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# Topological pressure for set-value map

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**Abstract**

In this paper we introduce the topological pressure for set-value map. We also study some properties of it.

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**Keywords:** Set-value map; topological entropy; topological pressure.

## 1 Introduction

In 1959, Kolmogorov and Sinai developed the metric entropy from Shannon's information theory into ergodic theory. Since then, the notion of entropy has played a vital role in dynamical systems. The subject of topological entropy was first introduced by Adler, Konheim and McAndrew [1] in 1965. They take the topological entropy as an invariant of topological conjugacy and a numerical measure for the complexity of a dynamical system. In 1971, Bowen [4] provided an equivalent definition in the context of metric spaces. Topological pressure is a generalization to topological entropy for a dynamical system. In 1973, Ruelle [20] first introduced the concept of topological pressure of additive

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potentials for expansive dynamical systems. Later in 1982, Walters [24] extended this concept to the compact space with the continuous transformation. The topological pressure can be stated as follows: Let $(X, d)$ be a compact metric space and $T : X \to X$ be continuous map. Recall that $C(X, \mathbb{R})$ is the Banach algebra of real-valued continuous functions of $X$ equipped with the supremum norm. For $f \in C(X, \mathbb{R})$ and $n \geq 1$, let $(S_n f)(x) = \sum_{i=0}^{n-1} f(T^i x)$. Define

$$P(T, f) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log Q_n(T, f, \varepsilon),$$

where

$$Q_n(T, f, \varepsilon) = \inf \Big\{ \sum_{x\in S} \mathrm{e}^{(S_n f)(T^i x)} : S \text{ is an } (n, \varepsilon)\text{-spanning set } \Big\}.$$

Walters also gave the conclusion that

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log Q_n(T, f, \varepsilon) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log P_n(T, f, \varepsilon),$$

where

$$P_n(T, f, \varepsilon) = \inf \Big\{ \sum_{x\in E} \mathrm{e}^{(S_n f)(T^i x)} : E \text{ is an } (n, \varepsilon)\text{-separated set } \Big\}.$$

Topological pressure was further developed by Pesin and Pitskel [17]. Furthermore, many nonlinear physical problems involve a complex discrete dynamical system. Topological pressure contains information on the dynamics of the system, and different energy functions decide different dynamics. Related studies include [8, 9, 10, 11, 16, 18, 22, 21].

The study of the set-valued dynamical systems has been increasing along these years following the success of the single-value case. For instance, Maschler and Peleg [13] investigated the existence of endpoints for set-valued dynamical systems through the notion of stable sets. In [2] the notion of invariant measure for set-valued dynamical systems, generalizing the same concept in the single-valued case was introduced. In 2015, Carrasco-Olivera, Alvan and Rojas[5] introduced two kinds of entropies by using separated and spanning sets for set-valued maps on metric spaces, which call separated topological entropy $h_{se}(f)$ and spanning topological entropy $h_{sp}(f)$. More specifically, let $X$ be a metric space and $f$ a set-valued map of $X$. Define

$$h_{se}(f) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{\log s_{n,\varepsilon}}{n} \quad \text{and} \quad h_{sp}(f) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{\log r_{n,\varepsilon}}{n},$$

where $s_{n,\varepsilon}$ is the supremum cardinality of all $(n, \varepsilon)$-separated sets and $r_{n,\varepsilon}$ is the minimum cardinality of all $(n, \varepsilon)$-spanning sets.
Further notions of invariant or coincidence measures are given in [14, 15, 23]. In 2015, Kelly and Tennant [12] study the dynamics of upper semi-continuous, set-valued functions. By Bowen's definition of topological entropy, they defined the topological entropy of set-valued functions denoted by $h(F)$. It can be stated as follows: Let $X$ be a compact metric space, $2^X$ the set of all non-empty compact subsets of $X$ and let $F : X \to 2^X$ be a set-valued mapping. Define

$$h(F) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log s_{n,\varepsilon},$$

where $s_{n,\varepsilon}$ is the supremum cardinality of all $(n, \varepsilon)$-separated sets. They also gave the conclusion that

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log s_{n,\varepsilon} = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log r_{n,\varepsilon},$$

where $r_{n,\varepsilon}$ is the minimum cardinality of all $(n, \varepsilon)$-spanning sets. Then they demonstrate some well-known results extend naturally to the more general setting while others do not.

Continuous time set-valued dynamical systems and its relationship with economic models were defined and studied in [6]. The von Neumann model [3] is an example of a set-valued process for which much is known about the existence of equilibrium activity rays, but somewhat less is known about the calculation of these rays. The equilibria for this model are proven to exist and, they may even be calculated [7] under certain added restrictions upon the technology matrices. Models of bounded rationality utilize recursive. Optimization, generating sets of possible time paths of uncertain dynamic properties emanating from each starting point. Simulation studies [25] where recursive mathematical programming is used are becoming more numerous, but without an analytic stability theory that recognizes unstable sets, an unstable set of equilibria may allow computer roundoff error to create gross inaccuracies in the predicted time paths. Thus there are three areas in the realm of economic inquiry for which set valued dynamical theory would prove useful.

Now, we will give a definition of topological pressure for set-valued maps and study some properties of it. And prove some theorems of it, which are right for single-valued functions.

