Theoretical Mathematics & Applications, vol.8, no.1, 2018, 1-8
ISSN: 1792-9687 (print), 1792-9709 (online)
Scienpress Ltd, 2018

Wirtinger's Integral Inequality on Time Scale

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Abstract

In this paper, we establish a Wirtinger-type inequality on an arbitrary time scale. We give, as special cases of the time scales, new Wirtinger-type inequality in the continuous and discrete cases, respectively.

Mathematics Subject Classification: 34N05; 26D10
Keywords: Wirtinger-type inequality; Time Scale; Dynamic Inequalities

1 Introduction

A time scale, (we denote it by the symbol $\mathbb{T}$) is an arbitrary nonempty closed subset of the real numbers. For $t \in \mathbb{T}$ we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) := \inf\{s \in T : s > t\}$. If $t < \sup T$ and $\sigma(t) = t$, then t is called right-dense, and if $t > \inf T$ and $\rho(t) = t$, then t is called left-dense. Graininess function $\mu : T \to [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$ (see [2], [3], [6]).

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Article Info: *Received* : February 6, 2017. *Revised* : May 5, 2017.
Published online : February 1, 2018.

A function $f:\mathbb{T}\rightarrow\mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f:\mathbb{T}\rightarrow\mathbb{R}$ will be denoted by $\mathbb{C}_{rd}=\mathbb{C}_{rd}(\mathbb{T})=\mathbb{C}_{rd}(\mathbb{T},\mathbb{R})$. The set of functions $f:\mathbb{T}\rightarrow\mathbb{R}$ that are differentiable and whose derivative is rd-continuous is denoted by $\mathbb{C}_{rd}^{1}=\mathbb{C}_{rd}^{1}(\mathbb{T})=\mathbb{C}_{rd}^{1}(\mathbb{T},\mathbb{R})$. We define the time scale interval $[a,b]_{\mathbb{T}}$ by $[a,b]_{\mathbb{T}}=[a,b]\cap\mathbb{T}$.

In 2000, Hilscher [8] proved a Wirtinger-type inequality on time scales in the form:

Theorem 1.1. *(Discrete Wirtinger Inequality, [8]) If M be positive and strictly monotone such that M^{Δ} exists and is rd-continuous, then*

$$\int_a^b \left|M^{\Delta}(t)\right| y^2(\sigma(t))\,\Delta t \leq \Psi^2 \int_a^b \frac{M(t)M(\sigma(t))}{|M^{\Delta}(t)|}\left(y^{\Delta}(t)\right)^2 \Delta t \tag{1}$$

for any y with $y(a)=y(b)=0$ and such that y^{Δ} exists and is rd-continuous, where

$$\Psi=\left(\sup_{t\in[a,b]\cap\mathbb{T}}\frac{M(t)}{M(\sigma(t))}\right)^{\frac{1}{2}}+\left[\left(\sup_{t\in[a,b]\cap\mathbb{T}}\frac{\mu(t)\left|M^{\Delta}(t)\right|}{M(\sigma(t))}\right)+\left(\sup_{t\in[a,b]\cap\mathbb{T}}\frac{M(t)}{M(\sigma(t))}\right)\right]^{\frac{1}{2}}. \tag{2}$$

In [4] authors extended the following theorem:

Theorem 1.2. *([4]) Suppose $\gamma\geq 1$ is an odd integer. For a positive $M\in C_{rd}^{1}(\mathfrak{T})$ satisfying either $M^{\Delta}>0$ or $M^{\Delta}<0$ on $\mathfrak{T}$, we have*

$$\int_a^b \frac{M^{\gamma}(t)M(\sigma(t))}{|M^{\Delta}(t)|^{\gamma}}\left(y^{\Delta}(t)\right)^{\gamma+1}\Delta t \geq \frac{1}{\Psi^{\gamma+1}(\alpha,\beta,\gamma)}\int_a^b \left|M^{\Delta}(t)\right| y^{\gamma+1}(t)\,\Delta t \tag{3}$$

