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Generic property of the historic set for ergodic automorphisms of abelian groups

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Abstract

In this paper, we study the generic property of the historic set for ergodic automorphisms of compact metric abelian groups. In particular, our result holds for the k-dimensional torus.

Mathematics Subject Classification: 37D35; 37D45
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1 Introduction

This article is devoted to the study of generic property of the historic set for ergodic automorphisms of the compact metric abelian groups. Before stating

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our results, we first give some notations and backgrounds about the historic set and the specification property for group automorphisms. By a topological dynamical system (TDS) (X, d, T), we mean that (X, d) is a compact metric space and T is a continuous map from X to X. Let $C(X, \mathbb{R})$ be the Banach algebra of real-valued continuous functions of X equipped with the supremum norm. For a continuous function $\varphi : X \to \mathbb{R}$, X can be divided into the following parts:

$$X = \bigcup_{\alpha \in \mathbb{R}} X(\varphi, \alpha) \cup \widehat{X}(\varphi, T),$$

where for $\alpha \in \mathbb{R}$,

$$X(\varphi, \alpha) = \left\{ x \in X : \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(T^i x) = \alpha \right\},$$

and

$$\widehat{X}(\varphi, T) = \left\{ x \in X : \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(T^i x) \text{ does not exist} \right\}.$$

The level set $X(\varphi, \alpha)$ is so-called multifractal decomposition set of ergodic average of φ in multifractal analysis. The set $\widehat{X}(\varphi, T)$ is called the historic set of ergodic average of φ. The historic set was introduced by Ruelle in [22]. $\widehat{X}(\varphi, T)$ is also called non-typical points set(see [5]), irregular set (see [25, 26]) and divergence points set(see [6, 16, 17]). By Birkhoff's ergodic theorem, $\widehat{X}(\varphi, T)$ is not detectable from the view of an invariant measure. However, the recent works [7, 10, 11] have shown that in many cases $\widehat{X}(\varphi, T)$ can have full Hausdorff dimension, that is, for any invariant measure μ, $\mu(\widehat{X}(\varphi, T)) = 0$, but $\dim_H(\widehat{X}(\varphi, T)) = \dim_H(X)$.

Barreira and Schmeling [5] confirmed this in the uniformly hyperbolic setting in the symbolic dynamical system. In 2005, Chen, Kupper and Shu [6] proved that either $\widehat{X}(\varphi, T)$ is empty, or it carries full entropy for the map with the specification property. Thompson [25] extended it to topological pressure for maps with the specification property. In [26], Thompson obtained the same result for maps with g-almost product property, which can be applied to every β-shift. Zhou and Chen [28] also investigated the multifractal analysis for the historic set for systems with g-almost product property.

The specification property for group automorphisms has been studied by several authors (see [1, 2, 13]). In [13], Lind showed that non-hyperbolic toral automorphisms do not obey specification property. According to [2], an automorphism of a solenoidal group has the specification property if and only if it is ergodic under Haar measure and has central spin, so ergodic group automorphisms do not always satisfy specification property. However, Dateyama [9] proved that every ergodic group automorphism satisfies a weak specification property which was introduced by Marcus [15].

In [27], Yamamoto studied the relationships between the specification property, g-almost product property and the almost weak specification property. In [12], Kwietniak, Oprocha and Rams proved that these properties are not equivalent to each other. They constructed a dynamical system with g-almost product property, which does not have the almost weak specification property.

In a metric space X, a subset B is *residual* when its complement is of the first category. In a complete metric space a set is residual if it contains a dense G_δ set (see [18]). From the topological point of view, a set is large if it is residual. During these years, some results show that some irregular sets can be residual. In 2008, Takens [23] indicated why historic behavior is often generic, in particular in the basins of attraction of hyperbolic attractions. He used a result that if a map $g : X \to X$ has an orbit $\{x_0, x_1, \ldots\}$ which is dense and has historic behavior, then there is a residual subset of X such that every orbit starting in that set has historic behavior. In [14], under the hypothesis that a continuous map f satisfies the specification property, Li and Wu prove that the set consisting of those points for which the Birkhoff ergodic average does not exist is either residual or empty.