In this paper, let $(X, F)$ be a topological dynamical system, $X$ a compact metric space, $2^X$ the set of all non-empty compact subsets of $X$, and let $F : X \to 2^X$ be a set-valued mapping.

## 2 Definitions of topological pressure for set-valued maps

In this section, we will give a definition of topological pressure for set-valued maps and study some properties of it.

In [12] Kelly and Tennant defined the topological entropy of set-valued maps as following: Let $X$ be a compact metric space, $2^X$ the set of all non-empty compact subsets of $X$ and let $F : X \to 2^X$ be a set-valued mapping. We call $(X, F)$ a topological dynamical system.

A forward orbit for the system $(X, F)$ is a sequence $(x_0, x_1, x_2, ...)$ in $X$ such that for any $i \geq 0$, $x_{i+1} \in F(x_i)$. Fix $n \in \mathbb{N}$, an $n$-orbits for the system $(X, F)$ is a finite sequence $(x_0, ..., x_{n-1})$ in $X$ such that for any $i = 0, ..., n-2$, $x_{i+1} \in F(x_i)$.

Given a set $A \subseteq X$, and $n \in \mathbb{N}$, we define the following orbit spaces:

$$Orb_n(A, F) = \big\{n\text{-orbits } (x_0, ..., x_{n-1}) : x_0 \in A\big\},$$
$$\overrightarrow{Orb}(A, F) = \big\{\text{forward orbits } (x_0, x_1, ...) : x_0 \in A\big\}.$$

Each of these is given the subspace topology inherited as a subset of the respective product space. Let $d$ be the metric on $X$, and suppose that the diameter of $X$ is equal to 1. For any $n \in \mathbb{N}$, we define a metric $D$ on $\prod_{i=0}^{n-1} X$ by

$$D(\mathbf{x}, \mathbf{y}) = \max_{0 \leq i \leq n-1} d(x_i, y_i).$$

If $\mathbb{A} \in \{\mathbb{Z}, \mathbb{Z}_{\geq 0}, \mathbb{Z}_{\leq 0}\}$, then we define a metric $\rho$ on $\prod_{i \in \mathbb{A}} X$ by

$$\rho(\mathbf{x}, \mathbf{y}) = \sup_{i \in \mathbb{A}} \frac{d(x_i, y_i)}{|i| + 1}.$$

Also, for any set $L \subseteq \mathbb{A}$, we define the projection map $\pi_L : \prod_{i \in \mathbb{A}} X \to \prod_{i \in L} X$ by $\pi_L = (x_i)_{i \in L}$.

**Definition 2.1.** *[12] We say that* $E \subseteq Orb_n(X, F)$ *is* $(n, \varepsilon)$*-separated , if for any*

$$(x_0, ..., x_{n-1}), (y_0, ..., y_{n-1}) \in E,$$

$d(x_i, y_i) \geq \varepsilon$ *for at least one* $i \in \{0, ..., n-1\}$*. Let* $s_n(\varepsilon)$ *denote the supremum cardinality of all* $(n, \varepsilon)$*-separated sets .*

**Definition 2.2.** *[12] Let* $(X, F)$ *be a topological dynamical system, and let* $\varepsilon > 0$, $n \in \mathbb{N}$*. An* $(n, \varepsilon)$*-separated set for* $F$ *is an* $\varepsilon$*-separated subset of* $Orb_n(X, F)$*. Let* $s_{n,\varepsilon}(F)$ *denote the supremum cardinality of all* $(n, \varepsilon)$*-separated sets of F.*

**Definition 2.3.** *[12] Let* $(X, F)$ *be a topological dynamical system,* $F$ *a set-valued map. Given* $\varepsilon > 0$*, the* $\varepsilon$*-entropy of* $F$ *is defined to be*

$$h(F, \varepsilon) = \limsup_{n\to\infty} \frac{1}{n} \log s_{n,\varepsilon},$$

*and the topological entropy of* $F$ *is defined to be*

$$h(F) = \lim_{\varepsilon\to 0} h(F, \varepsilon),$$

*where* $s_{n,\varepsilon}$ *is defined in Definition 2.2.*

**Definition 2.4.** *[12] We say that* $S \subseteq Orb_n(X, F)$ *is* $(n, \varepsilon)$*-spanning if for any* $(x_0, ..., x_{n-1}) \in Orb_n(X, F)$*, there is a* $(y_0, ..., y_{n-1}) \in S$ *such that* $d(x_i, y_i) < \varepsilon$ *, for all* $i \in \{0, ..., n-1\}$*. Let* $r_n(\varepsilon)$ *denote the minimum cardinality of all* $(n, \varepsilon)$*-spanning sets.*

**Definition 2.5.** *[12] Let* $(X, F)$ *be a topological dynamical system, and let* $\varepsilon > 0$, $n \in \mathbb{N}$*. An* $(n, \varepsilon)$*-spanning set for* $F$ *is an* $\varepsilon$*-spanning subset of* $Orb_n(X, F)$*. Let* $r_{n,\varepsilon}(F)$ *denote the minimum cardinality of all* $(n, \varepsilon)$*-spanning sets of F.*

It was shown in [19] that

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log s_{n,\varepsilon}(F)=\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log r_{n,\varepsilon}(F),$$

where $r_{n,\varepsilon}$ is defined in Definition 2.5. Let $C(X,\mathbb{R})$ denote the Banach algebra of real-valued continuous functions of $X$ equipped with the supremum norm. Then we give the definition of topological pressure of set-valued functions by using spanning sets or separated sets. For $f\in C(X,\mathbb{R})$ and $n\geq 1$, let $(S_nf)(\mathbf{x})=\sum\limits_{i=0}^{n-1}f(x_i)$.