for any $y\in C_{rd}^{1}(\mathfrak{T})$ with $y(a)=y(b)=0$, where $\Psi(\alpha,\beta,\gamma)$ is the largest root of

$$x^{\gamma+1}-2^{\gamma-1}(\gamma+1)\,\alpha x^{\gamma}-2^{\gamma-1}\beta=0, \tag{4}$$

whereby

$$\alpha:=\sup_{t\in\mathfrak{T}^k}\left(\frac{M(\sigma(t))}{M(t)}\right)^{\frac{\gamma}{\gamma+1}},\quad \beta:=\sup_{t\in\mathfrak{T}^k}\left(\frac{\mu(t)\left|M^{\Delta}(t)\right|}{M(t)}\right)^{\gamma}.$$

2 Main Results

Let us prove the following theorem:

Theorem 2.1. *Let $M \in \mathbb{C}_{rd}^{1}\left([a,b]_{\mathbb{T}}\right)^{k}$ be positive and strictly monotone such that satisfying either $M^{\Delta} > 0$ or $M^{\Delta} < 0$ on $\left([a,b]_{\mathbb{T}}\right)^{k}$. Then, for some integer $\eta \geq 1$ we have*

$$\int_{a}^{b} \left|M^{\Delta}(t)\right| y^{\eta+1}(\sigma(t))\,\Delta t \leq \Lambda^{\eta+1}(\omega,\xi_r,\psi) \int_{a}^{b} \frac{M^{\eta}(t)M(\sigma(t))}{\left|M^{\Delta}(t)\right|^{\eta}} \left(y^{\Delta}(t)\right)^{\eta+1} \Delta t \tag{5}$$

for any $y \in \mathbb{C}_{rd}^{1}\left([a,b]_{\mathbb{T}}\right)^{k}$, with $y(a) = y(b) = 0$, where $\Lambda(\omega,\xi_r,\psi)$ is the largest root of equality

$$x^{\eta+1} = 2^{\eta}\omega x^{\eta} + \sum_{r=1}^{\eta-1} 2^{\eta-(r+1)}\xi_r x^{r} + 2^{\eta-1}\psi, \tag{6}$$

whereby

$$\begin{aligned} \omega &= \sup_{t\in\left([a,b]_{\mathbb{T}}\right)^{k}} \left(\frac{M^{\sigma}}{M}\right)^{\frac{\eta}{\eta+1}}, \quad \psi = \sup_{t\in\left([a,b]_{\mathbb{T}}\right)^{k}} \left(\frac{\mu^{\frac{1}{\eta}}\left|M^{\Delta}\right|}{M}\right)^{\eta}, \\ \xi_r &= \sup_{t\in\left([a,b]_{\mathbb{T}}\right)^{k}} \left(\frac{\mu^{\frac{\eta+1}{r}} M^{\sigma}\left|M^{\Delta}\right|^{\frac{\eta(\eta-(r-1))}{r}-1}}{M^{\frac{\eta(\eta-(r-1))}{r}}}\right)^{\frac{\eta}{\eta+1}}, \quad r = 1,\ldots,\eta-1. \end{aligned} \tag{7}$$

We denote by

$$A = \int_{a}^{b} \left|M^{\Delta}(t)\right| y^{\eta+1}(\sigma(t))\,\Delta t, \quad B = \int_{a}^{b} \frac{M^{\eta}(t)M(\sigma(t))}{\left|M^{\Delta}(t)\right|^{\eta}} \left(y^{\Delta}(t)\right)^{\eta+1} \Delta t. \tag{8}$$