Motivated by the work of Li and Wu [14], etc, our purpose here is to study generic property of the historic set for ergodic automorphisms of the compact metric abelian groups. The main result of this paper is the following theorem.

Theorem 1.1. *Let X be a compact metric abelian group, σ be an automorphism of X and $\varphi \in C(X, \mathbb{R})$. σ is ergodic under the Haar measure. Then the historic set $\widehat{X}(\varphi, \sigma)$ is residual if it is not empty.*

As we know, every automorphism of a compact metric abelian group is ergodic under the Haar measure if and only if it satisfies almost weak specification (see [9]). To obtain the result, we first study the historic set for maps with almost weak specification. Let (X, d, T) be a TDS. For $\varphi \in C(X, \mathbb{R})$ and $n \geq 1$, let $S_n\varphi(x) := \sum_{i=0}^{n-1} \varphi(T^i x)$, and for $c > 0$, let $\mathrm{Var}(\varphi, c) := \sup\{|\varphi(x) - \varphi(y)| : d(x, y) \leq c\}$.

For every $\epsilon > 0$, $n \in \mathbb{N}$ and a point $x \in X$, we define

$$B_n(x, \epsilon) = \{y \in X : d(T^i x, T^i y) < \epsilon,\ \forall\ 0 \le i \le n-1\},$$

and

$$\overline{B_n}(x, \epsilon) = \{y \in X : d(T^i x, T^i y) \le \epsilon,\ \forall\ 0 \le i \le n-1\}.$$

Denote by $M(X)$, $M(X,T)$ and $E(X,T)$ the collection of all Borel probability measures on X, the collection of all T-invariant Borel probability measures and the collection of all ergodic T-invariant Borel probability measures, respectively. It is well known that $M(X)$ and $M(X,T)$ equipped with the weak* topology are both convex, compact spaces.

Definition 1.1. [9] *We say the map T has almost weak specification property if for any $\epsilon > 0$, there is a function $M_\epsilon : \mathbb{N} \to \mathbb{N}$ with $M_\epsilon(n)/n \to 0$ as $n \to \infty$, so that for any $k \ge 1$ and k points $x_1, \cdots, x_k \in X$ and for any sequence of integers $0 \le a_1 \le b_1 < a_2 \le b_2 < \cdots < a_k \le b_k$ with $a_i - b_{i-1} \ge M_\epsilon(b_i - a_i)(2 \le i \le k)$, there exists a point $x \in X$ such that*

$$d(T^{a_i+j}x, T^j x_i) < \epsilon \qquad (0 \le j \le b_i - a_i, 1 \le i \le k).$$

Theorem 1.2. *Let (X, d, T) be a TDS, T satisfies the almost weak specification property, $\varphi \in C(X, \mathbb{R})$. Then the historic set $\widehat{X}(\varphi, T)$ is residual if it is not empty.*

This article is organized as follows. Section 2 is devoted to the proof of Theorem 1.2, which implies Theorem 1.1. A specific example about the k-dimensional torus is given in Section 3.

2 Proof of Main Results

In this section, we will prove Theorem 1.2. It suffices to show that if $\widehat{X}(\varphi, T)$ is not empty, there exists some set $F \subset X$ satisfying the following properties: (1)$F \subset \widehat{X}(\varphi, T)$; (2)$F$ is dense in X; (3)F is a G_δ set.

We separate the proof into steps.

Step 1. Construction of a specific set F.