**Definition 2.6.** *For $f\in C(X,\mathbb{R})$, $n\geq 1$ and $\varepsilon\geq 0$ put*

$$Q_n(F,f,\varepsilon)=\inf\Big\{\sum_{\boldsymbol{x}\in S}\mathrm{e}^{(S_nf)(\boldsymbol{x})}:S \textit{ is an } (n,\varepsilon)\textit{-spanning set for } F\Big\}.$$

*Put*

$$Q(F,f,\varepsilon)=\limsup_{n\to\infty}\frac{1}{n}\log Q_n(F,f,\varepsilon).$$

*Define*

$$P(F,f):=\lim_{\varepsilon\to 0}Q(F,f,\varepsilon).$$

**Definition 2.7.** *The map $P(F,\cdot):C(X,\mathbb{R})\to\mathbb{R}\cup\{\infty\}$ defined above is called the topological pressure of $F$.*

Next we shall give an equivalent definition of pressure with separated sets.

**Definition 2.8.** *For $f\in C(X,\mathbb{R})$, $n\geq 1$ and $\varepsilon\geq 0$ put*

$$P_n(F,f,\varepsilon)=\sup\Big\{\sum_{\boldsymbol{x}\in E}\mathrm{e}^{(S_nf)(\boldsymbol{x})}:E \textit{ is an } (n,\varepsilon)\textit{-separated set for } F\Big\},$$

$$P(F,f,\varepsilon)=\limsup_{n\to\infty}\frac{1}{n}\log P_n(F,f,\varepsilon),$$

*We call $P^{'}(F,f)=\lim\limits_{\varepsilon\to 0}P(F,f,\varepsilon)$.*

**Proposition 2.9.** *For $f\in C(X,\mathbb{R})$, $n\geq 1$ and $\varepsilon\geq 0$,*

$$\begin{aligned}P(F,f)&=\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log Q_n(F,f,\varepsilon)\\&=\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log P_n(F,f,\varepsilon).\end{aligned}$$

*Proof.* Let $n \in \mathbb{N}$ and $\varepsilon > 0$. First, we show that

$$Q_n(F, f, \varepsilon) \leq P_n(F, f, \varepsilon).$$

Let $E \subseteq Orb_n(X, F)$ be an $(n, \varepsilon)$-separated set for $F$ of maximal cardinality. Then for any $\mathbf{x} \in Orb_n(X, F)$, there exists $\mathbf{y} \in E$, such that $D(\mathbf{x}, \mathbf{y}) < \varepsilon$, i.e.,

$$\max_{0 \leq i \leq n-1} d(x_i, y_i) < \varepsilon.$$

So $d(x_i, y_i) < \varepsilon$, for any $i = 0, ..., n-1$.

Then $E$ be an $(n, \varepsilon)$-spanning set for $F$. By definition of topological pressure,

$$Q_n(F, f, \varepsilon) \leq P_n(F, f, \varepsilon).$$

It follows that

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log Q_n(F, f, \varepsilon) \leq \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log P_n(F, f, \varepsilon).$$

Next, we show the opposite inequality.

Let $n \in \mathbb{N}$ and $\varepsilon > 0$. Since $f \in C(X, \mathbb{R})$, so, if $\delta > 0$ is such that $d(x, y) < \frac{\varepsilon}{2}$, implies $|f(x) - f(y)| < \delta$. Then $\mathrm{e}^{n\delta} Q_n(F, f, \frac{\varepsilon}{2}) \geq P_n(F, f, \varepsilon)$.

To see this, let $E$ be an $(n, \varepsilon)$-separated set of $F$, $S$ be an $(n, \frac{\varepsilon}{2})$-spanning set of $F$. Define $\varphi : E \to S$ by choosing, for each $\mathbf{x} \in E$, some point $\mathbf{y} = \varphi(\mathbf{x}) \in S$ with $D(\mathbf{x}, \mathbf{y}) < \frac{\varepsilon}{2}$.

Then $\varphi$ is injective, so

$$\begin{aligned}
\sum_{\mathbf{y} \in S} \mathrm{e}^{(S_n f)(\mathbf{y})} &\geq \sum_{\mathbf{y} \in \varphi(E) \subset S} \mathrm{e}^{(S_n f)(\mathbf{y})} \\
&= \sum_{\mathbf{x} \in E} \mathrm{e}^{(S_n f)(\varphi(\mathbf{x})) - (S_n f)(\mathbf{x})} \mathrm{e}^{(S_n f)(\mathbf{x})} \\
&\geq \min_{\mathbf{x} \in E} \mathrm{e}^{(S_n f)(\varphi(\mathbf{x})) - (S_n f)(\mathbf{x})} \sum_{\mathbf{x} \in E} \mathrm{e}^{(S_n f)(\mathbf{x})} \\
&= \min_{\mathbf{x} \in E} \mathrm{e}^{\sum_{i=0}^{\infty} (f(\varphi(x_i)) - f(x_i))} \sum_{\mathbf{x} \in E} \mathrm{e}^{(S_n f)(\mathbf{x})} \\
&\geq \mathrm{e}^{-\sum_{i=0}^{\infty} |f(\varphi(x_i)) - f(x_i)|} \sum_{\mathbf{x} \in E} \mathrm{e}^{(S_n f)(\mathbf{x})} \\
&\geq \mathrm{e}^{-n\delta} \sum_{\mathbf{x} \in E} \mathrm{e}^{(S_n f)(\mathbf{x})}.
\end{aligned}$$

