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Using homotopy analysis method for solving Volterra integral equations of the second kind

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Abstract

This paper employs an analytical method, where linear and nonlinear integral equations are solve using the homotopy analysis method. The Volterra integral equations of the second kind are considered by this method. Here an infinite solution series which converges to the exact solution of considered equations are realized. In this method one is allowed to choose an initial guess and iteratively deforms the considered equations with an initial guess to obtain the exact solution.

Keywords: Exact solution; Integral equations; Volterra integral equations; Homotopy Analysis Method.

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1 Introduction

Liao in 1992 proposed this analytical method known as homotopy analysis method [5] .In this method, the exact solution is obtained as the summation of an infinite series which usually converges quickily to the exact solution. Liao in [7] introduced an auxiliary parameter h, as a convergence control parameter. Let's consider linear and nonlinear Volterra integral equations

$$y(x) = g(x) + \lambda \int_r^x H(x,t)dt \quad (1)$$

In the equation (1) the upper limit is variable, as the equation is Volterra integral equation, the kernel H(x,t) [1] and g(x) are known functions, whereas y is to be determined, λ is a complex number [3].

2 Description of the method

Let the following equation

$$N[r(x)]=0 \quad (2)$$

These are parameters in equation (2) N nonlinear operator, r(x) which is unknown function and x independent variable [8]. Consider $r_0(x)$ that represents an initial guess of the exact solution r(x), $h \neq 0$ an auxiliary parameter, $H(x) \neq 0$ an auxiliary function, and L an auxiliary linear operator with the property L[r(x)]=0 when r(x)= 0. Then using $q \in [0,1]$ as an embedding parameter [8], homotopy is constructed as [6] and [4]

$$(1 - q)L[\phi(x : q) - r_0(x)] - qhH(x)N[\phi(x : q)] = 0 \quad (3)$$

It should be emphasized that ,there is great laxity in selecting an initial guess $r_0(x)$, the auxiliary linear operator L, the non-zero auxiliary parameter h,[6] and the auxiliary function H(x).We have the so-called zero-order deformation equation[8]

$$(1- q)L[\phi(x : q) - r_0(x)] = qhH(x)N[\phi(x : q)] \quad (4)$$

When q=0, the equation (4) becomes ϕ (x:0)= r_0(x). And when q=1, since h≠0 and H(x) ≠0, the equation (3) is equivalent to ϕ (x:1)= r(x) .For (3) and (4) as the embedding parameter [8] increases from 0 to 1, ϕ (x:q) varies continuously from the initial approximation r_0(x) to the exact solution r(x)[6]. Where this continuous variation is known as deformation in homotopy [1] and[4].

As apply to Taylor's theorem, ϕ(x;q) can be expanded in an exponential series of q as follows

$$\phi(x:q) = r_0(x)+\sum_{n=1}^{\infty} r_n(x)q^n \quad (5)$$

where

$$r_n(x)=\frac{1}{n!}\frac{\delta^n \phi(x:q)}{\delta q^n}\,|_{q=0} \quad (6)$$

If the initial guess r_0(x), the auxiliary linear parameter L, the non-zero auxiliary parameter h and the auxiliary function H(x) are selected properly, it fascilitates the convergence of power series (5) of ϕ(x:q) [8] at a point where q=1. Where these assumptions of the solution series are made

$$r(x)= \phi(x:1) = r_0(x)+ r_n(x) \quad (7)$$

where the vector is defined

$$r_n(x)=r_0(x), r_1(x), r_2(x),\ldots r_n(x) \quad (8)$$

$$L[r_n(x) - \chi_n r_{n-1}(x)]=hH(x)R_n(\vec{r}_{n-1}(x)), \qquad r_n(0)=0 \quad (9)$$

Where

$$R_n(r(x)=\frac{1}{(n-1)!}\frac{\delta^{n-1} N[\phi(x:q)]}{\delta q^{n-1}}\,|_{q=0} \quad (10)$$

And

$$\chi_n= \begin{cases} 0, & for\ n \le 1 \\ 1, & for\ n \ge 1 \end{cases}$$

As in [2] the equation (9) is governed by L, and $R_n(\vec{r}_{n-1}(x)$ as presented by (10) for any N, which is the nonlinear operator. And MATLAB is used to obtain

$r_n(x)$ the solution. The solution r(x) depends on L,h,H(x) and $r_0(x)$see[6]. If $\sum_{n=1}^{\infty} r_n(x)$ moves to a limit as n→∞, converges to the exact solution [8].

