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Boundary values of r_r, r_r^*, R_r, R_r^* sets of certain classes of graphs

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Abstract

Let $G(V,E)$ be an undirected, finite, connected and a simple graph and $S = \{s_1, s_2, s_3, \ldots, s_k\}$ be a subset of V. For each $u \in V$, we associate a vector $\Gamma(u/S) = (d(u/s_1), d(u/s_2), d(u/s_3), \ldots, d(u/s_k))$, where $d(u/v) = \frac{\sum_{u_i \in N[u]} d(u_i,v)}{deg(u)+1}$. The subset S is said to be rational resolving set if $\Gamma(u/S) \neq \Gamma(v/S)$ for all $u, v \in V - S$ and is denoted by r_r set. A rational resolving set S with minimum cardinality is called rational metric basis or an rmb set and its cardinality is called rational metric dimension, denoted by $rmd(G)$ or $l_{r_r}(G)$. The maximum cardinality of a minimal r_r set of graph G is called upper r_r number of G and is denoted by $u_{r_r}(G)$. A subset S of $V(G)$ is said to be an r_r^* set if S is r_r set and $\bar{S} = \{V - S\}$ is also an r_r set. The minimum and maximum cardinality of minimal r_r^* set of graph G are respectively called lower and upper r_r^* number of G, denoted by $l_{r_r^*}(G)$ and $u_{r_r^*}(G)$. A subset S of $V(G)$ is said to be an R_r set if S is an r_r set and $\bar{S} = \{V - S\}$ is

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not an r_r set. The minimum and maximum cardinality of minimal R_r set of G are called respectively lower and upper R_r number of G and are denoted by $l_{R_r}(G)$ and $u_{R_r}(G)$. A subset S of $V(G)$ is said to be an R_r^* set if both S and $\bar{S} = \{V - S\}$ are not r_r sets. The minimum and maximum cardinality of minimal R_r^* set of G are called respectively lower and upper R_r^* number of G, denoted by $l_{R_r^*}(G)$ and $u_{R_r^*}(G)$. In this paper we are obtaining the lower and upper r_r, r_r^*, R_r, R_r^* numbers of certain classes of graphs.

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1 Introduction

Many networks are represented by a graph, in which vertex play an important role and it depends on its neighbors. To determine the position of a vertex in the network, we need to select the landmarks in such a way that the distance of the vertex from the landmark and the distances of its neighborhood vertices from the landmark are considered. Here $N(u) = \{x : ux \in E(G)\}$, called open neighborhood of the vertex u, $N[u] = N(u) \cup u$ is called closed neighborhood of the vertex u and $d(u, v)$ is the length of the shortest path between u and v. A subset S of the vertex set V of a connected graph G is said to be a resolving set of G if for every pair of vertices u, $v \in V - S$ there exists a vertex $w \in S$ such that $d(u, w) \neq d(v, w)$. The minimum cardinality of a resolving set S of G is called metric dimension of a graph G and is denoted by $\beta(G)$. Metric dimension was defined by F. Harary et al. [2] and P.J. Slater [8].

For the entire survey, we refer the latest survey article by Joseph A. Gallian [5]. All the graphs considered here are undirected, finite, connected and simple. Throughout this paper P_n denote a path on n vertices with a vertex set $V = \{v_i : 1 \leq i \leq n\}$ and edge set $E = \{v_iv_{i+1} : 1 \leq i < n\}$. Similarly C_n denote a cycle on n vertices with a vertex set $V = \{v_i : 1 \leq i \leq n\}$ and edge set $E = \{v_iv_{i+1}\} \cup \{v_1v_n\}$. We use the standard terminology, the terms not defined here may be found in [1, 3, 4].