Since $E, S$ are arbitrary, it follows that $\mathrm{e}^{n\delta}Q_n(F,f,\frac{\varepsilon}{2}) \geq P_n(F,f,\varepsilon)$.

So

$$\limsup_{n\to\infty}\frac{1}{n}\log Q_n(F,f,\frac{\varepsilon}{2})+\delta \geq \limsup_{n\to\infty}\frac{1}{n}\log P_n(F,f,\varepsilon).$$

$\delta \to 0$ as $\varepsilon \to 0$, so

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log Q_n(F,f,\frac{\varepsilon}{2}) \geq \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log P_n(F,f,\varepsilon).$$

To sum up,

$$\begin{aligned} P(F,f) &= \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log Q_n(F,f,\varepsilon) \\ &= \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log P_n(F,f,\varepsilon). \end{aligned}$$

□

## 2.1 Properties

Now we study the properties of $P(F,\cdot): C(X,\mathbb{R}) \to \mathbb{R}\cup\{\infty\}$.

**Theorem 2.10.** *Let $(X,F)$ be a topological dynamical system. If $f,g \in C(X,\mathbb{R})$, $\varepsilon > 0$ and $c \in \mathbb{R}$ then the following are true.*

*(1) $P(F,0) = h(F)$.*

*(2) $f \leq g$ implies $P(F,f) \leq P(F,g)$. In particular $h(F)+\inf f \leq P(F,f) \leq h(F)+\sup f$.*

*(3) $P(F,\cdot)$ is either finite valued or constantly $\infty$.*

*(4) $|P(F,f,\varepsilon)-P(F,g,\varepsilon)| \leq \|f-g\|$, and so if $P(F,\cdot)<\infty, |P(F,f)-P(F,g)| \leq \|f-g\|$.*

*(5) $P(F,\cdot,\varepsilon)$ is convex, and so if $P(F,\cdot)<\infty$ then $P(F,\cdot)$ is convex.*

*(6) $P(F,f+c) = P(F,f)+c$.*

*(7) $P(F,f+g) \leq P(F,f)+P(F,g)$.*

*(8) $P(F,cf) \leq cP(F,f)$ if $c \geq 1$ and $P(F,cf) \geq cP(F,f)$ if $c \leq 1$.*

*(9) $|P(F,f)| \leq P(F,|f|)$.*

*Proof.* (1) By Definition 2.8

$$P_n(F,f,\varepsilon) = \sup\Big\{\sum_{\mathbf{x}\in E}\mathrm{e}^{(S_nf)(\mathbf{x})} : E \text{ is an } (n,\varepsilon)\text{-separated set } \Big\}.$$

Then

$$P_n(F,0,\varepsilon)=S_{n,\varepsilon}(F).$$

It follows that

$$P(F,0)=\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log S_{n,\varepsilon}(F)=h(F).$$

(2) As $f\le g$, by Definition 2.6, we have $Q_n(F,f,\varepsilon)\le Q_n(F,g,\varepsilon)$. Thus, $P(F,f)\le P(F,g)$.

Moreover,

$$\begin{aligned}Q_n(F,f,\varepsilon)&=\inf\Big\{\sum_{\mathbf{x}\in S}\mathrm{e}^{(S_nf)(\mathbf{x})}:S\text{ is an }(n,\varepsilon)\text{ -spanning set }\Big\}\\&\le\inf\Big\{\sum_{\mathbf{x}\in S}\mathrm{e}^{n\sup f}:S\text{ is an }(n,\varepsilon)\text{ -spanning set }\Big\}\\&=\mathrm{e}^{n\sup f}r_{n,\varepsilon}(F).\end{aligned}$$

So

$$\begin{aligned}P(F,f)&=\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log Q_n(F,f,\varepsilon)\\&\le\lim_{\varepsilon\to 0}\sup f+\lim_{\varepsilon\to 0}\limsup_{n\to\infty}r_{n,\varepsilon}(F)\\&=\sup f+h(F),\end{aligned}$$

*i.e.*, $P(F,f)\le\sup f+h(F)$.

Similarly, we have

$$h(F)+\inf f\le P(F,f).$$

From the above

$$h(F)+\inf f\le P(F,f)\le h(F)+\sup f.$$

(3) From (2) and (1) we have

$$h(F)+\inf f\le P(F,f)\le h(F)+\sup f,$$

so $P(F,f)=\infty$ if, and only if $h(F)=\infty$.

(4)Before the proof, we have $\frac{\sup a_j}{\sup b_j}\le\sup\big(\frac{a_j}{b_j}\big)$.