3 HAM's solution to Volterra integral equations

Let consider the equation

$$h(t)u(t)=g(x)+\lambda\int_r^x H(t,x)\,u(t)dt \tag{11}$$

where equation (11)[2] is Volterra integral equation of the second kind if h(t)=1 is substituted into equation(11),then

$$U(t)=g(t)+\lambda\int_r^x H(t,x)u(x)dx \qquad b\le t\le c \tag{12}$$

construct the zeroth-order deformation [8] for this kind of integral equations as

$$(1-p)(u(t,p,h)-g(x))=hp(u(t,p,h)-g(t)-\int_r^x H(x,t)u(x,p,h)dx) \tag{13}$$

For p=0 and p=1,implies

u(t,0,h)=g(t)

u(t,1,h)=u(t)

For Maclaurin series of u(t,p,h) corresponding to p, then

$$U(t,p,h)=u(t,0,h)+\sum_{n=1}^{+\infty}\frac{u_0^{[n]}(t,h)}{n!}p^n \tag{14}$$

Which

$$u_0^{[n]}(t,h)=\frac{\delta^n u(t,p,h)}{\delta p^n}\Big|_{p=0} \tag{15}$$

Substituting p=1 into (14) give

$$u(t)=g(t)+\sum_{n=1}^{+\infty}\frac{u_0^{[n]}(t,h)}{n!} \tag{16}$$

where the nth-order deformation equation is obtained as

$$L[u_0^{[n]}(t,h) - \chi_n u_0^{[n]}(t,h)] = hR_n(\vec{u}_{n-1}) \tag{17}$$

And the solution of the nth-order deformation equation for n≥1 yields

$$u_0^{[1]}(t,h) = -h\int_r^x H(t,x)g(x)dx \tag{18}$$

And

$$\frac{u_0^{[n]}(t,h)}{n!} = \frac{u_0^{[n-1]}(t,h)}{(n-1)!} + h\frac{(u_0^{[n-1]}(t,h))}{(n-1)!} - h\int_r^x H(x,t)\frac{u_0^{[n-1]}(x,h)}{(n-1)!}dx \tag{19}$$

The solution of the problem looks similar to that of homotopy perturbation method when one choose h = -1[8], [4] and [1].

Applying the HAM

Here application of the HAM to Volterra integral equations are considered.

4 The Volterra integral equation of the second kind [2]

Let look at Volterra integral equation of the second kind, which reads

$$\phi(x) = g(x) + \int_a^x H(x;\ t)\phi(t)dt \tag{20}$$

where H(x,t) is the kernel of the integral equation

Example 1: Consider this Volterra integral equation

$$\phi(x) = x + \int_0^x (3t - x)\,\phi(t)dt \tag{21}$$

Then choose

$$\phi_0(x) = x \tag{22}$$

where the linear operator

$$L[\phi(x,p)] = \phi(x,p) \tag{23}$$

the nonlinear operator is define below

$$N[\phi(x,p)]= \phi(x,p)-x-\int_0^x (3t-x)\phi(t)dt \qquad (24)$$

where the nth-order deformation equation is

$$L[\phi_n-\chi_n\phi_{n-1}]=hR_n(\vec{\phi}_{n-1}) \qquad (25)$$

And

$$R_n(\vec{\phi}_{n-1})= \phi_{n-1}(x)-(1-\chi_n)x+\int_0^x (3t-x)\phi_{n-1}(t)dy \qquad (26)$$

where the solution of the nth-order deformation equation (25)