2 Boundary values of r_r, r_r^*, R_r, R_r^* sets of certain classes of Graphs

Rational metric dimension of graphs were originally proposed by A. Raghavendra, B. Sooryanarayana, C. Hegde [11] . Consider a graph $G(V, E)$. For $u \in V$, associate a vector $\Gamma(u/S) = (d(u/s_1), d(u/s_2)..., d(u/s_k))$ with respect to $S = \{s_1, s_2, ..., s_k\}$ of V, where $d(u/v) = \frac{\sum_{u_i \in N[u]} d(u_i,v)}{deg(u)+1}$. Then subset S is said to be a rational resolving set if $\Gamma(x/S) \neq \Gamma(y/S)$ for all $x, y \in V - S$ and is denoted by r_r set. The minimum cardinality of a rational resolving set S is called rational metric dimension and is denoted by $rmd(G)$ or $l_{r_r}(G)$. A rational resolving set S with minimum cardinality is called rational metric basis or an *rmb* set. An r_r set of G is said to be minimal if no subset of it is a r_r set. Clearly minimum cardinality of a minimal r_r set is $l_{r_r}(G)$, called lower r_r number of G. Now we define the following. The maximum cardinality of a minimal r_r set of graph G is called upper r_r number of G and is denoted by $u_{r_r}(G)$. A subset S of $V(G)$ is said to be an r_r^* set if S is r_r set and $\bar{S} = \{V - S\}$ is also an r_r set. The minimum cardinality of an r_r^* set of graph G is called lower r_r^* number of G and is denoted by $l_{r_r^*}(G)$ and the maximum cardinality of a minimal r_r^* set of graph G is called upper r_r^* number of G and is denoted by $u_{r_r^*}(G)$. A subset S of $V(G)$ is said to be an R_r set if S an r_r set and $\bar{S} = \{V - S\}$ is not an r_r set. The minimum and maximum cardinality of minimal R_r sets of G are called respectively lower and upper R_r number of G and are denoted by $l_{R_r}(G)$ and $u_{R_r}(G)$. A subset S of $V(G)$ is said to be an R_r^* set if both S and $\bar{S} = \{V - S\}$ are not r_r sets. The minimum and maximum cardinality of minimal R_r^* sets of G are called respectively lower and upper R_r^* number of G and are denoted by $l_{R_r^*}(G)$ and $u_{R_r^*}(G)$. (Suppose p and q represent some graph theoretical properties like domination, resolving, rmd etc, then a subset S of $V(G)$ is said to be *pq* set if S is both p set and q set. If S is an arbitrary set, need not be minimal having the property p then minimum and maximum cardinality of S is denoted by $\hat{l}_p(G)$ and $\hat{u}_p(G)$.)

Remark 2.1. *For a path P_n,*

$$d(v_j/v_i) = \begin{cases} \frac{1}{2} & \textit{if} \quad i = j = 1 \textit{ or } i = j = n \\ \frac{2}{3} & \textit{if} \quad i = j \neq 1 \textit{ or } i = j \neq n \\ \mid j - i \mid & \textit{if} \quad i \neq j \textit{ and } (j \neq 1 \textit{ or } j \neq n) \\ \frac{2|j-i|-1}{2} & \textit{if} \quad i \neq j \textit{ and } (j = 1 \textit{ or } j = n). \end{cases}$$

Lemma 2.2. *For a Path P_n, a singleton set $\{v_i\}$ is an rmb set if and only if either v_i is an end vertex or a support vertex in P_n.*

Proof. From Ragavendra et al [11], $rmd(P_n) = 1$, $\{v_1\}$, $\{v_n\}$ are *rmb* sets. Now $\{v_2\}$ is an *rmb* set, because from the Remark 2.1,

$$d(v_j/v_2) = \begin{cases} \frac{1}{2} & \text{if} \quad j = 1 \\ j - 2 & \text{if} \quad 3 \leq i \leq n - 1 \\ \frac{2n-3}{2} & \text{if} \quad j = n. \end{cases}$$