Fix $\epsilon > 0$. Let $\{n_k\}_{k\geq 0}$ be a sequence of positive integers with $n_0 = 1$, $\{p_k\}_{k\geq 1}$ be the sequence of integers defined by $p_k = M_{2^{-k}\epsilon}(n_k)$. Let $\{W_k\}_{k\geq 0}$ be a sequence of finite sets in X with $W_0 = \{x_0\} \subset X$. Assume that

$$d_{n_k}(x,y) \geq 8\epsilon \quad \text{(for any } \ x,y \in W_k, \quad x \neq y).$$

Let $\{N_k\}_{k\geq 0}$ be another sequence of positive integers with $N_0 = 1$. Using these data, we are going to construct a subset of Cantor type, which will be denoted by $F = F(\epsilon, \{x_0\}, \{W_k\}, \{n_k\}, \{N_k\})$.

We enumerate the points in the set W_i and consider the set $W_i^{N_i}$. Let $\underline{x}_i = (x_1^i, \cdots, x_{N_i}^i) \in W_i^{N_i}$. For any $(\underline{x}_1, \cdots, \underline{x}_k) \in W_1^{N_1} \times \cdots \times W_k^{N_k}$, by almost weak specification property, we have

$$\begin{aligned} B(\underline{x}_1) = B_{n_0}(x_0, \frac{\epsilon}{2}) \cap T^{-p_1} B_{n_1}(x_1^1, \frac{\epsilon}{2}) \cap T^{-(n_1+p_1)-p_1} B_{n_1}(x_2^1, \frac{\epsilon}{2}) \cap \cdots \cap \\ T^{-(N_1-1)(n_1+p_1)-p_1} B_{n_1}(x_{N_1}^1, \frac{\epsilon}{2})) \neq \emptyset. \end{aligned}$$

Let $l_1 = N_0 n_0 + N_1(p_1 + n_1)$, then

$$\begin{aligned} B(\underline{x}_1, \cdots, \underline{x}_k) = B(\underline{x}_1) \cap \left(\bigcap_{i=1}^{N_2} T^{-t_1-(i-1)(p_2+n_2)-p_2} B_{n_2}(x_i^2, \frac{\epsilon}{2^2}) \right) \cap \cdots \cap \\ \left(\bigcap_{i=1}^{N_k} T^{-t_1-\sum_{j=2}^k N_j(p_j+n_j)-(i-1)(p_k+n_k)-p_k} B_{n_k}(x_i^k, \frac{\epsilon}{2^k}) \right) \neq \emptyset. \end{aligned}$$

Let $l_k = l_1 + \sum_{i=2}^k N_i(p_i + n_i), k \geq 2$. We define F_k by

$$F_k = \bigcup \{B(\underline{x}_1, \cdots, \underline{x}_k) : (\underline{x}_1, \cdots, \underline{x}_k) \in W_1^{N_1} \times \cdots \times W_k^{N_k}\}.$$

Obviously, $F_{k+1} \subset F_k$.

Lemma 2.1. *Let $\underline{x}$ and $\underline{y}$ be distinct elements of $W_1^{N_1} \times \cdots \times W_k^{N_k}$. Then $z_1 = z(\underline{x})$ and $z_2 = z(\underline{y})$ are $(l_k, 6\epsilon)$ separated points.*

Proof. Since $\underline{x} \neq \underline{y}$, there exist i, j such that $x_j^i \neq y_j^i$. We may assume $i \geq 2$, then

$$\begin{aligned} d_{n_i}(x_j^i, T^{t_{i-1}+(j-1)(p_i+n_i)+p_i} z_1) < \epsilon, \\ d_{n_i}(y_j^i, T^{t_{i-1}+(j-1)(p_i+n_i)+p_i} z_2) < \epsilon. \end{aligned}$$

Together with $d_{n_i}(x^i_j, y^i_j) \geq 8\epsilon$, we have

$$\begin{aligned} d_{l_k}(z_1, z_2) \geq & d_{n_i}(T^{t_{i-1}+(j-1)(p_i+n_i)+p_i} z_1, T^{t_{i-1}+(j-1)(p_i+n_i)+p_i} z_2) \\ \geq & d_{n_i}(x^i_j, y^i_j) - d_{n_i}(x^i_j, T^{t_{i-1}+(j-1)(p_i+n_i)+p_i} z_1) - d_{n_i}(y^i_j, T^{t_{i-1}+(j-1)(p_i+n_i)+p_i} z_2) \\ \geq & 8\epsilon - \epsilon - \epsilon = 6\epsilon. \end{aligned}$$

□

Finally, define

$$F(\epsilon, \{x_0\}) := F(\epsilon, \{x_0\}, \{W_k\}, \{n_k\}, \{N_k\}) = \bigcap_{k=0}^{\infty} F_k.$$

Remark 2.1 $d(x_0, y) < \epsilon$ for any $y \in F(\epsilon, \{x_0\})$.

