+\beta}{1-\lambda-\beta}\right) d(x_n, x_{n-1}, z)$$
$$\Rightarrow d(x_{n+1}, x_n, z) \leq \rho d(x_n, x_{n-1}, z)$$
$$if\ \ \rho = \frac{K+\mu+\beta}{1-\lambda-\beta} < 1\ i.e. if\ \ K+\mu+\lambda+2\beta < 1$$

Also,

$$\Rightarrow d(x_n, x_{n-1}, z) \leq \rho d(x_{n-1}, x_{n-2}, z) \leq \ldots$$
$$\leq \rho^n d(x_1, x_0, z)$$

Now for some $m, n \in N, n > m$.

$$d(x_n, x_m, z) \leq d(x_n, x_m, x_{n-1}) + d(x_n, x_{n-1}, z) + d(x_{n-1}, x_m, z)$$
$$\leq \rho^{n-1} d(x_1, x_0, z) + d(x_n, x_{n-1}, z) + d(x_{n-1}, x_m, z)$$
$$\leq (\rho^{n-1} + \rho^{n-2}) d(x_1, x_0, z) + d(x_{n-2}, x_m, z)$$
$$\leq \left(\rho^{n-1}\rho^{n-2} + \ldots + \rho^m\right) d\left(x_1, x_0, z\right)$$
$$\leq \rho^m \left(1 + \rho + \rho^2 \ldots + \rho^{n-m-1}\right) d\left(x_1, x_0, z\right)$$
$$\leq \frac{\rho^m}{1-\rho} d\left(x_1, x_0, z\right)$$

$$\Rightarrow \quad \| d(x_n, x_m, z) \| \le \frac{\rho^m}{1-\rho} \| d(x_1, x_0, z) \|$$

$$\Rightarrow \quad \| d(x_n, x_m, z) \| \to o \text{ as } n,m \to \infty \qquad \text{by remark (2.5)}$$

$$\Rightarrow \quad \lim_{n,m\to\infty} d(x_n, x_m, z) = o$$

Thus, $\{x_n\}$ is a Cauchy sequence in $T(X)$ *or* $S(X)$ which has a convergent subsequence. Let $\{x_{n(k+1)}\} \subset \{x_n\}$ be the convergent subsequence such that $x_{n(k+1)} \to x \text{ as } k \to \infty$

$$\therefore \lim_{k\to\infty} x_{n(k+1)} = \lim_{k\to\infty} S\, x_{n(k)} \Rightarrow \quad Sx = x$$

To prove $x = Tx$. *If not*, $x \ne Tx$.

Now, $d(x, Tx, z)$

$$= d\left(Sx, \lim_{n\to\infty} Tx_n, z\right)$$

$$= \lim_{n\to\infty} d(Sx, Tx_n, z)$$

$$= \lim_{n\to\infty} d\left\{Kd(x, x_n, z) + \lambda d(Sx, x, z) + \mu d(Tx_n, x_n, z) + \beta \max\left(d(Sx, x_n, z), d(Tx_n, x, z)\right)\right\}$$

$$\le \left(Kd(x,x,z) + \lambda d(sx,x,z) + \mu d(Tx,x,Z) + \beta \max\left(d(Sx,x,z), d(Tx,x,z)\right)\right)$$

$$\therefore \ d(x, Tx, z) \le (\mu + \beta) d(Tx, x, z)$$

$$\Rightarrow T(x, Tx, z) = 0 \qquad by\ remark[2.5]$$

$$\Rightarrow Tx = x$$

Thus, $Sx = Tx = x \Rightarrow$ The coincident points of S and T is the common fixed point of the mappings S and T.

We prove that the fixed point is unique .

If possible, let z' be another fixed point of S and T, then $z = z'$.

Now,

$$d(z, z', a)$$

$$= d(Sz, Tz', a)$$

$$\le Kd(z, z', a) + \lambda d(sz, z, a) + \mu d(Tz', z', a) + \beta \max\left\{d(Sz, z', a), d(Tz, z', a)\right\}$$

$$\le Kd(z, z', a) + \lambda d(z, z, a) + \mu d(z', z', a) + \beta \max\left\{d(z, z', a), d(z', z', a)\right\}$$

$$\le (K + \beta) d(z, z', a)$$

$$i.e, d(z, z', a) \le (k + \beta) d(z, z' a)$$

$$i.e, d(z, z', a) = o \qquad by\ remark(2.5)$$

$$\therefore z = z'$$

Thus, the fixed point s of S and T is unique.

Theorem 2.7 : Let (X, d) be a complete cone 2-metric space and let P be a normal cone with normal constant K suppose the mappings $S, T : X \to X$ satisfy the contractive conditions :

(2.6.1) $d(SxTy,z) \le Kd(x,y,z) + \lambda d(Sx,x,z) + \mu d(Ty,y,z) + \beta max\{d(Sx,y,z),d(Ty,x,z)\}$

where K,λ,μ and β are constants and $o \le K,\lambda,u,\beta < 1$ and

(2.6.2) $K+\lambda+u+2\beta<1,$with $K+\beta<1,\mu+\beta<1$

Thus S and T have coincident points which are the unique common fixed point of the mappings S and T

Proof : For $x_0 \in X$, We define a sequence $\{x_n\}$ as

$x_n = Tx_{n-1}$ if n is odd.