Since

$$\begin{aligned}P(F,f,\varepsilon)-P(F,g,\varepsilon)&=\limsup_{n\to\infty}\frac{1}{n}\log P_n(F,f,\varepsilon)-\limsup_{n\to\infty}\frac{1}{n}\log P_n(F,g,\varepsilon)\\&=\limsup_{n\to\infty}\frac{1}{n}\log\frac{P_n(F,f,\varepsilon)}{P_n(F,g,\varepsilon)},\end{aligned}$$

and

$$\begin{aligned}\frac{P_n(F,f,\varepsilon)}{P_n(F,g,\varepsilon)} &\le \sup\Big\{\frac{\sum_{\mathbf{x}\in E}\mathrm{e}^{(S_nf)(\mathbf{x})}}{\sum_{\mathbf{x}\in E}\mathrm{e}^{(S_ng)(\mathbf{x})}} : E \text{ is an } (n,\varepsilon) \text{ -separated set } \Big\}\\ &\le \sup\Big\{\max_{\mathbf{x}\in E}\frac{\mathrm{e}^{(S_nf)(\mathbf{x})}}{\mathrm{e}^{(S_ng)(\mathbf{x})}} : E \text{ is an } (n,\varepsilon) \text{ -separated set } \Big\}\\ &\le \mathrm{e}^{n\|f-g\|}.\end{aligned}$$

So

$$P(F,f,\varepsilon)-P(F,g,\varepsilon)\le\|f-g\|.$$

Similarly,

$$\frac{P_n(F,g,\varepsilon)}{P_n(F,f,\varepsilon)}\le \mathrm{e}^{n\|f-g\|},$$

it follows that

$$P(F,g,\varepsilon)-P(F,f,\varepsilon)\le\|f-g\|.$$

Then we have

$$|P(F,f,\varepsilon)-P(F,g,\varepsilon)|\le\|f-g\|.$$

It clearly that if $P(F,\cdot)<\infty$, then

$$|P(F,f)-P(F,g)|\le\|f-g\|.$$

(5)By Hölder's inequality, if $q\in[0,1]$ and $E$ be a finite subset of $Orb_n(X,F)$, we have

$$\sum_{\mathbf{x}\in E}\mathrm{e}^{q(S_nf)(\mathbf{x})+(1-q)(S_ng)(\mathbf{x})}\le\Big(\sum_{\mathbf{x}\in E}\mathrm{e}^{(S_nf)(\mathbf{x})}\Big)^q\Big(\sum_{\mathbf{x}\in E}\mathrm{e}^{(S_ng)(\mathbf{x})}\Big)^{1-q}.$$

Therefore

$$P_n(F,qf+(1-q)g,\varepsilon)\le P_n(F,f,\varepsilon)^qP_n(F,g,\varepsilon)^{1-q}.$$

and (5) follows.

(6) By definition of topological pressure,

$$P_n(F,f+c,\varepsilon)=\sup\Big\{\sum_{\mathbf{x}\in E}\mathrm{e}^{(S_nf)(\mathbf{x})+nc} : E \text{ is an } (n,\varepsilon)\text{-separated set } \Big\}.$$

So

$$\begin{aligned}P(F,f+c)&=\lim_{\varepsilon\to0}\limsup_{n\to\infty}\frac{1}{n}\log P_n(F,f+c,\varepsilon)\\ &=P(F,f)+c.\end{aligned}$$

(7) Let $E$ is a $(n,\varepsilon)$-separated set of $F$. Since

$$\sup_E \sum_{\mathbf{x}\in E} \mathrm{e}^{(S_nf)(\mathbf{x})+(S_ng)(\mathbf{x})} \le \big(\sup_E \sum_{\mathbf{x}\in E} \mathrm{e}^{(S_nf)(\mathbf{x})}\big)\big(\sup_E \sum_{\mathbf{x}\in E} \mathrm{e}^{(S_ng)(\mathbf{x})}\big).$$

So

$$P_n(F,f+g,\varepsilon) \le P_n(F,f,\varepsilon)P_n(F,g,\varepsilon).$$

Thus

$$P(F,f+g) \le P(F,f)+P(F,g).$$

(8) If $a_1,...,a_k$ are positive numbers with $\sum\limits_{i=1}^{k} a_i = 1$ then $\sum\limits_{i=1}^{k} a_i^c \le 1$ if $c \ge 1$, and $\sum\limits_{i=1}^{k} a_i^c \ge 1$ if $c \le 1$. Then, if $E$ is a finite subset of $Orb_n(X,F)$ we have

$$\sum_{\mathbf{x}\in E} \mathrm{e}^{(S_nf)(\mathbf{x})} \le \big(\sum_{\mathbf{x}\in E} \mathrm{e}^{(S_nf)(\mathbf{x})}\big)^c \text{ if } c \ge 1,$$

and

$$\sum_{\mathbf{x}\in E} \mathrm{e}^{(S_nf)(\mathbf{x})} \ge \big(\sum_{\mathbf{x}\in E} \mathrm{e}^{(S_nf)(\mathbf{x})}\big)^c \text{ if } c \le 1.$$