Thus $\Gamma(v_i/\{v_2\}) \neq \Gamma(v_j/\{v_2\})$ for every $i \neq j$. Similarly by symmetry $\{v_{n-1}\}$ is also an *rmb* set. But for a vertex v_i which is not an end vertex or a support vertex in P_n, singleton set $\{v_i\}$ is not an *rmb* set, because $d(v_{i-1}/v_i) = 1 = d(v_{i+1}/v_i)$ which imply $\Gamma(v_{i-1}/\{v_i\}) = \Gamma(v_{i+1}/\{v_i\})$ □

Theorem 2.3. *For a Path P_n, a subset $S = \{v_i, v_j\}$, $\forall\ i, j$ with $3 \leq i < j \leq n - 2$ of $V(P_n)$ is a minimal r_r set.*

Proof. Let $S = \{v_i, v_j\}$, $3 \leq i < j \leq n - 2$ be a subset of $V(P_n)$. Let x, y be any two vertices of P_n.
Since $3 < i < n - 2$, from Lemma 2.2, $\{v_i\}$ is not an r_r set, which imply $d(x/v_i) = d(y/v_i)$, for some x, y of $V(P_n)$. Let $d(x/v_i) = d(y/v_i)$ for $x = v_l$ and $y = v_m$ for some l, m with $1 \leq l, m \leq n$. Without loss of generality, consider $l < m$. Consider the following cases.

Case 1: $l = 1$.

From Remark 2.1, $d(v_l/v_j) = \frac{2|j-l|-1}{2}$ and $d(v_m/v_j) =\mid j - m \mid$ as $l < m$. Therefor $d(v_l/v_j) \neq d(v_m/v_j)$ as $d(v_m/v_j)$ is an integer whereas $d(v_l/v_j)$ is not an integer. Hence $\Gamma(x/S) \neq \Gamma(y/S)$

Case 2: $l \neq 1$

From Remark 2.1, $d(v_l/v_j) = \mid j - l \mid$ and $d(v_m/v_j) = \mid j - m \mid$ as $l \neq 1$ $\Rightarrow m \neq 1$.

Suppose $d(v_l/v_j) = d(v_m/v_j)$, then $\mid j - m \mid = \mid j - l \mid$ which imply $j - m = -(j - l)$, because $j - m \neq j - l$ as $l \neq m$. But $j - m = -(j - l)$ $\Rightarrow 2j = m + l$. Similarly we have $d(v_l/v_i) = d(v_m/v_i) \Rightarrow 2i = m + l$. Combining we have $2j = m + l$ and $2i = m + l$ imply $i = j$ which is not possible. Therefore $d(v_l/v_i) \neq d(v_m/v_i)$ and hence $\Gamma(x/S) \neq \Gamma(y/S)$. Other cases follow by symmetry.

Therefore $\forall\ i, j$ with $3 \leq i < j \leq n - 2$, $\{V_i\}$, $\{V_j\}$ are not r_r sets, but $S = \{v_i, v_j\}$ of $V(P_n)$ is an r_r set which imply $S = \{v_i, v_j\}$ is a minimal r_r set. □

Corollary 2.4. *For a Path P_n, $n \geq 2$, any k-element subset S of $V(P_n)$ for $k \geq 2$ is an r_r set,but not minimal, because either S contain end vertices or support vertices or a subset $\{v_i, v_j\}$ with $3 \leq i < j \leq n - 2$.*

Corollary 2.5. *For a Path P_n, $n \geq 6$, a subset $\{v_i, v_j\}$ of $V(P_n)$ with $3 \leq i < j \leq n-2$ is a minimal r_r set with maximum cardinality from Corollary 2.4.*

Theorem 2.6. *For a Path P_n, $l_{r_r}(P_n) = 1$ for $n \geq 1$ and*

$$u_{r_r}(P_n) = \begin{cases} 1 & if \quad n \leq 5 \\ 2 & if \quad n \geq 6 \end{cases}$$

Proof. $\{v_1\}$ is one of the r_r set with minimum cardinality. Therefore $l_{r_r}(P_n) = 1$.

To find $u_{r_r}(P_n)$, consider the following cases.

Case 1: $n \leq 4$.