$x_{n+1} = Sx_n$ if n is even.

Then continuing as in the proof of Theorem 2.6 are can prove that the sequence $\{x_n\}$ is Cauchy

Since X is complete the sequence $\{x_n\}$ converges to a unique limit i.e. if $z \in X$ such that $x_n \to z$ as $n \to \infty$

Then $Sz = Tz = z,$ Such a fixed point is unique.

The details of the proof are omitted.

Theorem 2.8 : Let (X,d) be a complete cone -2 metric space and Let P be a normal cone with a normal constant K Suppose the mapping $T : X \to X$ satisfies for all $x,y,z \in X : d(Tx,Ty,Z) \le Kd(x,y,z) + \lambda d(Tx,x,z) + \mu d(Ty,y,z) + \beta max\{d(Tx,y,z),d(Ty,x,z)\}$

For some constants $K,\lambda,\mu \& \beta$ such that $o \le k,\lambda,\mu,\beta < 1$ and

$$K+T+\mu+2\beta<1,\lambda+\mu+\beta<1,K+\beta<1$$

Then T has a unique fixed point in X.

Proof: For some $x_o \in X$ define a sequence $\{x_n\}$ in X by

$x_1 = Tx_o$

$x_2 = Tx_1 = T^2x_o ....,x_{n+1} = Tx_n = T^{(n+1)}x_o$

Now,

$d\left(x_{n+1},x_n,z\right)=d\left(Tx_n,Tx_{n-1},z\right)$
$\leq Kd\left(x_n,x_{n-1},Z\right)+\lambda d\left(Tx_n,x_n,z\right)+\mu d\left(Tx_{n-1},x_{n-1},z\right)+\beta max\left\{d\left(Tx_n,x_{n-1},z\right),d\left(Tx_{n-1},x_n,z\right)\right\}$
$\Rightarrow d\left(x_{n+1},x_n,x\right)\leq Kd\left(x_n,x_{n-1},x\right)+\lambda d\left(x_{n+1},x_n,z\right)+\mu d\left(x_n,x_{n-1},z\right)+\beta max\left\{d\left(x_{n+1},x_{n-1},Z\right),d\left(x_n,xn,z\right)\right\}$
$\Rightarrow\left(1-\lambda-\beta\right)d\left(x_{n+1},x_n,z\right)\leq\left(K+\mu+\beta\right)d\left(x_n,x_{n-1},z\right)$
$\Rightarrow d\left(x_{n+1},x_n,z\right)\leq\rho d\left(x_n,x_{n-1},x\right),if\ \rho=\left(\frac{k+u+\beta}{1-\lambda-\beta}\right)<1$
$i.e.\ K+u+\lambda+2\beta<1$
$Also, d\left(x_n,x_{n-1},x\right)\leq\rho d\left(x_{n-1},x_{n-2},x\right)$
$Thus,\ d\left(x_{n+1},x_n,z\right)\leq\rho^n d\left(x_1,x_o,x\right)$

For some $m,n\in N\ and\ n>m$
We can prove that
$d\left(x_n,x_m,z\right)\to o$ as $n,m\to\infty$
$\therefore\{x_n\}$ is a Cauchy sequence in X and hence if converges to some $x\in X$.
To prove $x=Tx=\lim_{n\to\infty}Tx_n$. Assume that $x\neq Tx$.
Now,

$d\left(Tx,x,z\right)$
$\leq d\left(Tx,x,Tx_n\right)+d\left(Tx,Tx_n,z\right)+d\left(Tx_n,Tx_n,z\right)$
$\leq d\left(Tx,Tx_n,x\right)+d\left(Tx,Tx_n,z\right)+d\left(Tx_n,x,z\right)$
$\leq\left[Kd\left(x,x_n,x\right)+\lambda d\left(Tx,x,x\right)+\mu d\left(Tx_n,x_n,x\right)+\beta max\left\{d\left(Tx,x_n,x\right),d\left(Tx_n,x,x\right)\right\}\right]$
$+\left[Kd\left(x,x_n,z\right)+\lambda d\left(Tx,x,x\right)+\mu d\left(Tx_n,x_n,z\right)\right]$
$+\beta\ max\left\{d\left(Tx,x_n,z\right),d\left(Tx_n,x,z\right)\right\}+d\left(Tx_n,x,z\right)$
$\leq\mu d\left(Tx_n,x_n,x\right)+\beta.d\left(Tx,x_n,x\right)+Kd\left(x,x_n,z\right)$
$+\lambda d\left(Tx,x,z\right)+\mu d\left(Tx_n,x_n,z\right)$
$+\beta\ max\left\{d\left(Tx,x_n,z\right),d\left(Tx_n,x,z\right)\right\}+d\left(Tx_n,x,z\right)$