Now we introduce some notations. Let

$$\mathcal{L}_\varphi = \{\alpha \in \mathbb{R} : X(\varphi, \alpha) \neq \emptyset\}.$$

Note that $\mid \alpha \mid \leq \|\varphi\|$ for any $\alpha \in \mathcal{L}_\varphi$, where $\|\varphi\| = \max_{x\in X} \mid \varphi(x) \mid$. For $\alpha \in \mathcal{L}_\varphi$, $\delta > 0$, and $n \in \mathbb{N}$, let

$$P(\alpha, \delta, n) = \{x \in X : \ \mid \frac{S_n\varphi(x)}{n} - \alpha \mid < \delta\},$$

where $S_n\varphi(x)$ is defined as the first section. Clearly, for $\alpha \in \mathcal{L}_\varphi$ and any $\delta > 0$, the set $P(\alpha, \delta, n)$ is not empty for sufficiently large n.

Lemma 2.2. *When $\widehat{X}(\varphi, T)$ is not empty, $\mathcal{L}_\varphi$ is not equal to a single point.*

Proof. By Birkhoff's ergodic theorem, there exists an ergodic invariant probability measure μ such that

$$\frac{1}{n} S_n\varphi(x) \to \int \varphi d\mu, \qquad \text{as } n \to \infty$$

for μ-a.e. $x \in X$. Let $c = \int \varphi d\mu$. Because $\widehat{X}(\varphi, T)$ is not empty, $\frac{1}{n} S_n\varphi$ does not converge to a constant. Hence there exists an $\epsilon > 0$ and sequences $n_k \to \infty$ and $x \in X$ such that

$$\mid \frac{1}{n_k} S_{n_k}\varphi(x) - c \mid > \epsilon.$$

Let $v_k = \delta_{x,n_k} = \frac{1}{n_k}\sum\limits_{i=0}^{n_k-1} \delta_{f^i(x)}$ and μ_2 be a limit point of the sequence v_k. Then $\mu_2 \in M(X,T)$ and $\int \varphi d\mu_2 \neq c$. Hence $\inf\limits_{\mu\in M(X,T)} \int \varphi d\mu < \sup\limits_{\mu\in M(X,T)} \int \varphi d\mu$.

Next we show $\inf\limits_{\mu\in E(X,T)} \int \varphi d\mu < \sup\limits_{\mu\in E(X,T)} \int \varphi d\mu$. Suppose $\mu \in M(X,T)$, by Choquet representation theorem, there is a unique measure τ on the Borel subsets of the compact metrisable space $M(X,T)$ such that $\tau(E(X,T)) = 1$ and for any $f(x) \in C(X,\mathbb{R})$,

$$\int_X f(x)d\mu(x) = \int_{E(X,T)} (\int_X f(x)d\mu(x))d\tau(m).$$

Then for any $\epsilon > 0$, there exists a Borel probability measure $\mu' \in E(X,T)$ such that $\int_X \varphi(x)d\mu'(x) > \int_X \varphi(x)d\mu(x) - \epsilon$. So we have

$$\sup_{\mu\in M(X,T)} \int \varphi d\mu = \sup_{\mu\in E(X,T)} \int \varphi d\mu.$$

Using the same method, we can get a Borel probability measure $\mu'' \in E(X,T)$ such that $\int_X \varphi(x)d\mu''(x) < \int_X \varphi(x)d\mu(x) + \epsilon$, then

$$\inf_{\mu\in M(X,T)} \int \varphi d\mu = \inf_{\mu\in E(X,T)} \int \varphi d\mu.$$

Thus $\inf\limits_{\mu\in E(X,T)} \int \varphi d\mu < \sup\limits_{\mu\in E(X,T)} \int \varphi d\mu$.