From Lemma 2.2, every singleton subset of $V(P_n)$ is a minimal r_r set which imply $u_{r_r}(P_n) = 1$.

Case 2: $n = 5$.

From Lemma 2.2, every singleton subset of $V(P_5)$ except $\{v_3\}$ is a minimal r_r set and hence no 2-element subset of $V(P_5)$ is an r_r set which imply $u_{r_r}(P_5) = 1$

Case 3: $n \geq 6$.
From Corollary 2.5, a subset $\{v_i, v_j\}$ with $3 \leq i < j \leq n-2$ is a minimal r_r set with maximum cardinality Therefore $u_{r_r}(P_n) = 2$.

□

Theorem 2.7. *For a Path P_n, $l_{r_r^*}(P_n) = 1$ for $n > 1$ and*

$$u_{r_r^*}(P_n) = \begin{cases} 1 & \text{if} \quad n \leq 5 \\ 2 & \text{if} \quad n \geq 6 \end{cases}$$

Proof. $S = \{v_1\}$ is an r_r set and $\bar{S} = V - S = \{v_2, v_3, ..., v_n, \}$ is also an r_r set as it contain the end vertex v_n. Hence S is r_r^* set with minimum cardinality. Therefor $l_{r_r^*}(P_n) = 1$.

To find $u_{r_r^*}(P_n)$, consider the following cases.

Case 1: $n \leq 5$.
From Lemma 2.2, a singleton subset $S = \{v_1\}$ or $\{v_2\}$ or $\{v_{n-1}\}$ or $\{v_n\}$ is an r_r set and for any S, $\bar{S} = V - S$ is also an r_r set and no k-element subset for $k \geq 2$ of $V(P_n)$ is an r_r set which imply S is a minimal r_r^* set with maximum cardinality. Therefore $u_{r_r^*}(P_n) = 1$.

Case 2: $n \geq 6$.

From Corollary 2.5, a subset $S = \{v_i, v_j\}$ with $3 \leq i < j \leq n-2$ of $V(P_n)$ and $\bar{S} = V - S$ are r_r sets and no k-element subset for $k \geq 3$ of $V(P_n)$ is an r_r set which imply S is an minimal r_r^* set with maximum cardinality. Therefore $u_{r_r^*}(P_n) = 2$.

□

Theorem 2.8. *For a Path P_n,*

$$l_{R_r}(P_n) = u_{R_r}(P_n) = \begin{cases} 0 & \text{if} \quad 1 < n \leq 4 \\ n-1 & \text{if} \quad n \geq 5 \end{cases}$$

Proof. To find $l_{R_r}(P_n)$, consider the following cases.

Case 1: $1 < n \leq 4$.

From Lemma 2.2, every singleton subset of $V(P_n)$ is an r_r set which imply for any k with $1 \leq k \leq 3$, a k-element subset S of $V(P_n)$ is an r_r set and for any S, $\bar{S} = V - S$ is also an r_r set which imply S is not an R_r set and therefore $l_{R_r}(P_n) = u_{R_r}(P_n) = 0$.

Case 2: $n \geq 5$.

From Lemma 2.2, every k-element subset of $V(P_n)$ for $k \geq 2$ is an r_r set and every singleton subset $\{v_i\}$, $3 \leq i \leq n-2$ of $V(P_n)$ is not an r_r set, which imply a subset S of $V(P_n)$ is an R_r set, only if $\bar{S} = V - S$ is a singleton subset $\{v_i\}$, $3 \leq i \leq n-2$ of $V(P_n)$. Therefore $l_{R_r}(P_n) = u_{R_r}(P_n) = n - 1$.