Using the integration by parts, whereby $y(a) = y(b) = 0$, left side of inequality (2.1) become

$$\begin{aligned} A &= \int_{a}^{b} \left|M^{\Delta}(t)\right| y^{\eta+1}(t)\,\Delta t = \pm \int_{a}^{b} M^{\Delta}(t)\, y^{\eta+1}(t)\,\Delta t \\ &= \pm \left\{ \left[M(t)\, y^{\eta+1}(t)\right]_{a}^{b} - \int_{a}^{b} M^{\sigma}(t) \left(y^{\eta+1}(t)\right)^{\Delta} \Delta t \right\} \\ &\leq \int_{a}^{b} M^{\sigma}(t) \left|y^{\eta+1}\right|^{\Delta}(t)\,\Delta t = \int_{a}^{b} M^{\sigma} \left|\sum_{r=0}^{\eta} y^{r} \left(y^{\sigma}\right)^{\eta-r}\right| \left|y^{\Delta}\right| \Delta t \\ &= \int_{a}^{b} M^{\sigma} \left|\left(y^{\sigma}\right)^{\eta} + y\left(y^{\sigma}\right)^{\eta-1} + y^{2}\left(y^{\sigma}\right)^{\eta-2} + \ldots + y^{\eta-1}\left(y^{\sigma}\right) + y^{\eta}\right| \left|y^{\Delta}\right| \Delta t \end{aligned}$$

$$
\begin{aligned}
&= \int_a^b M^\sigma \left| \left(y+\mu y^\Delta\right)^\eta + y\left(y+\mu y^\Delta\right)^{\eta-1} + ... + y^{\eta-1}\left(y+\mu y^\Delta\right) + y^\eta \right| \left|y^\Delta\right| \Delta t \\
&\leq \int_a^b M^\sigma \{2^{\eta-1}|y|^\eta \left|y^\Delta\right| + 2^{\eta-1}\mu \left|y^\Delta\right|^{\eta+1} + 2^{\eta-2}|y|^\eta \left|y^\Delta\right| + 2^{\eta-2}\mu |y| \left|y^\Delta\right|^\eta + ...+ \\
&+ |y|^\eta \left|y^\Delta\right| + \mu |y|^{\eta-1}\left|y^\Delta\right|^2 + |y|^\eta \left|y^\Delta\right|\}\Delta t \\
&= \int_a^b \{2^\eta M^\sigma |y|^\eta \left|y^\Delta\right| + 2^{\eta-2}M^\sigma \mu |y| \left|y^\Delta\right|^\eta + 2^{\eta-3}M^\sigma\mu |y|^2 \left|y^\Delta\right|^{\eta-1} + \\
&... + M^\sigma \mu |y|^{\eta-1}\left|y^\Delta\right|^2 + 2^{\eta-1}M^\sigma\mu \left|y^\Delta\right|^{\eta+1}\}\Delta t \\
&= 2^\eta \int_a^b \left(\frac{M^\eta M^\sigma}{|M^\Delta|^\eta}\left|y^\Delta\right|^{\eta+1}\right)^{\frac{1}{\eta+1}} \left(\frac{M^\eta M^\sigma}{M}|y|^{\eta+1}\right)^{\frac{\eta}{\eta+1}} \Delta t + \\
&2^{\eta-2}\int_a^b \left(\frac{M^\eta M^\sigma}{|M^\Delta|^\eta}\left|y^\Delta\right|^{\eta+1}\right)^{\frac{\eta}{\eta+1}} \left(\frac{\mu^{\eta+1}M^\sigma\left|M^\Delta\right|^{\eta^2-1}\left|M^\Delta\right|}{M^{\eta^2}}|y|^{\eta+1}\right)^{\frac{1}{\eta+1}} \Delta t \\
&+2^{\eta-3}\int_a^b \left(\frac{M^\eta M^\sigma}{|M^\Delta|^\eta}\left|y^\Delta\right|^{\eta+1}\right)^{\frac{\eta-1}{\eta+1}} \left(\frac{\mu^{\frac{\eta+1}{2}}M^\sigma\left|M^\Delta\right|^{\frac{\eta(\eta-1)}{2}-1}\left|M^\Delta\right|}{M^{\frac{\eta(\eta-1)}{2}}}|y|^{\eta+1}\right)^{\frac{2}{\eta+1}} \Delta t + ... \\
&+2\int_a^b \left(\frac{M^\eta M^\sigma}{|M^\Delta|^\eta}\left|y^\Delta\right|^{\eta+1}\right)^{\frac{3}{\eta+1}} \left(\frac{\mu^{\frac{\eta+1}{\eta-2}}M^\sigma\left|M^\Delta\right|^{\frac{3\eta}{\eta-2}-1}\left|M^\Delta\right|}{M^{\frac{3\eta}{\eta-2}}}|y|^{\eta+1}\right)^{\frac{\eta-2}{\eta+1}} \Delta t \\
&+\int_a^b \left(\frac{M^\eta M^\sigma}{|M^\Delta|^\eta}\left|y^\Delta\right|^{\eta+1}\right)^{\frac{2}{\eta+1}} \left(\frac{\mu^{\frac{\eta+1}{\eta-1}}M^\sigma\left|M^\Delta\right|^{\frac{2\eta}{\eta-1}-1}\left|M^\Delta\right|}{M^{\frac{2\eta}{\eta-1}}}|y|^{\eta+1}\right)^{\frac{\eta-1}{\eta+1}} \Delta t \\
&+2^{\eta-1}\int_a^b \left(\frac{M^\eta M^\sigma}{|M^\Delta|^\eta}\left|y^\Delta\right|^{\eta+1}\right) \left(\frac{\mu\left|M^\Delta\right|^\eta}{M^\eta}\right) \Delta t.
\end{aligned}
$$