Letting $n\to\infty$ and lim $x_n=x$, we have
$d\left(Tx,x,z\right)\leq\mu d\left(Tx,x,z\right)+\beta d\left(Tx,x,x\right)+Kd\left(x,x,z\right)$
$+\lambda d\left(Tx,x,z\right)+\mu d\left(Tx,x,z\right)+\beta\ max\left\{d\left(Tx,x,z\right),d\left(Tx,x,z\right)\right\}$
$\Rightarrow d\left(Tx,x,z\right)\leq\left(\lambda+\mu+\beta\right)d\left(Tx,x,z\right)$
$\Rightarrow d(Tx,x,z)=0\ as\ \lambda+\mu+\beta<1$
$\Rightarrow Tx=x$
i.e. x is as fixed point of T.
This fixed point is unique. Let, if possible, $Ty=y$ for some $y\in X$.
Now,
$d(x,y,z)$

$$
\begin{aligned}
&= d(Tx,Ty,z) \\
&\le Kd(x,y,z) + \lambda d(Tx,x,z) + \mu d(Ty,y,z) \\
&+\beta \max\left\{d\left(Tx,y,z\right), d\left(Ty,x,z\right)\right\} \quad \text{using (2.8.1)} \\
&\le Kd(x,y,z) + \lambda d(x,x,z) + \mu d(y,y,z) \\
&+\beta \max\left\{d\left(x,y,z\right), d(y,x,z)\right\} \\
&\le \left(K+\beta\right) d(x,y,z)
\end{aligned}
$$

i.e. $d(x,y,z) \le \left(K+\beta\right) d\left(x,y,z\right), K+\beta < 1.$

$\therefore d(x,y,z) = 0 \forall z \in X$ using remark (2.5)

i.e. $x = y$

$\therefore$ The fixed point of T is unique in $\left(X,d\right)$.

Corollary Theorem 2.9:

Let (X,d) be a complete cone 2-metric space and P be a normal cone with a normal constant K. Suppose $T : X \to X$ satisfies, $d(Tx,Ty,c) \le Kd(x,y,c) + \lambda d(Tx,x,c) + \mu d(Ty,y,c)$ for some $k,\lambda,\mu \in [0,1)$ with $k+\lambda+\mu < 1,$ then T has a unique fixed point on X and for every $x \in X$, the sequence $\left\{T^n x\right\}$ converages to the fixed point. This is the main Theorem 2.1 of [15] as a particular case of our theorem 2.7.

Proof: Putting $\beta = 0$ in Theorem 2.8 above, we get a sequence $\left\{T^n x_0\right\} \subset X$ which is a Cauchy sequence and hence converges to a point $x \in X \; as \; X$ is x_0 - orbitally complete. The uniqueness of such fixed point follows easily.

Example: Let $E = R^2$ and $P = \{(\mathrm{x},\mathrm{y}) \in R^2 : \mathrm{x},\mathrm{y} \ge 0\}$ be a normal cone in *E*. Let X be $\mathrm{X} = \{(\mathrm{x},0) : 0 \le \mathrm{x} \le 1\} \cup \{(0,\mathrm{y}) : 0 \le \mathrm{y} \le 1\}$. Define a mapping d by $\mathrm{d} : \mathrm{X} \times \mathrm{X} \times \mathrm{X} \to \mathrm{E}$ $\mathrm{d}(\mathrm{a}_1,\mathrm{a}_2,\mathrm{a}_3) = \mathrm{d}_1(\mathrm{b}_1,\mathrm{b}_2), \mathrm{a}_i \in X, \forall i = 1,2,3 \,\&\, \mathrm{b}_1,\mathrm{b}_2 \in \{\mathrm{a}_1,\mathrm{a}_2,\mathrm{a}_3\}$ such that $\|\mathrm{b}_1 - \mathrm{b}_2\| = \min\{\|a_1 - a_2\|, \|a_2 - a_3\|, \|a_3 - a_1\|\}$ and

$$\mathrm{d}_1((x,0),(\mathrm{y},0)) = \left(\frac{3}{2}|x-y|, |x-y|\right)$$

$$\mathrm{d}_1((0,\mathrm{x}),(0,\mathrm{y})) = \left(|x-y|, \frac{1}{2}|x-y|\right),$$

$$\mathrm{d}_1((x,0),(0,\mathrm{y})) = \mathrm{d}_1((0,y),(x,0)) = \left(\frac{3}{2}x + y, x + \frac{1}{2}y\right)$$

Then (X, d) is a complete cone 2-metric space. Define the mappings $S, T : X \to X$ by

$$S(x, 0) = T(x, 0) = (0, \frac{1}{2}x)$$

$$S(0, y) = T(0, y) = (\frac{1}{12}y, 0)$$

then the mappings S and T satisfy the contractive condition

$$d(Sx, Ty, z) \le kd(x, y, z) + \lambda d(Sx, x, z) + \mu d(Ty, y, z) + \beta \max\{d(Sx, y, z), d(Ty, x, z)\}$$

for all $x, y, z \in X$ with the constants assuming the values $k = \frac{1}{4}, \lambda = \mu = \frac{1}{6}$ and $\beta = \frac{1}{12}$.

The mappings S and T have a coincident point (0,0) which is the unique common fixed points. One can check that S and T are not contractive in the 2-metric space on X.

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