$$\phi_n(x)= \chi_n\phi_{n-1}(x)+hL^{-1}[R_n(\vec{\phi}_{n-1})] \qquad (27)$$

Finally,

$$\phi(x)=\phi_0(x)+\sum_{n=1}^{+\infty}\phi_n(x) \qquad (28)$$

where

$$\phi_0(x)=x$$

$$\phi_1(x)=-\int_0^x (3t-x)\,\phi_0(t)dt = -h\frac{3}{3!}x^3$$

$$\phi_2(x)=-\int_0^x (3t-x)\,\phi_1(t)dt = h\frac{9}{5!}x^5$$

$$\phi_3(x)=-\int_0^x (3t-x)\,\phi_2(t)dt = -h\frac{27}{7!}x^7$$

$$\phi_4(x)=-\int_0^x (3t-x)\,\phi_3(t)dt = h\frac{81}{9!}x^9$$

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Hence

$\phi(x)= \phi_0(x) + \phi_1(x)+\phi_2(x)+\phi_3(x)+\phi_4(x) + \cdots$

$=x - h\frac{3}{3!}x^3 + h\frac{9}{5!}x^5 - h\frac{27}{7!}x^7 + h\frac{81}{9!}x^9$

If h= -1

$= x + \frac{3}{3!}x^3 - \frac{9}{5!}x^5 + \frac{27}{7!}x^7 - \frac{81}{9!}x^9$

$$=\sum_{n=0}^{+\infty}\frac{(-3)^n}{(2n+1)!}x^{(2n+1)} \quad (29)$$

Which is the exact solution of equation (21)

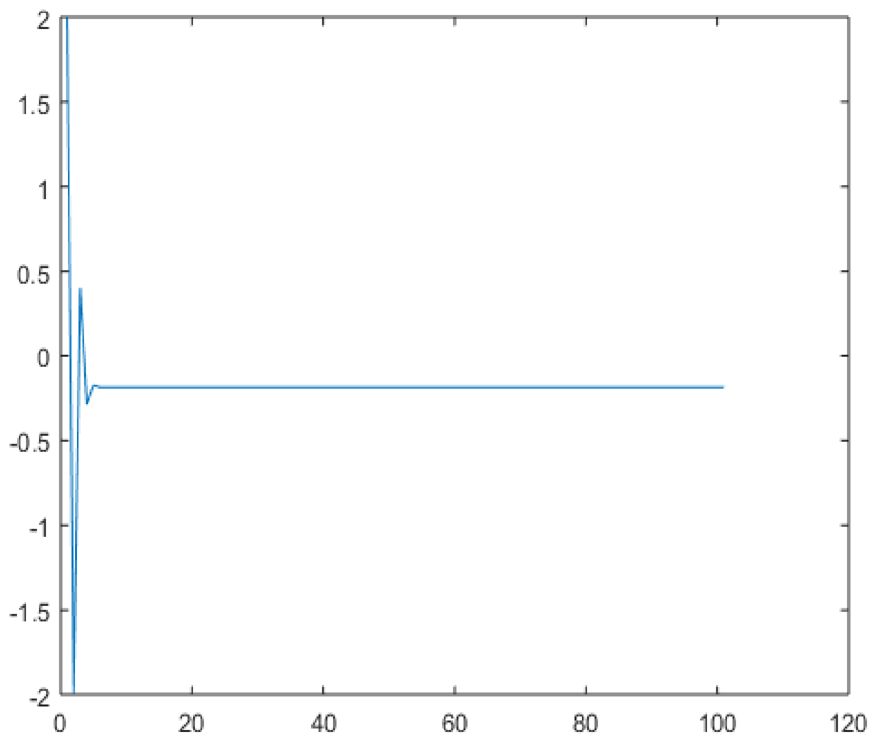

Figure 4:1 Example 1.Exact solution to equation (21) [3].

The following algorithm produces Figure 4.1 using the Matlab software.

```
function [x,sumc] = solplot1(x,n)
      sumc(1) = x;
   for i=1:n
      num = (-3)^i;
      den = factorial(2*i+1);
      rsult = num/den; rsult = rsult*x^(2*i+1);
      sumc(i+1) = sumc(i) + rsult;
   end
    plot(1:n+1,sumc)
    %plot(1:n,sumc(2:end))
end
```

