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Common fixed points for contraction admitting g-center map

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Abstract

In this work we extend the concept of admitting center map to Lipschitz admitting g-center map and obtain Banach contraction principle for the class of such an map. Also we prove some common fixed point theorems for Banach operator pair. At the end we obtain some applications of the results.

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1 Introduction

The technique of employing the asymptotic centers in fixed point theory was first considered by Edelstein [1], and the compactness assumption given on asymptotic centers was introduced by Kirk and Massa [2]. In 2007, García et al. introduced admitting center mappings in Banach space [3]. The study of the existence of fixed points for those mappings is very useful in solving the problems of equations in science and applied science. Our aim here is to study the class of all Banach operator pair mappings contraction admitting a center. Let C be a subset of a Banach space X. We call $y_0 \in X$ a center for the mapping $T : C \to X$ if

$$\|Tx - y_0\| \leq \|x - y_0\|, \tag{1}$$

for any $x \in C$. The mapping $T : C \to X$ is called J-type, whenever it is continuous and it has a center $y_0 \in X$. In this case, by $Z(T)$ we denote the set of all centers of T, that is,

$$Z(T) := \{y_0 \in X \ : \ \|Tx - y_0\| \leq \|x - y_0\|, \quad \text{for all } x \in C\}.$$

The inequality (1) may be satisfied even for a nonexpansive fixed point free mapping. Now, we extend the concept of center to g-center, where g is a map on C into X.

Definition 1.1. *Let C be a subset of a metric space (X, d) and f, g be two maps from C into X. The map f is said to satisfy Lipschitz admitting g-center condition on C, if there exists a constant $L > 0$ and exists $y_0 \in X$, such that for all $x \in C$, we have*

$$d(f(x), y_0) \leq Ld(g(x), y_0). \tag{2}$$

If g is an identity map then f is said to be Lipschitzian admitting center map. The smallest L for which (2) holds is called the Lipschitz admitting g-center constant. In this case, we say that f is an L-Lipschitz admitting g-center map or simply a Lipschitzian admitting g-center map with Lipschitz constant L. Otherwise, it is called non-Lipschitzian admit g-center map. The L-Lipschitz admit g-center map f is said to be contraction admitting g-center, if (2) is satisfied for all $x \in C$, which $g(x) \neq f(x)$ and $L < 1$. The mapping f is called admitting g-center if $L = 1$. The mapping f is said to be strictly admitting

g-center if for all $x \in C$ that $g(x) \neq f(x)$, we have

$$d(f(x), y_0) < d(g(x), y_0),$$

and the point $y_0 \in X$ is said to be a center for the pair (f, g).

Example 1.2. *Let $C = X = [0,1]$ with euclidean metric. Let $f,\ g : [0,1] \to [0,1]$ be given function defined by $f(x) = \frac{x^2}{1+x^2}$ and $g(x) = \frac{x}{1+x^2}$. Set $y_0 = 0$. Clearly $f,\ g$ satisfy (2) with $L = 1$, and so f is admitting g-center.*

Proposition 1.3. *Let C be a subset of a metric space (X, d) and $f, g : C \to X$ be given maps. Let L be a positive real constant. Then the set of all centers of pair (f, g), that is*

$$Z_g(f) = \{y_0 \in X : \quad d(f(x), y_0) \leq Ld\,(g(x), y_0)\,, \quad \text{for all} \ \ x \in C\}.$$

is closed.

Proof. If $Z_g(f) = \emptyset$, the result is obvious. So, suppose that $Z_g(f) \neq \emptyset$. Let $z \in \overline{Z(f)}$. Then there exists a sequence $z_n \in Z(f)$ such that $z_n \to z$ as $n \to \infty$. Consider

$$\begin{aligned} d(f(x), z) = d(f(x), \lim z_n) = \lim d(f(x), z_n) &\leq \lim Ld(g(x), z_n) \\ &= Ld(g(x), \lim z_n) \\ &= Ld(g(x), z). \end{aligned}$$