So

$$P_n(F,cf,\varepsilon) \le \big(P_n(F,f,\varepsilon)\big)^c \text{ if } c \ge 1,$$

and

$$P_n(F,cf,\varepsilon) \ge \big(P_n(F,f,\varepsilon)\big)^c \text{ if } c \le 1.$$

Therefore

$$P(F,cf) \le cP(F,f) \text{ if } c \ge 1,$$

and

$$P(F,cf) \ge cP(F,f) \text{ if } c \le 1.$$

(9) Since $-|f| < f < |f|$, by (2), we have

$$P(F,-|f|,\varepsilon) \le P(F,f,\varepsilon) \le P(F,|f|,\varepsilon).$$

From (8) we have

$$-P(F,|f|) \le P(F,-|f|),$$

so

$$|P(F,f)| \le P(F,|f|).$$

□

# 3 Topological pressure of iterates

In this section, we will demonstrate the relationship between the topological pressure of a set-valued map and the topological pressure of its iterates. Before that, we show the following well-known result for continuous functions on compact metric space.

**Theorem 3.1.** *Let $X$ be a compact metric space, $T : X \to X$ be continuous, and let $f \in C(X, \mathbb{R})$. Then, if $k > 0$,*

$$P(T^k, S_k f) = kP(T, f).$$

The result does not hold for set-valued map. However, in Theorem 3.4, we show that for any topological dynamical system $(X, F)$ and any $k \in \mathbb{N}, P(F, f) \leq P(F^k, S_k f) \leq kP(F, f)$. First, we give the following lemma.

**Lemma 3.2.** *Let $(X, F)$ be a topological dynamical system, $f \in C(X, \mathbb{R})$, $n \in \mathbb{N}, \varepsilon > 0$, and $E$ be an $(n, \varepsilon)$-separated set for $F$. Let $k, m \in \mathbb{N}$, such that $(m-1)k < n \leq mk$, and let $L = n - (m-1)k$.*

*Let*

$$A_i = \{i, i+k, i+2k, ..., i+(m-1)k\},$$

*for any $i = 0, ..., L-1$,*

*and let*

$$A_i = \{i, i+k, i+2k, ..., i+(m-2)k\},$$

*for any $i = L, ..., k-1$.*

*If, for each $i = 0, ..., k-1$, take $E_i$ is the largest $\frac{\varepsilon}{2}$-separated subset of $\pi_{A_i}(E)$ , then*

$$\sum_{\boldsymbol{x} \in E} \mathrm{e}^{(S_n f)(\boldsymbol{x})} \leq \prod_{i=0}^{k-1} \Big( \sum_{\boldsymbol{x} \in E_i} \mathrm{e}^{(S_{\theta(i)} f)(\boldsymbol{x})} \Big), \quad \textit{where}$$

$$\theta(i) = \begin{cases} m, & i = 0, ..., L-1, \\ m-1, & i = L, ..., k-1. \end{cases}$$

*Proof.* Define $\Gamma \subseteq X^n$ is the set

$$\Gamma = \bigcap_{i=0}^{k-1} \pi_{A_i}^{-1}(E_i).$$

Then

$$\sum_{\mathbf{x}\in\Gamma} \mathrm{e}^{(S_n f)(\mathbf{x})} \leq \prod_{i=0}^{k-1} \big( \sum_{\mathbf{x}\in E_i} \mathrm{e}^{(S_{\theta(i)} f)(\mathbf{x})} \big), \quad \text{where}$$

$$\theta(i) = \begin{cases} m, & i = 0, ..., L-1, \\ m-1, & i = L, ..., k-1. \end{cases}$$

We next show that $\sum\limits_{\mathbf{x}\in E} \mathrm{e}^{(S_n f)(\mathbf{x})} \leq \sum\limits_{\mathbf{x}\in\Gamma} \mathrm{e}^{(S_n f)(\mathbf{x})}$ by demonstrating that

$$\sum_{\mathbf{x}\in E\backslash\Gamma} \mathrm{e}^{(S_n f)(\mathbf{x})} \leq \sum_{\mathbf{x}\in \Gamma\backslash E} \mathrm{e}^{(S_n f)(\mathbf{x})}.$$

Suppose $\mathbf{x} \in E\backslash\Gamma$. For each $j = 0, ..., k-1$, consider the point $\pi_{A_j}(\mathbf{x})$, and define

$$\Gamma_j(\mathbf{x}) = \big\{ \mathbf{y} \in E_j : D(\mathbf{y}, \pi_{A_j}(\mathbf{x})) < \frac{\varepsilon}{2} \big\}.$$

Then there exists some $0 \leq j \leq k-1$ such that $\pi_{A_j}(\mathbf{x}) \notin E_j$, since $\mathbf{x}$ is not in $\Gamma$. Hence $\pi_{A_j}(\mathbf{x}) \notin \Gamma_j(\mathbf{x})$. Since $E_j$ is the largest $\frac{\varepsilon}{2}$-separated subset of $\pi_{A_j}(E)$, then we have $\Gamma_j(\mathbf{x}) \neq \emptyset$ for each $0 \leq j \leq k-1$.