Take $\mu_1, \mu_2 \in E(X,T)$ such that $\int \varphi d\mu_1 < \int \varphi d\mu_2$. We can find $x_i \in X$ such that $\frac{1}{n} S_n\varphi(x_i) \to \int \varphi d\mu_i$ as $n \to \infty$ for $i = 1, 2$. Let $\alpha_i = \int \varphi d\mu_i, i = 1, 2$, then the result is deserved.

□

Take $\alpha, \beta \in \mathcal{L}_\varphi$ with $\alpha \neq \beta$. Let $\{\delta_k\}$ be a positive real number sequence satisfying $\delta_k \to 0$ as $k \to \infty$. Choose $\delta, \epsilon > 0$ so small that

$$|\alpha - \beta| > 4\delta, \qquad \mathrm{Var}(\varphi, \epsilon) < \frac{\delta}{4}.$$

Choose an increasing integer sequence $\{n_k\}_{k\geq 1}$ with $n_0 = 1$ such that $P(\alpha, \delta_{2j-1}, n_{2j-1}) \neq \emptyset$ and $P(\beta, \delta_{2j}, n_{2j}) \neq \emptyset$ for $j = 1, 2, \ldots$. Let $D = \{d_1, d_2, \ldots, d_i, \ldots\} \subset X$ be a countable dense set. Fix $d_i \in D$ and $W_0 = \{d_i\}$. For $j \geq 1$, let W_{2j-1} be the $(n_{2j-1}, 8\epsilon)$-separated set in $P(\alpha, \delta_{2j-1}, n_{2j-1})$ and W_{2j} the $(n_{2j}, 8\epsilon)$-separated set in $P(\beta,\delta_{2j},n_{2j})$. Finally, choose a sequence $\{N_k\}_{k\geq 0}$ increasing to ∞ sufficiently quickly so that

$$\lim_{k\to\infty}\frac{n_{k+1}+p_{k+1}}{N_k}=0,\qquad \lim_{k\to\infty}\frac{N_0n_0+N_1(n_1+p_1)+\cdots+N_k(n_k+p_k)}{N_{k+1}}=0. \tag{1}$$

By the construction presented in the former section, we obtain a set

$$F(\epsilon,\{d_i\}):=F(\epsilon,\{d_i\},\{W_k\},\{n_k\},\{N_k\}).$$

Write

$$F(\epsilon)=\bigcup_{i=1}^{\infty}F(\epsilon,\{d_i\}).$$

Finally, let

$$F=\bigcup_{j=1}^{\infty}F(\frac{1}{j})=\bigcup_{j=1}^{\infty}\bigcup_{i=1}^{\infty}F(\frac{1}{j},\{d_i\},\{W_k\},\{n_k\},\{N_k\}).$$

Step 2. We prove $F\subset\widehat{X}(\varphi,T)$. It suffices to prove that $F(\epsilon,\{d_i\})\subset\widehat{X}(\varphi,T)$ for any $\epsilon>0$ and any $d_i\in D$.

Lemma 2.3. *For any $q\in F(\epsilon,\{d_i\})$, $\lim\limits_{k\to\infty}\frac{1}{l_k}\sum\limits_{i=0}^{l_k-1}\varphi(T^iq)$ does not exist.*

Proof. Choose $q\in F(\epsilon,\{d_i\})$ and let $q_k=T^{l_{k-1}}(q)$. Then there exists $(x_1^k,\cdots x_{N_k}^k)\in W_k^{N_k}$ such that

$$q_k\in\bigcap_{i=1}^{N_k}T^{-(i-1)(p_k+n_k)-p_k}\overline{B_{n_k}}(x_i^k,\frac{\epsilon}{2^k}).$$