for all $x \in C$. Consequently, $z \in Z_g(f)$ and so $Z_g(f)$ is closed. □

2 Extension of Banach Contraction Principle

The Banach contraction principle is one of the earliest and most important results in fixed point theory. Because of its application in many field such as physics, computer sciences, chemistry, biology, and many branches of mathematics, a lot of authors have improved, generalized and extended this classical result in nonlinear analysis; see, e.g., [4, 5, 6, 7, 8, 9, 10, 11, 12] and the references therein. Let (X, d) be a complete metric space. The map $T : X \to X$, is said to be a contraction mapping, if for all $x, y \in X$,

$$d(Tx, Ty) \leq kd(x, y), \quad \text{where } 0 < k < 1. \tag{3}$$

According to Banach contraction principle, any map T satisfying (3) has a unique fixed point.
Generalisation of the above principle has been a heavily investigated branch of research. In [4], Boyd and Wong proved that the constant k in (3) can be replaced by the use of an upper semicontinuous function. In [13, 14], generalized Banach contraction conjecture has been established. In [5], Suzuki has proved a generalization of Banach contraction principle which characterizes metric completeness. The Banach contraction principle has also been extended to probabilistic metric spaces [15].

Theorem 2.1. *Let (X, d) be a complete metric space and let K be a closed subset of X, $F : K \to K$ be a continuous contraction L-Lipschitz admitting center. Let $y_0 \in X$ be a center of F. Then F has a fixed point $u \in K$. Also, for each $x \in K$ we have $\lim_{n\to\infty} F^n(x) = u$, with*

$$d(F^n(x), u) \leq \frac{L^{n-1}}{1-L} d(y_0, F(x)).$$

Proof. Let $x \in X$. We first show that $F^n(x)$ is a Cauchy sequence. For $n \in \mathbb{N} \cup \{0\}$ and $y_0 \in X$, we have

$$d(F^n(x), y_0) \leq Ld(F^{n-1}(x), y_0) \leq \cdots \leq L^{n-1} d(F(x), y_0)$$

Thus for $m > n$ where $n \in \{0, 1, \ldots\}$,

$$\begin{aligned}
&d(F^n(x), F^m(x)) \\
&\leq d(F^n(x), F^{n+1}(x)) + d(F^{n+1}(x), F^{n+2}(x)) + \cdots + d(F^{m-1}(x), F^m(x)) \\
&\leq d(F^n(x), y_0) + d(F^{n+1}, y_0) + \cdots + d(F^{m-1}(x), y_0) + d(F^m(x), y_0) \\
&\leq L^{n-1} d(y_0, F(x)) + \cdots + L^{m-1} d(y_0, F(x)) \\
&\leq L^{n-1} d(y_0, F(x))[1 + L + L^2 + \cdots] \\
&= \frac{L^{n-1}}{1-L} d(y_0, F(x))
\end{aligned}$$

Hence for $m > n$, $n \in \{0, 1, \ldots\}$, we have

$$d(F^n(x), F^m(x)) \leq \frac{L^{n-1}}{1-L} d(y_0, F(x)). \tag{4}$$

This shows that $\{F^n(x)\}$ is a Cauchy sequence in K. Since K is complete there exists $u \in K$ with $\lim_{n\to\infty} F^n(x) = u$. As F is continuous, we have $u =$

$\lim_{n\to\infty} F^{n+1}(x) = \lim_{n\to\infty} F(F^n(x)) = F(u)$. So u is a fixed point of F. Now, let m tends to infinity, (4) implies

$$d(F^n(x), u) \leq \frac{L^{n-1}}{1-L} d(y_0, F(x)).$$

□

3 Common fixed-points for Banach operator pair

For two self-maps f and g of the metric space (X, d), we denote by $F(f)$ the set of all fixed points of f, by $C(f, g)$ the set of all coincidence points of f and g, and by $F(f, g)$ the set of all common fixed points of f and g.
The notion of Banach operator pair have been studied by many authors and applied to various problems.