Applying Hölder inequality on each summand of the above inequality, except the last one, it follows

$$
\begin{aligned}
A &\leq 2^\eta \left\{\int_a^b \left(\frac{M^\eta M^\sigma}{|M^\Delta|^\eta}\left|y^\Delta\right|^{\eta+1}\right)\Delta t\right\}^{\frac{1}{\eta+1}} \left\{\int_a^b \left(\frac{M^\eta M^\sigma}{M}|y|^{\eta+1}\right)\Delta t\right\}^{\frac{\eta}{\eta+1}} \\
&+2^{\eta-2}\left\{\int_a^b \left(\frac{M^\eta M^\sigma}{|M^\Delta|^\eta}\left|y^\Delta\right|^{\eta+1}\right)\Delta t\right\}^{\frac{\eta}{\eta+1}} \left\{\int_a^b \left(\frac{\mu^{\eta+1}M^\sigma\left|M^\Delta\right|^{\eta^2-1}\left|M^\Delta\right|}{M^{\eta^2}}|y|^{\eta+1}\right)\Delta t\right\}^{\frac{1}{\eta+1}} \\
&+... + \left\{\int_a^b \left(\frac{M^\eta M^\sigma}{|M^\Delta|^\eta}\left|y^\Delta\right|^{\eta+1}\right)\Delta t\right\}^{\frac{2}{\eta+1}} \left\{\int_a^b \left(\frac{\mu^{\frac{\eta+1}{\eta-1}}M^\sigma\left|M^\Delta\right|^{\frac{2\eta}{\eta-1}-1}\left|M^\Delta\right|}{M^{\frac{2\eta}{\eta-1}}}|y|^{\eta+1}\right)\Delta t\right\}^{\frac{\eta-1}{\eta+1}} \\
&+2^{\eta-1}\int_a^b \left(\frac{M^\eta M^\sigma}{|M^\Delta|^\eta}\left|y^\Delta\right|^{\eta+1}\right)\left(\frac{\mu\left|M^\Delta\right|^\eta}{M^\eta}\right)\Delta t \\
&= 2^\eta \omega B^{\frac{1}{\eta+1}}A^{\frac{\eta}{\eta+1}} + 2^{\eta-2}\xi_1 B^{\frac{\eta}{\eta+1}}A^{\frac{1}{\eta+1}} + 2^{\eta-3}\xi_2 B^{\frac{\eta-1}{\eta+1}}A^{\frac{2}{\eta+1}} + ... \\
&+2\xi_{\eta-2}B^{\frac{3}{\eta+1}}A^{\frac{\eta-2}{\eta+1}} + \xi_{\eta-1}B^{\frac{2}{\eta+1}}A^{\frac{\eta-1}{\eta+1}} + 2^{\eta-1}\psi B,
\end{aligned}
\tag{9}
$$

i.e.