We first show

$$\left|\frac{1}{N_{2k-1}(p_{2k-1}+n_{2k-1})}S_{N_{2k-1}(p_{2k-1}+n_{2k-1})}\varphi(q_{2k-1})-\alpha\right|\to 0.$$

Let $t_i=(i-1)(p_{2k-1}+n_{2k-1})+p_{2k-1}$, we have

$$\begin{aligned}
&\left|S_{N_{2k-1}(p_{2k-1}+n_{2k-1})}\varphi(q_{2k-1})-N_{2k-1}(p_{2k-1}+n_{2k-1})\alpha\right|\\
\leq&\left|\sum_{i=1}^{N_{2k-1}}S_{n_{2k-1}}\varphi(T^{t_i}q_{2k-1})-N_{2k-1}n_{2k-1}\alpha\right|+2N_{2k-1}p_{2k-1}\|\varphi\|\\
\leq&\sum_{i=1}^{N_{2k-1}}\left|S_{n_{2k-1}}\varphi(T^{t_i}q_{2k-1})-S_{n_{2k-1}}\varphi(x_i^{2k-1})\right|+\sum_{i=1}^{N_{2k-1}}\left|S_{n_{2k-1}}\varphi(x_i^{2k-1})-n_{2k-1}\alpha\right|\\
&+2N_{2k-1}p_{2k-1}\|\varphi\|\\
\leq&n_{2k-1}N_{2k-1}\{\mathrm{Var}(\varphi,\frac{\epsilon}{2^{2k-1}})+\delta_{2k-1}\}+2N_{2k-1}p_{2k-1}\|\varphi\|.
\end{aligned}$$

Since

$$\mathrm{Var}(\varphi, \frac{\epsilon}{2^{2k-1}}) < \frac{\delta}{4}, \quad \lim_{k\to\infty} \delta_k = 0 \text{ and } \lim_{k\to\infty} \frac{p_{2k-1}}{n_{2k-1}} = \lim_{k\to\infty} \frac{M_\epsilon(n_{2k-1})}{n_{2k-1}} = 0,$$

for sufficiently large k, we have

$$\left| \frac{1}{N_{2k-1}(p_{2k-1}+n_{2k-1})} S_{N_{2k-1}(p_{2k-1}+n_{2k-1})}\varphi(q_{2k-1}) - \alpha \right| \leq \frac{\delta}{2}.$$

One can readily verify that $\frac{N_{2k-1}(p_{2k-1}+n_{2k-1})}{l_{2k-1}} \to 1$ as $k \to \infty$. Thus for sufficiently large k, we have $\left|\frac{N_{2k-1}(p_{2k-1}+n_{2k-1})}{l_{2k-1}} - 1\right| \leq \frac{\delta}{4\|\varphi\|}$. We obtain that

$$\begin{aligned}
&\left| \frac{1}{l_{2k-1}} S_{l_{2k-1}}\varphi(q) - \frac{1}{N_{2k-1}(p_{2k-1}+n_{2k-1})} S_{N_{2k-1}(p_{2k-1}+n_{2k-1})}\varphi(q_{2k-1}) \right| \\
\leq& \left| \frac{1}{l_{2k-1}} S_{l_{2k-1}-N_{2k-1}(p_{2k-1}+n_{2k-1})}\varphi(q) \right| \\
&+ \left| \frac{S_{N_{2k-1}(p_{2k-1}+n_{2k-1})}\varphi(q_{2k-1})}{N_{2k-1}(p_{2k-1}+n_{2k-1})} \left(\frac{N_{2k-1}(p_{2k-1}+n_{2k-1})}{l_{2k-1}} - 1 \right) \right| \\
\leq& \frac{l_{2k-2}}{l_{2k-1}} \|\varphi\| + \frac{\delta}{4} \\
\leq& \frac{\delta}{2}.
\end{aligned}$$

Hence for sufficiently large k,

$$\left| \frac{1}{l_{2k-1}} \sum_{i=0}^{l_{2k-1}-1} \varphi(T^i q) - \alpha \right| \leq \delta < \frac{|\ \alpha - \beta\ |}{4}.$$

In a similar way, we can also prove the following estimate. For sufficiently large k,

$$\left| \frac{1}{l_{2k}} \sum_{i=0}^{l_{2k}-1} \varphi(T^i q) - \beta \right| \leq \delta < \frac{|\ \alpha - \beta\ |}{4},$$

the desired result follows. □

Step 3. We show that F is dense in X and is a G_δ set.