Definition 3.1. *The ordered pair (f, g) of two self-maps f and g of a metric space (X, d) is called a Banach operator pair, if the set $F(g)$ of all fixed points of g is f-invariant, namely $f(F(g)) \subseteq F(g)$.*

Theorem 3.2. *Suppose f and g are two self-maps of a closed subset C of the metric space (X, d), such that (f, g) is a Banach operator pair on C and f is a continuous admitting g-center on C, i.e. there exists $y_0 \in X$, such that*

$$d(f(x), y_0) \leq kd(g(x), y_0), \tag{5}$$

for all $x \in C$, for some $k \in (0, 1)$. If g is continuous, $F(g)$ is nonempty and $\mathrm{cl}(f(C))$ is complete, then $F(f, g) \neq \emptyset$.

Proof. The sets $F(g)$ and $F(f, g)$ are considered as subsets of C. We apply Theorem 2.1 to f on $F(g)$. By assumptions, $f(F(g)) \subseteq F(g)$, $F(g)$ is nonempty closed subset of C and $\mathrm{cl}(f(F(g))) \subseteq \mathrm{cl}(f(C))$ is complete. The inequality (5) follows that

$$d(f(x), y_0) \leq kd(g(x), y_0) = kd(x, y_0), \qquad \text{for all } x \in F(g)$$

Then Theorem 2.1 implies that there is a fixed point $x_0 \in F(g)$ of f and consequently $F(f, g) \neq \emptyset$. □

We need the concepts of q-starshaped and demiclosed at zero in the next theorem.

Definition 3.3. *Let D be a subset of a normed space E.The set D is said to be q-starshaped if there exists $q \in D$ such that $kx + (1 - k)q \in D$ for all $x \in D$ and $k \in [0, 1]$.*
Also a map $J : D \to E$ is said to be demiclosed at zero if, whenever $\{x_n\}$ is a sequence in D such that $x_n \xrightarrow{w} z \in D$ and $Jx_n \xrightarrow{s} 0$, then $Jz = 0$.

Theorem 3.4. *Let S be a weakly compact subset of a normed space X which is starshaped with respect to $p \in S$, and let f and g are two self-maps of S such that (f, g) is a Banach operator pair on S, f is admitting g-center on S, with center p and $p \in F(g)$. If g is both weakly continuous and strongly continuous on S, $F(g)$ is starshaped with respect to p, $\mathrm{cl}(f(S))$ is complete, and if $g - f$ is demiclosed on S, then $F(f, g) \neq \emptyset$.*

Proof. Let $\{k_n\}$ be a sequence of real numbers such that $0 < k_n < 1$ and $k_n \to 1$ as $n \to \infty$. Define a sequence $\{f_n\}$ of self-maps on S by putting $f_n(x) = k_n f(x) + (1 - k_n)p$, for all $x \in S$. Then the following facts are easy to be verified:

(a) for each n, the map $f_n : S \to S$, since S is starshaped with respect to $p \in S$;

(b) since f is admitting g-center on S, it follows that for each n and $x \in S$ and $y_0 = p$,

$$\begin{aligned}
\|f_n(x) - y_0\| &= \|k_n f(x) + (1 - k_n)p - y_0\| \\
&= \|k_n(f(x) - y_0) + (k_n - 1)y_0 + (1 - k_n)p\| \\
&= k_n\|f(x) - y_0\| \\
&\le k_n\|g(x) - y_0\|,
\end{aligned}$$

i.e., each f_n is contraction admitting g-center on S;

(c) for each n, (f_n, g) is a Banach operator pair on S; indeed, since (f, g) is a Banach operator pair, for $x \in F(g)$ we have $f(x) \in F(g)$, and hence $f_n(x) = k_n f(x) + (1 - k_n)p \in F(g)$ by the fact that $F(g)$ is starshaped with respect to $p \in F(g)$;

(d) the completeness of $\mathrm{cl}(f(S))$ implies the completeness of each $\mathrm{cl}(f_n(S))$, and the weak compactness of S implies that S is closed.