$$A \leq 2^\eta \omega B^{\frac{1}{\eta+1}} A^{\frac{\eta}{\eta+1}} + \sum_{r=1}^{\eta-1} 2^{\eta-(r+1)} \xi_r B^{\frac{\eta-(r-1)}{\eta+1}} A^{\frac{r}{\eta+1}} + 2^{\eta-1} \psi B. \tag{10}$$

After some calculations one obtains it holds the following inequality

$$\begin{array}{l} \left(\frac{A}{B}\right)^{\frac{1}{\eta+1}} \leq 2^\eta \omega + 2^{\eta-2} \xi_1 \left(\frac{B}{A}\right)^{\frac{\eta-1}{\eta+1}} + 2^{\eta-3} \xi_2 \left(\frac{B}{A}\right)^{\frac{\eta-2}{\eta+1}} + ... \\ +2\xi_{\eta-2} \left(\frac{B}{A}\right)^{\frac{2}{\eta+1}} + \xi_{\eta-1} \left(\frac{B}{A}\right)^{\frac{1}{\eta+1}} + 2^{\eta-1} \psi \left(\frac{B}{A}\right)^{\frac{\eta}{\eta+1}}, \end{array}$$

$$\left(\frac{A}{B}\right)^{\frac{1}{\eta+1}} \leq 2^\eta \omega + \sum_{r=1}^{\eta-1} 2^{\eta-(r+1)} \xi_r \left(\frac{B}{A}\right)^{\frac{\eta-r}{\eta+1}} + 2^{\eta-1} \psi \left(\frac{B}{A}\right)^{\frac{\eta}{\eta+1}}.$$

By introducing $C = \left(\frac{A}{B}\right)^{\frac{1}{\eta+1}}$, we get

$$C \leq 2^\eta \omega + \sum_{r=1}^{\eta-1} 2^{\eta-(r+1)} \xi_r C^{r-\eta} + 2^{\eta-1} \psi \left(\frac{B}{A}\right)^{-\eta},$$

i.e.

$$C^{\eta+1} \leq 2^\eta \omega C^\eta + \sum_{r=1}^{\eta-1} 2^{\eta-(r+1)} \xi_r C^r + 2^{\eta-1} \psi, \tag{11}$$

whence follows the desired inequality,

$$A \leq \Lambda^{\eta+1} (\omega, \xi_r, \gamma) \leq B.$$

3 Application

Corollary 3.1. *In the case of* $\mathbb{T} = \mathbb{R}$, *the inequality (1.3) reduces to*

$$\int_a^b |M'(t)| \, y^{\eta+1}(t) \, dt \leq (2^\eta)^{\eta+1} \int_a^b \frac{M^{\eta+1}(t)}{|M'(t)|^\eta} (y'(t))^{\eta+1} dt. \tag{12}$$

Proof: In the case of $\mathbb{T} = \mathbb{R}$ it is $f^\Delta(t) = f'(t)$, $\sigma(t) = t$ and $\mu(t) = 0$, so $\omega = 1$, $\xi_r = 0$ and $\psi = 0$. By substitute this values in the equalities (2.2) we obtain $x^{\eta+1} = 2^\eta x^\eta$. i.e. $x^\eta (x - 2^\eta) = 0$. Since $\int_a^b f(t) \Delta t = \int_a^b f(t) \, dt$, follows inequality (3.1).

Remark 3.2. *Specially, in the case of $\eta = 1$, the largest root of the (1.3) is 2, so the inequality (1.3) becomes*

$$\int_a^b |M'(t)|\, y^2((t))\, dt \leq 4 \int_a^b \frac{M^2(t)}{|M'(t)|} (y'(t))^2\, dt, \tag{13}$$

what was proved in [6].