Now define

$$\Gamma(\mathbf{x}) = \bigcap_{i=0}^{k-1} \pi_{A_i}^{-1}[\Gamma_i(\mathbf{x})].$$

Then for any $\mathbf{z} \in \Gamma(\mathbf{x}), D(\mathbf{x}, \mathbf{z}) < \frac{\varepsilon}{2}$. Hence, since $E$ is $(n, \varepsilon)$-separated, and $\mathbf{x} \in E$, $\mathbf{z} \notin E$. Because this is right for all $\mathbf{z} \in \Gamma(\mathbf{x})$, it follows that $\Gamma(\mathbf{x}) \bigcap E = \emptyset$. Moreover, as for each $0 \leq j \leq k-1, |\Gamma_j(\mathbf{x})| \geq 1$, we have $|\Gamma(\mathbf{x})| \geq 1$. Hence, there exists at least one point $\mathbf{z} \in \Gamma(\mathbf{x})\backslash E \subseteq \Gamma\backslash E$, for any point $\mathbf{x} \in E\backslash\Gamma$.

Next we show that, if $\mathbf{x}, \mathbf{y} \in E\backslash\Gamma$, then $\Gamma(\mathbf{x}) \bigcap \Gamma(\mathbf{y}) = \emptyset$. This is due to the fact that if there existed a sequence $\mathbf{z}$ in $\Gamma(\mathbf{x}) \bigcap \Gamma(\mathbf{y})$, then $D(\mathbf{x}, \mathbf{y}) \leq D(\mathbf{x}, \mathbf{z}) + D(\mathbf{y}, \mathbf{z}) < \varepsilon$ which contradicted with $E$ being a $(n, \varepsilon)$-separated set.

Define a map $\varphi : E\backslash\Gamma \to \Gamma\backslash E$ such that for each $\mathbf{x} \in E\backslash\Gamma$, there is $\mathbf{z} = \varphi(\mathbf{x}) \in \Gamma\backslash E$ with $D(\mathbf{x}, \mathbf{z}) < \frac{\varepsilon}{2}$. Then $\varphi$ is injective.

Let $\varepsilon > 0$, there exists $\delta > 0$ with that $d(x,y) < \frac{\varepsilon}{2}$ implies $|f(x) - f(y)| < \delta, f \in (C, R)$.

Thus,

$$\begin{aligned}
\sum_{\mathbf{x}\in E\backslash\Gamma} \mathrm{e}^{(S_n f)(\mathbf{x})} &= \sum_{\mathbf{x}\in E\backslash\Gamma} \mathrm{e}^{(S_n f)(\mathbf{x})-(S_n f)(\varphi(\mathbf{x}))+(S_n f)(\varphi(\mathbf{x}))} \\
&= \sum_{\mathbf{x}\in E\backslash\Gamma} \mathrm{e}^{(S_n f)(\mathbf{x})-(S_n f)(\varphi(\mathbf{x}))}\mathrm{e}^{(S_n f)(\varphi(\mathbf{x}))} \\
&\leq \sum_{\mathbf{x}\in E\backslash\Gamma} \mathrm{e}^{\sum\limits_{i=0}^{n-1}|f(x_i)-f(\varphi(x_i))|}\mathrm{e}^{(S_n f)(\varphi(\mathbf{x}))} \\
&\leq \sum_{\mathbf{z}\in \Gamma\backslash E} \mathrm{e}^{n\delta}\mathrm{e}^{(S_n f)(\mathbf{z})} \\
&= \sum_{\mathbf{z}\in \Gamma\backslash E} \mathrm{e}^{n\delta+(S_n f)(\mathbf{z})}.
\end{aligned}$$

Since $\delta$ is arbitrary,

$$\sum_{\mathbf{x}\in E\backslash\Gamma} \mathrm{e}^{(S_n f)(\mathbf{x})} \leq \sum_{\mathbf{z}\in \Gamma\backslash E} \mathrm{e}^{(S_n f)(\mathbf{z})}.$$

So

$$\begin{aligned}
\sum_{\mathbf{x}\in E} \mathrm{e}^{(S_n f)(\mathbf{x})} &\leq \sum_{\mathbf{x}\in \Gamma} \mathrm{e}^{(S_n f)(\mathbf{x})} \\
&\leq \prod_{i=0}^{k-1}\Big(\sum_{\mathbf{x}\in E_i} \mathrm{e}^{(S_{\theta(i)} f)(\mathbf{x})}\Big),
\end{aligned}$$

*i.e.*,

$$\sum_{\mathbf{x}\in E} \mathrm{e}^{(S_n f)(\mathbf{x})} \leq \prod_{i=0}^{k-1}\Big(\sum_{\mathbf{x}\in E_i} \mathrm{e}^{(S_{\theta(i)} f)(\mathbf{x})}\Big), \quad \text{where}$$

$$\theta(i) = \begin{cases} m, & i = 0, ..., L-1, \\ m-1, & i = L, ..., k-1. \end{cases}$$

□

**Lemma 3.3.** *Let* $(X, F)$ *be a topological dynamical system, and let* $f \in C(X, \mathbb{R})$, $k \in \mathbb{N}$. *Then for all* $n \in \mathbb{N}$ *and* $\varepsilon > 0$, *if* $m \in \mathbb{N}$ *is chosen such that* $(m-1)k < n \leq mk$, *Then*

$$P_n(F, f, \varepsilon) \leq \big(P_m(F^k, S_k f, \varepsilon)\big)^k.$$

*Proof.* Let $n \in \mathbb{N}$ and $\varepsilon > 0$, and fix $m \in \mathbb{N}$ such that $(m-1)k < n \leq mk$, let $E$ is an $(n, \varepsilon)$-separated set for $F$ of maximal cardinality, and let $L = n-(m-1)k$.