Now by Theorem 3.2, for each n, there exists a point $x_n \in S$ such that $x_n \in F(f_n, g)$. Since the weak compactness of S and the weak continuity of g imply the weak compactness of $F(g) \subseteq S$, there exists a subsequence $\{x_{n_i}\}$ which converges weakly to some $x_0 \in F(g)$. In what follows, we shall show that there is also $x_0 \in C(f, g)$. First it is noted that the weak compactness of S implies that S is weakly bounded, and thus strongly (norm) bounded; therefore $\{f(x_{n_i})\} \subseteq S$ is bounded. Since

$$(g - f)(x_{n_i}) = (k_{n_i} f(x_{n_i}) + (1 - k_{n_i})p) - f(x_{n_i}) = (1 - k_{n_i})\,(p - f(x_{n_i}))\,.$$

We have $\|(g - f)(x_{n_i})\| \leq (1 - k_{n_i})(\|p\| + \|f(x_{n_i})\|)$, and hence by $k_{n_i} \to 1$, it follows that

$$\|(g - f)(x_{n_i})\| \to 0 \quad \text{as} \quad i \to \infty. \tag{6}$$

If $g - f$ is demiclosed on S, then from (6) and that $\{x_{n_i}\}$ converges weakly to x_0 we have $(g - f)(x_0) = 0$, i.e. $g(x_0) = f(x_0)$, and thus $x_0 \in F(f, g)$. □

Theorem 3.5. *Let S be a closed subset of a normed space X, which is starshaped with respect to $p \in S$, and let f and g be two self-maps of S such that (f, g) is a Banach operator pair on S, f is admitting g-center on S, with center p and $p \in F(g)$. If f, g are (strongly) continuous on S, the set $F(g)$ is starshaped with respect to p and* $\mathrm{cl}(f(S))$ *is compact, then $F(f, g) \neq \emptyset$.*

Proof. We define the sequence $\{f_n\}$ of maps on S as in the proof of Theorem 3.4, and the assertions from (a) to (d) still hold, with the observation that the compactness of $\mathrm{cl}(f(S))$ implies its completeness, and also the completeness of each $\mathrm{cl}(f_n(S))$. As before, by Theorem 3.2, there exists a point $x_n \in S$ for each n such that $x_n \in F(f_n, g)$. Also in view of the compactness of $\mathrm{cl}(f(S))$, there exists a subsequence $\{x_{n_i}\}$ such that $\{f(x_{n_i})\}$ converges in norm to some $x_0 \in \mathrm{cl}(f(S))$. Since

$$x_{n_i} = f_{n_i}(x_{n_i}) = k_{n_i} f(x_{n_i}) + (1 - k_{n_i})p \to x_0,$$

and $g(x_{n_i}) = x_{n_i}$, the continuity of f and g implies that $g(x_0) = x_0$, and $f(x_{n_i}) \to f(x_0)$, i.e. $f(x_0) = x_0$. Therefore $x_0 \in F(f, g)$. □

4 Applications

Let C be a subset of the normed space $(X, \|.\|)$. For $x_0 \in X$, we denote by $P_C(x_0)$ the set of all best C-approximants to x_0, i.e.

$$P_C(x_0) = \{y \in C : \ d(x_0, C) = \|y - x_0\|\},$$

where $d(x_0, C) = \inf_{z\in C} \|z - x_0\|$.

Theorem 4.1. *Let f and g be self-maps of the normed space $(X, \|.\|)$ and C be a subset of X with $f(\partial C \cap C) \subseteq C$, and $\mathrm{cl}(f(C))$ be complete. Let $x_0 \in F(f, g)$ such that $P_C(x_0)$ is nonempty, weakly compact and starshaped with respect to $p \in F(g)$. If (f, g) is a Banach operator pair on $P_C(x_0)$, f is admitting g-center on $P_C(x_0) \cup \{x_0\}$, with center x_0, and if g is both weakly and strongly continuous on $P_C(x_0)$, $F(g)$ is starshaped with respect to p and $g(P_C(x_0)) \subseteq P_C(x_0)$, and $g - f$ is demiclosed on $P_C(x_0)$, then $P_C(x_0) \cap F(f, g)$ is nonempty.*