Example 2: Consider the following Volterra integral equation

$$\phi(x)=2x - x^2 - \int_0^x \phi(t)dt \tag{30}$$

To solve equation (30), choose

$$\phi_0(x)=2x - x^2 \tag{31}$$

And the linear operator

$$L[\phi (x; p)] = \phi (x; p) \tag{32}$$

And the nonlinear operator is defined as

$$N[\phi (x; p)] = \phi (x; p) - 2x + x^2 + \int_0^x \phi(t)dt \tag{33}$$

Let's construct the nth-order deformation equation

$$L[\phi_n-\chi_n\phi_{n-1}]=hR_n(\vec{\phi}_{n-1}) \tag{34}$$

And

$$R_n(\vec{\phi}_{n-1})= \phi_{n-1}(x) - (1-\chi_n)2x + x^2 + \int_0^x \phi(t)dt \tag{35}$$

The solution of the nth-order deformation equation (34)

$$\phi_n(x)= \chi_n\phi_{n-1}(x)+hL^{-1}[R_n(\vec{\phi}_{n-1})] \tag{36}$$

Finally, the solution of equation (30) is

$$\phi(x)=\phi_0(x)+\sum_{n=1}^{+\infty}\phi_n(x) \tag{37}$$

where

$\phi_0(x)= 2x - x^2$

$\phi_1(x)=\int_0^x \phi_0(t)dt = \int_0^x (2t - t^2)dt = h\,(x^2 - \frac{1}{3}x^3)$

$\phi_2(x)=\int_0^x \phi_1(t)dt = \int_0^x \left(t^2 - \frac{t^3}{3}\right) dt = h(\frac{1}{3}x^3 - \frac{x^4}{12})$

$\phi_3(x)=\int_0^x \phi_2(t)dt = \int_0^x \left(\frac{1}{3}t^3 - \frac{t^4}{12}\right) dt = h(\frac{x^4}{12} - \frac{x^5}{60})$

$\phi_4(x)=\int_0^x \phi_3(t)dt = \int_0^x \left(\frac{t^4}{12} - \frac{t^5}{60}\right) dt = h(\frac{x^5}{60} - \frac{x^6}{360})$

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Hence

$\phi(x) = \phi_0(x) + \phi_1(x)+\phi_2(x)+\phi_3(x)+\phi_4(x) + \cdots$

$= 2x - x^2 + h(x^2 - \frac{1}{3}x^3) + h(\frac{1}{3}x^3 - \frac{x^4}{12}) + h(\frac{x^4}{12} - \frac{x^5}{60}) + h(\frac{x^5}{60} - \frac{x^6}{360})$

If h= -1

$=2x - x^2 - x^2 +\frac{1}{3}x^3 - \frac{1}{3}x^3 + \frac{x^4}{12} - \frac{x^4}{12} + \frac{x^5}{60} - \frac{x^5}{60} + \frac{x^6}{360}$

$$= \sum_{n=0}^{+\infty}\phi_n(x) = 2x - 2x^2 \tag{38}$$

Which is the exact solution to equation (30), [3].

Example 3: Let consider the following Volterra equation

$$\phi(x)= x + \int_0^x \phi^2(t)dt \tag{39}$$

To solve equation(39),let

$$\phi_0(x) = x \tag{40}$$

choose linear operator

$$L[\phi(x;p)] = \phi(x;p) \tag{41}$$

Thus the nonlinear operator is

$$N[\phi(x;p)] = \phi(x;p) - x - \int_0^x \phi^2(t)dt \tag{42}$$

And the nth-order deformation equation is as follows

$$L[\phi_n(x) - \chi_n\phi_{n-1}] = hR_n(\vec{\phi}_{n-1}) \tag{43}$$

And

$$R_n(\vec{\phi}_{n-1}) = \phi_{n-1}(x) - (1-\chi_n)\, x - \int_0^x \phi^2(t)dt \tag{44}$$

where the solution of the nth-order deformation equation (43)

$$\phi_n(x) = \chi_n\phi_{n-1}(x) + hL^{-1}[R_n(\vec{\phi}_{n-1})] \tag{45}$$

Finally, the solution of equation (39) is

$$\phi(x) = \phi_0(x) + \sum_{n=1}^{+\infty} \phi_n(x) \tag{46}$$

where

$$\phi_0(x) = x$$

$$\phi_1(x) = -\int_0^x \phi_0^2(t)dt = -[\frac{1}{3!}t^3]_0^x = -h\frac{1}{3!}x^3$$

$$\phi_2(x) = -\int_0^x \phi_1^2(t)dt = -[\frac{1}{7!}t^7]_0^x = -h\frac{1}{7!}x^7$$

$$\phi_3(x) = -\int_0^x \phi_2^2(t)dt = -[\frac{1}{15!}t^{15}]_0^x = -h\frac{1}{15!}x^{15}$$

$$\phi_4(x) = -\int_0^x \phi_3^2(t)dt = -[\frac{1}{31!}t^{31}]_0^x = -h\frac{1}{31!}x^{31}$$