Let

$$A_i = \big\{i, i+k, i+2k, ..., i+(m-1)k\big\},$$

for any $i = 0, ..., L-1$,

and let

$$A_i = \big\{i, i+k, i+2k, ..., i+(m-2)k\big\},$$

for any $i = L, ..., k-1$. If, for each $i = 0, ..., k-1$, $E_i$ is taken to be the largest $\frac{\varepsilon}{2}$-separated subset of $\pi_{A_i}(E)$. By Lemma 3.2,

$$\sum_{\mathbf{x}\in E} \mathrm{e}^{(S_n f)(\mathbf{x})} \leq \prod_{i=0}^{k-1} \big(\sum_{\mathbf{x}\in E_i} \mathrm{e}^{(S_{\theta(i)} f)(\mathbf{x})}\big), \quad \text{where}$$

$$\theta(i) = \begin{cases} m, & i = 0, ..., L-1, \\ m-1, & i = L, ..., k-1. \end{cases}$$

Moreover, $E_i$ is an $(m, \frac{\varepsilon}{2})$-separated set of $F^k$, for $i = 0, ..., L-1$, and $E_i$ is an $(m-1, \frac{\varepsilon}{2})$-separated set of $F^k$, for $i = L, ..., k-1$. Without loss of generality, let $\sum_{\mathbf{x}\in E_j} \mathrm{e}^{(S_m f)(\mathbf{x})} = \max_{0\leq i\leq k-1} \big\{ \sum_{\mathbf{x}\in E_i} \mathrm{e}^{(S_{\theta(i)} f)(\mathbf{x})}\big\}$, for any $0 \leq j \leq k-1$.

Therefore,

$$\sum_{\mathbf{x}\in E} \mathrm{e}^{(S_n f)(\mathbf{x})} \leq \big(\sum_{\mathbf{x}\in E_j} \mathrm{e}^{(S_m f)(\mathbf{x})}\big)^k.$$

It follows that

$$P_n(F, f, \varepsilon) \leq \big(P_m(F^k, S_k f, \varepsilon)\big)^k.$$

□

**Theorem 3.4.** *Let $(X, F)$ be a topological dynamical system, and let $f \in C(X, \mathbb{R})$, $k \in \mathbb{N}$. Then*

$$P(F, f) \le P(F^k, S_k f) \le kP(F, f).$$

*Proof.* First, we show that

$$P(F^k, S_k f) \le kP(F, f).$$

Let $n \in \mathbb{N}$, and $E$ is an $(n, \varepsilon)$-separated set of $F^k$ with the largest cardinality. For each $(x_0, ..., x_{n-1}) \in E$, take $(y_0, ..., y_{nk-1}) \in Orb_{nk}(X, F)$ such that for each $i = 0, ..., n-1$, $y_{ik} = x_i$, let $E' = \big\{(y_0, ..., y_{nk-1}) \in Orb_{nk}(X, F) : y_{ik} = x_i, i = 0, ..., n-1\big\}$.

Then for every $\mathbf{y}, \mathbf{y}' \in E'$, $D(\mathbf{y}, \mathbf{y}') = \max\limits_{0 \le i \le nk-1} (y_i, y_i^{'}) \ge \varepsilon$. So $E'$ be an $(nk, \varepsilon)$-separated set of $F$.

By Definition 2.8, $P_n(F^k, S_k f, \varepsilon) \le P_{nk}(F, f, \varepsilon)$. So

$$\limsup_{n\to\infty} \frac{1}{n} \log P_n(F^k, S_k f, \varepsilon) \le \limsup_{n\to\infty} \frac{k}{nk} \log P_{nk}(F, f, \varepsilon).$$

Then $P(F^k, S_k f) \le kP(F, f)$.

Next we show that $P(F, f) \le P(F^k, S_k f)$.

By Lemma 3.3 , if $n \in \mathbb{N}$, and $m \in \mathbb{N}$ is chosen such that $(m-1)k < n \le mk$, then

$$P_n(F, f, \varepsilon) \le \big(P_m(F^k, S_k f, \varepsilon)\big)^k.$$

So

$$\frac{1}{n} \log P_n(F, f, \varepsilon) \le \frac{mk}{n} \frac{1}{m} \log P_m(F^k, S_k f, \varepsilon).$$

Since $m \to \infty$ as $n \to \infty$, we have

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n} \log P_n(F, f, \varepsilon) &\le \limsup_{m\to\infty} \frac{mk}{n} \frac{1}{m} \log P_m(F^k, S_k f, \varepsilon) \\ &= \limsup_{m\to\infty} \frac{\beta}{m} \log P_m(F^k, S_k f, \varepsilon)\end{aligned}$$

where $\beta = \frac{mk}{n}$.

As $(m-1)k < n \le mk$, it follows that $\beta \to 1$ as $n \to \infty$. Then, $P(F, f) \le P(F^k, S_k f)$. □

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