Proof. If x_0 is in C, then $x_0 \in P_C(x_0) \cap F(f, g)$ and so the assertion holds. So we assume that $x_0 \notin C$. In this case we have $P_C(x_0) \subseteq \partial C \cap C$, and hence f maps $P_C(x_0)$ into C by assumption. Since $f(x_0) = g(x_0) = x_0$ and f is admitting g-center on $P_C(x_0) \cup \{x_0\}$, with center x_0 it follows that for each $y \in P_C(x_0)$,

$$\|f(y) - x_0\| \leq \|g(y) - x_0\| = d(x_0, C).$$

Since $g(P_C(x_0)) \subseteq P_C(x_0)$, it implies that $f(y) \in P_C(x_0)$. Therefore f is a self-map of $P_C(x_0)$. It is clear that the completeness of $\mathrm{cl}(f(C))$ implies the completeness of $\mathrm{cl}(f(P_C(x_0)))$, and $P_C(x_0) \cap F(g)$ is starshaped with respect to p. Applying Theorem 3.4 to f and g on $S = P_C(x_0)$ proves that $P_C(x_0) \cap F(f, g) \neq \emptyset$. □

Theorem 4.2. *Let f and g be self-maps of the normed space X, and C be a subset of X with $f(\partial C \cap C) \subseteq C$. Let $x_0 \in F(f, g)$ such that $\mathrm{cl}(f(D))$ is compact, where*

$$D := \{y \in P_C(x_0) : g(y) \in P_C(x_0)\}.$$

If (f, g) is a Banach operator pair on D, f is admitting g-center on $D \cup \{x_0\}$, with center x_0, and if g is continuous on $\mathrm{cl}(D)$ and $D \cap F(g)$ is starshaped with respect to p, then $P_C(x_0) \cap F(f, g) \neq \emptyset$.

Proof. If x_0 is in C, then we are done. So we assume that x_0 is not in C and also by assumption, f maps $P_C(x_0)$ into C. It is noted that $D \cap F(g) \neq \emptyset$, at least $p \in D \cap F(g)$. We shall show that $D \cap F(g)$ is closed. In fact, for $\{y_n\} \in D \cap F(g)$ such that $y_n \to y_0$, we have $y_0 \in \text{cl}(D) \subseteq P_C(x_0)$, since $P_C(x_0)$ is closed and from the continuity of g at y_0, we have $y_n = g(y_n) \to g(y_0)$, which implies that $g(y_0) = y_0 \in P_C(x_0)$. Therefore $y_0 \in D \cap F(g)$.

It is clear that g is a self-map of $D \cap F(g)$. We claim that f also maps $D \cap F(g)$ into itself. To show this, let $y \in D \cap F(g)$. By repeating the process of the proof of Theorem 4.1, we obtain $f(y) \in P_C(x_0)$, and since (f, g) is a Banach operator pair on D, we have $g(f(y)) = f(y) \in P_C(x_0)$. Thus $f(y) \in D \cap F(g)$.

Finally we note that $\text{cl}(f(D \cap F(g))) \subseteq \text{cl}(f(D))$ is compact, by assumption. Then applying Theorem 3.5 on $S = D \cap F(g)$ yields the desired conclusion. □

Example 4.3. *Let $X = \mathbb{R}$, be the Banach space of real numbers with $\|x\| = |x|$, and $[a, b] \subset \mathbb{R}$. Let $f : [a, b] \to [a, b]$ be a continuous function and $y_0 \in X$ be a center of f, with $k < 1$. Suppose we want to find the solution of the equation $f(x) = x$. Since $y_0 \in X$ is a center of f, we have*

$$|f(x) - y_0| \leq k|x - y_0|,$$

for all $x \in [a, b]$, and this is definition of contraction admitting center.
Thus f is a continuous contraction admitting center map on $[a, b]$ into itself. Since $[a, b]$ is a closed subset of $X = \mathbb{R}$, by Theorem 2.1, there exists a fixed point $u \in [a, b]$, i.e., $f(u) = u$. Therefore u is the solution of the equation $f(x) = x$.

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