□

Theorem 2.9. *For a Path P_n, $n > 1$, $l_{R_r^*}(P_n) = u_{R_r^*}(P_n) = 0$*

Proof. For any k-element subset S of $V(P_n)$ with $1 \leq k < n-1$, either S or $V - S$ contain atleast one end vertex which imply either S or $V - S$ is always an r_r set. Therefore there exists no R_r^* set for P_n and hence $l_{R_r^*}(P_n) = u_{R_r^*}(P_n) = 0$. □

Theorem 2.10. *For a complete graph K_n, $n > 2$, (when $n = 2$, $K_n = P_n$)*

(i) $l_{r_r}(K_n) = u_{r_r}(K_n) = n - 1$

(ii) $l_{r_r^*}(K_n) = u_{r_r^*}(K_n) = 0$

(iii) $l_{R_r}(K_n) = u_{R_r}(K_n) = n - 1$

(iv) $l_{R_r^*}(K_n) = u_{R_r^*}(K_n) = 2$

Proof. From Ragavendra et al [11], $rmd(K_n) = n-1$ and any $(n-1)$-element subset S of $V(K_n)$ is a minimal r_r set.

(i) $rmd(K_n) = n - 1 \Rightarrow l_{r_r}(K_n) = n - 1$ and there exists no minimal r_r set with cardinality greater than $n - 1$ which imply $u_{r_r}(K_n) = n - 1$.

(ii) Since from (i), any r_r set contain minimum $n-1$ elements, for any subset S of $V(K_n)$, both S and $\bar{S} = V - S$ cannot contain minimum $n-1$ elements. Hence there exist no r_r^* set for K_n and therefore $l_{r_r^*}(K_n) = u_{r_r^*}(K_n) = 0$.

(iii) Since from (i), any minimal r_r set S contain minimum $n-1$ elements, imply $\bar{S} = V - S$ contain exactly one element and hence $\bar{S}$ is not an r_r set. Therefore S is a minimal R_r set with minimum and maximum cardinality which imply $l_{R_r}(K_n) = u_{R_r}(K_n) = n-1$.

(iv) Since from (i), any subset of $V(K_n)$ containing $n-1$ elements is an r_r set, if S is a singleton subset of $V(K_n)$, then $\bar{S} = V - S$ contain $n-1$ elements which imply S is a non r_r set and $\bar{S} = V - S$ is an r_r set so that S is not an R_r^* set. But if S is 2-element subset of $V(K_n)$, then $\bar{S} = V - S$ contain $n-2$ elements which imply both S and $\bar{S} = V - S$ are non r_r sets so that S is an R_r^* set and is minimal. Therefore $l_{R_r^*}(K_n) = u_{R_r^*}(K_n) = 2$.

□

Theorem 2.11. *For a star graph $K_{1,n}$, $n > 2$, (when $n = 2$, $K_{1,n} = P_{n+1}$)*

(i) $l_{r_r}(K_{1,n}) = u_{r_r}(K_{1,n}) = n-1$

(ii) $l_{r_r^*}(K_{1,n}) = u_{r_r^*}(K_{1,n}) = 0$

(iii) $l_{R_r}(K_{1,n}) = u_{R_r}(K_{1,n}) = n-1$

(iv) $l_{R_r^*}(K_{1,n}) = u_{R_r^*}(K_{1,n}) = 2$

Proof. From Ragavendra et al [11], $rmd(K_{1,n}) = n-1$ and any $(n-1)$-element subset S of $V(K_{1,n})$ containing only pendent vertices is a minimal r_r set.

(i) $rmd(K_{1,n}) = n-1 \Rightarrow l_{r_r}(K_{1,n}) = n-1$ and there exists no minimal r_r set with cardinality greater than $n-1$ which imply $u_{r_r}(K_{1,n}) = n-1$.

(ii) Since any r_r set must contain minimum $n-1$ elements, both S and $\bar{S} = V - S$ cannot contain minimum $n-1$ elements. Hence there exists no r_r^* set for $K_{1,n}$ and therefore $l_{r_r^*}(K_{1,n}) = u_{r_r^*}(K_{1,n}) = 0$.