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Hence

$\phi(x) = \phi_0(x) + \phi_1(x) + \phi_2(x) + \phi_3(x) + \phi_4(x) + \cdots$

$= x + -h\frac{1}{3!}x^3 + -h\frac{1}{7!}x^7 + -h\frac{1}{15!}x^{15} + -h\frac{1}{31!}x^{31} + \ldots$

If h= -1

$= x + \frac{1}{3!}x^3 + \frac{1}{7!}x^7 + \frac{1}{15!}x^{15} + \frac{1}{31!}x^{31} + \ldots$

$$= \sum_{n=0}^{+\infty} \phi_n(x) \qquad (47)$$

Which is the exact solution to equation (39).

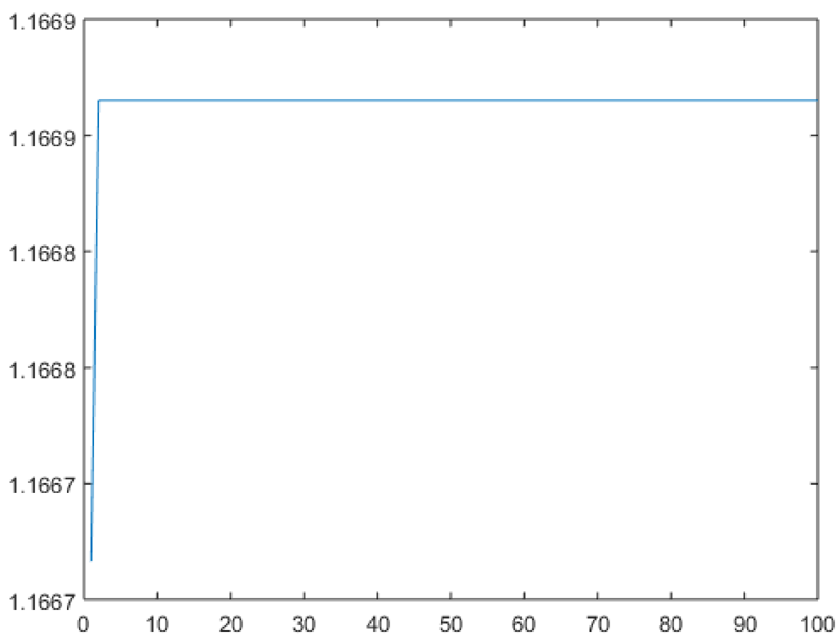

Figure 4.2: Example 3. Exact solution to equation (39) [3].

The following algorithm produces Figure 4.2 using the Matlab software.

```
function [x,sumc] = solplot4(x,n)
   sumc(1) = x; m=1;
   for i=1:n
      m = 2*m+1;
      num = 1;
      den = factorial(m);
      rsult = (num*x^m)/den;
      sumc(i+1) = sumc(i) + rsult;
   end
    plot(1:n,sumc(2:end))
end
```

Example 4: Let consider the Volterra integral equations

$$\phi(x)= x + \frac{1}{2}\int_0^x \phi^2(t)dt \qquad (48)$$

To solve equation (48), choose

$$\phi_0(x) = x \qquad (49)$$

the linear operator

$$L[\phi (x; p)] = \phi(x; p) \qquad (50)$$

Now define the nonlinear operator is

$$N[\phi (x; p)] = \phi (x; p) - x - \frac{1}{2}\int_0^x \phi^2(t)dt \qquad (51)$$

And the nth-order deformation equation is

$$L[\phi_n - \chi_n\phi_{n-1}] = hR_n(\vec{\phi}_{n-1}) \qquad (52)$$

And

$R_n(\vec{\phi}_{n-1}) = \phi_{n-1}(x) - (1-\chi_n)\, x - \frac{1}{2}\int_0^x \phi^2(t)dt$ (53)

where the solution of the nth-order deformation equation (52)