Theorem 3.3. *Let $\mathbb{T} = h\mathbb{Z}$. For a positive sequence $\{M_n\}_{0\leq n\leq N+1}$ satisfying either $\Delta M > 0$ or $\Delta M < 0$ on $[0, N] \cap h\mathbb{Z}$, we have*

$$\sum_{n=0}^{N} |\Delta_h M_n|\, y_n^{\eta+1} \leq \Omega^\eta(\omega, \xi_r, \psi) \sum_{n=0}^{N} \frac{M_n^\eta M_{n+1}}{|\Delta_h M_n|^\eta} (\Delta_h y_n)^{\eta+1},$$

for any sequence $\{y_n\}_{0\leq n\leq N+1}$ with $y_0 = y_{N+1} = 0$, where $\Omega(\omega, \xi_r, \psi)$ is the smallest root of the inequality

$$(1 + 2\omega)\, 2^{\eta-1} x^\eta = \sum_{r=1}^{\eta-1} 2^{\eta-(r+1)} \xi_r x^r + 2^{\eta-1}\psi, \tag{14}$$

when

$$\begin{aligned} \omega &= \sup_{0\leq n\leq N} \left(\frac{M_{n+h}}{M_n}\right)^{\frac{\eta}{\eta+1}}, \\ \xi_r &= \sup_{0\leq n\leq N} \left(\frac{h^{\frac{\eta+1}{r}} M_{n+h} |\Delta_h M_n|^{\frac{\eta(\eta-(r-1))}{r}-1}}{M_n}\right)^{\frac{\eta}{\eta+1}}, \quad r = 1, ..., \eta - 1, \\ \psi &= \sup_{0\leq n\leq N} \left(\frac{h^{\frac{1}{\eta}} |\Delta_h M_n|}{M_n}\right)^\eta. \end{aligned} \tag{15}$$

Proof. Starting from the inequality

$$(1 + C)^{\eta+1} \leq C^{\eta+1} + (\eta + 1)\, C^\eta + 2^{\eta-1} C^\eta$$

it is obtained

$$C^{\eta+1} \geq (1 + C)^{\eta+1} - (\eta + 1)\, C^\eta - 2^{\eta-1} C^\eta.$$

Involving this result in (1.2) proves it holds

$$(1 + C)^{\eta+1} - (\eta + 1)\, C^\eta - 2^{\eta-1} C^\eta - 2^\eta \omega C^\eta - \sum_{r=1}^{\eta-1} 2^{\eta-(r+1)} \xi_r C^r - 2^{\eta-1}\psi \leq 0.$$

Since

$$(1 + C)^{\eta+1} \geq (\eta + 1)\, C^\eta,$$

last inequality becomes

$$(1+2\omega)\,2^{\eta-1}C^{\eta} \geq \sum_{r=1}^{\eta-1} 2^{\eta-(r+1)}\xi_r C^r + 2^{\eta-1}\psi.$$

Since, for $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$ is $\sigma(t) = t+h$, $\mu(t) = h$, $f^{\Delta}(t) = \Delta_h f(t) = \frac{f(t+h)-f(t)}{h}$, $\int\limits_a^b f(t)\,\Delta t = \sum\limits_{t\in[0,N]\cap h\mathbb{Z}} \mu(t)\,f(t)$, so that

$$A = \sum_{n=0}^{N} |\Delta_h M_n|\, y_n^{\eta+1}, \qquad B = \sum_{n=0}^{N} \frac{M_n^{\eta} M_{n+1}}{|\Delta_h M_n|^{\eta}} (\Delta_h y_n)^{\eta+1},$$

whence follows the desired inequality.

4 Conclusion

In this paper, we present some new Wirtinger-type inequalities on time scales for function f^k. As special cases, some new continuous and discrete Wirtinger-type inequalities are given.

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