To prove that F is dense in X, it suffices to show that $F \cap B(x, r) \neq \emptyset$ for every $x \in X$ and $r > 0$. Given $x \in X$ and $r > 0$, there exist $j \in \mathbb{N}$ with $\frac{2}{j} < r$ and $d_i \in D$ such that $d(x, d_i) < \frac{1}{j}$. Choose any point $y \in F(\frac{1}{j}, d_i) \subset F$, it follows from Remark 2.1 that $d(y, d_i) < \frac{1}{j}$. Hence

$$d(x, y) < d(x, d_i) + d(d_i, y) < \frac{2}{j} < r.$$

This implies that $F \cap B(x, r) \neq \emptyset$.

Clearly, the sets F_k's are open sets, $F(\epsilon, d_i) = \bigcap_{k \geq 0} F_k$. Because the intersection of countable G_δ sets is also a G_δ set, it is obvious that $F(\epsilon, d_i)$ is a G_δ set for any $\epsilon > 0$ and any $d_i \in D$.

3 An Example

In this section, we will give a specific case of Theorem 1.1.

Take $\mathbb{T}^k = \mathbb{R}^k/\mathbb{Z}^k$ to be the k-torus. Let $A = (a_{ij})$ be a $k \times k$ matrix with entries in $\mathbb{Z}$ and with $detA \neq 0$. We can define a linear map $\mathbb{R}^k \to \mathbb{R}^k$ by $(x_1, \ldots, x_k)' \mapsto A(x_1, \ldots, x_k)'$, where the mark $\prime$ denotes the transposition of a vector. Since A is an integer matrix, it maps $\mathbb{Z}^k$ to itself. We know that A allows us to define a map

$$T = T_A : \mathbb{R}^k/\mathbb{Z}^k \to \mathbb{R}^k/\mathbb{Z}^k;$$

$$(x_1, \ldots, x_k)' \mapsto A(x_1, \ldots, x_k)' \ \ mod\ 1.$$

Definition 3.1. *Let $A = (a_{ij})$ denote a $k \times k$ matrix with integer entries such that $detA \neq 0$. We call the map $T_A : \mathbb{R}^k/\mathbb{Z}^k \to \mathbb{R}^k/\mathbb{Z}^k$ a linear toral endomorphism.*

The map T is not invertible in general. However, if $detA = \pm 1$, then A^{-1} exists and is an integer matrix. Hence we have a map T^{-1} given by

$$T^{-1}(x_1, \ldots, x_k)' \mapsto A(x_1, \ldots, x_k)' \ \ mod\ 1.$$

Definition 3.2. *Let $A = (a_{ij})$ denote a $k \times k$ matrix with integer entries such that $detA = \pm 1$. We call the map $T_A : \mathbb{R}^k/\mathbb{Z}^k \to \mathbb{R}^k/\mathbb{Z}^k$ a linear toral automorphism.*

Now T denotes an ergodic automorphism of the torus $\mathbb{T}^k$. Marcus [15] proved that T satisfies the almost weak specification. Hence from Theorem 1.1, we can deserve the following result.

Theorem 3.1. *Let $\mathbb{T}^k = \mathbb{R}^k/\mathbb{Z}^k$ be the k-dimensional torus. T denotes an ergodic automorphism of the torus $\mathbb{T}^k$ and $\varphi \in C(\mathbb{T}^k, \mathbb{R})$. Then the historic set $\widehat{\mathbb{T}^k}(\varphi, T)$ is residual if it is not empty.*

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