$\phi_n(x) = \chi_n \phi_{n-1}(x) + hL^{-1}[R_n(\vec{\phi}_{n-1})]$ (54)

Finally,equation (48)

$\phi(x) = \phi_0(x) + \sum_{n=1}^{+\infty} \phi_n(x)$ (55)

where

$$\phi_0(x) = x$$

$\phi_1(x) = -\frac{1}{2}\int_0^x \phi_0^2(t)dt = -\frac{1}{2}[\frac{t^3}{3}]_0^x = -h\frac{1}{6}x^3$

$\phi_2(x) = -\frac{1}{2}\int_0^x \phi_1^2(t)dt = -\frac{1}{2}[\frac{t^{10}}{10}]_0^x = -h\frac{1}{20}x^{10}$

$\phi_3(x) = -\frac{1}{2}\int_0^x \phi_2^2(t)dt = -\frac{1}{2}[\frac{t^{101}}{101}]_0^x = -h\frac{1}{202}x^{101}$

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Hence

$\phi(x) = \phi_0(x) + \phi_1(x) + \phi_2(x) + \phi_3(x) + \ldots$

$= x + -h\frac{1}{6}x^3 + -h\frac{1}{20}x^{10} + -h\frac{1}{202}x^{101}$

If h= - 1

$= x + \frac{1}{6}x^3 + \frac{1}{20}x^{10} + \frac{1}{202}x^{101}$

Which is the exact solution to equation (48).

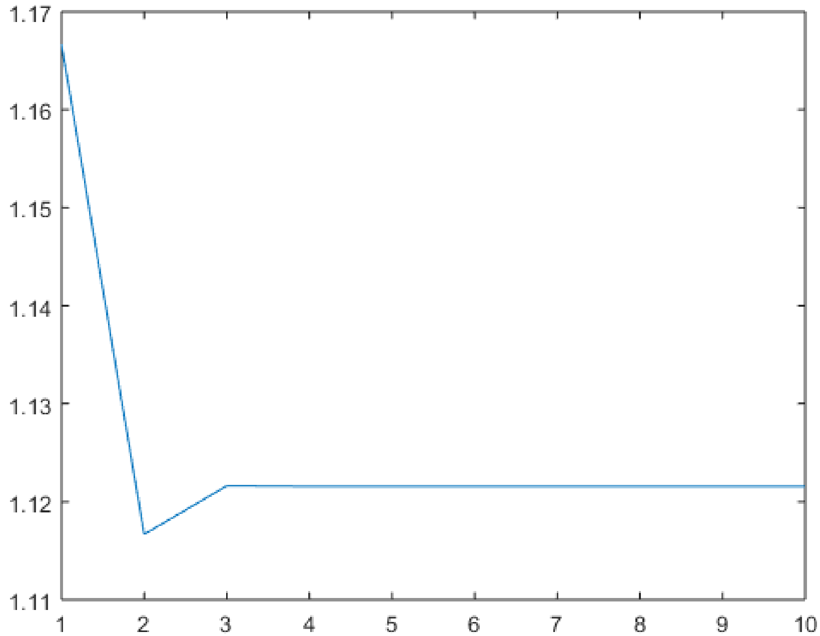

Figure 4.3: Example 4. Exact solution to equation (48).

The following algorithm produces Figure 4.3 using the Matlab software.

```
function [x,sumc] = solplot5(x,n)
   sumc(1) = x; m=sqrt(2);
   for i=1:n
      m = m^2 + 1;
      num = x^m;
      den = 2*m;
      rsult = (-1)^(i+1) * (num/den);
      sumc(i+1) = sumc(i) + rsult;
   end
    plot(1:n,sumc(2:end))
end
% % Script to run
```

Solplot1(0.5,100)

Solplot2(1,100)

Solplot3(1,100)

\end{verbatim}[3]

5 Conclusion

Volterra integral equation of the second kind has been solved successfully by Homotopy analysis method (HAM).The exact solutions obtained by the analytical solution of the considered equations showed that HAM is a powerful method for solving Volterra integral equations.

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