(iii) Any r_r set S contain minimum $n-1$ elements, imply $\bar{S} = V - S$ contain maximum 2 elements and hence $\bar{S}$ is not an r_r set. Also any r_r set of $V(K_{1,n})$ with greater cardinality cannot be minimal. Therefore any r_r set with $n-1$ elements is a minimal R_r set with minimum and maximum cardinality which imply $l_{R_r}(K_{1,n}) = u_{R_r}(K_{1,n}) = n - 1$.

(iv) Since for R_r^* set, both S and $\bar{S}$ should not contain $n-1$ pendent vertices, any 2-element subset of $V(K_{1,n})$ containing only pendent vertices is a minimal R_r^* set with minimum and maximum cardinality. Therefore $l_{R_r^*}(K_{1,n}) = u_{R_r^*}(K_{1,n}) = 2$.

□

Theorem 2.12. *For a cycle C_n, $n > 3$, (when $n = 3$, $C_n = K_n$)*

(i) $l_{r_r}(C_n) = u_{r_r}(C_n) = 2$

(ii) $l_{r_r^*}(C_n) = u_{r_r^*}(C_n) = 2$

(iii)

$$l_{R_r}(C_n) = u_{R_r}(C_n) = \begin{cases} n-1 & \text{if} \quad n \text{ is odd or } n = 4 \\ n-2 & \text{if} \quad n \text{ is even and } n \neq 4 \end{cases}$$

(iv)

$$l_{R_r^*}(C_n) = u_{R_r^*}(C_n) = \begin{cases} 2 & \text{if} \quad n = 4 \\ 0 & \text{if} \quad n > 4 \end{cases}$$

Proof. From Ragavendra et al [11], $rmd(C_n) = 2$. Any 2-element subset S of $V(C_n)$ (non diagonal elements when n is even) is a minimal r_r set.

(i) $rmd(C_n) = 2 \Rightarrow l_{r_r}(C_n) = 2$. Also any k-element subset of $V(C_n)$ for $k \geq 3$ contain 2-element subset which is an r_r set, which imply any 2-element subset of $V(C_n)$ is a minimal r_r set with maximum cardinality. Hence $u_{r_r}(C_n) = 2$

(ii) Since an r_r set of C_n must contain minimum 2 elements, any S of $V(C_n)$ with both S and $\bar{S} = V - S$ containing minimum 2 elements (non diagonal elements when n is even) is an r_r^* set, out of which exactly 2-element subset S is a minimal r_r^* set with minimum and maximum cardinality. Hence $l_{r_r^*}(C_n) = u_{r_r^*}(C_n) = 2$

(iii) Consider the following cases.

Case i When n is odd or $n = 4$
Every k-element subset of $V(C_n)$ for $k \geq 2$ is an r_r set and every singleton subset $\{v_i\}$ of $V(C_n)$ is not an r_r set, which imply a subset S of $V(C_n)$ is an R_r set, only if $\bar{S} = V - S$ is a singleton subset, that is S contain minimum $n-1$ elements. Therefore $l_{R_r}(C_n) = u_{R_r}(C_n) = n-1$.

Case ii When n is even and $n \neq 4$
Since any two diagonally opposite vertices of $V(C_n)$ is a non r_r set, choose S of $V(C_n)$ such that $\bar{S} = V - S$ contain two diagonally opposite vertices of $V(C_n)$. Then S is minimal R_r set with minimum and maximum cardinality $n-2$. Therefore $l_{R_r}(C_n) = u_{R_r}(C_n) = n-2$.

(iv) Consider the following cases.

Case i When $n = 4$
$S = \{v_1, v_3\}$ and $\bar{S} = V - S = \{v_2, v_4\}$ are not r_r sets which imply S is an R_r^* set and hence $l_{R_r^*}(C_n) = u_{R_r^*}(C_n) = 2$.

Case ii When $n > 4$
For any subset S of $V(C_n)$, either S or $V - S$ contain atleast two elements (non diagonal elements when n is even) which imply either S or $V - S$ is always an r_r set. Therefore there exists no R_r^* set for C_n and hence $l_{R_r^*}(C_n) = u_{R_r^*}(C_n) = 0$.

□

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