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A family of symmetric implicit higher order methods for the solution of third order initial value problems in ordinary differential equations

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Abstract

A family of higher order implicit methods with $k-$steps is constructed which was used to solve initial value problems of third order ordinary differential equations directly without reducing them to first order systems. Implicit methods with step numbers k=3, 4, 5 are considered. For these methods, we discussed the local truncation error with the basic properties. Analysis of the basic properties of the methods shows that the methods are consistent, convergent and zero – stable. The results obtained from numerical experiment shows that the methods are more efficient and accurate than some existing methods.

Mathematics Subject Classifications: 65L05; 65L06; 65L20

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1 Introduction

Recently, researchers have beamed their light on the methods of solution of higher order initial value problems.

In this paper we considered the method of approximate solution of the general third order initial value problem of the form:

$$y^{(3)} = f\left(x, y, y', y''\right), y^{(i)} = y_i, i = 0,1,2 \tag{1}$$

Where x_n , is the initial point, y_n is the solution at x_n, f is continuous within the interval of integration

The study of (1) is of interest to researcher because of its wide application in engineering, and other real life problems.

For instance, the Blassius equation in fluid dynamics given as

$$2y''' + yy'' = 0,\ y(0) = y'(0) = 0,\ y''(0) = 1 \tag{2}$$

is an application problem

The conventional method of solving (1) is to reduce it to a system of first order differential equations (see [1, 2]). It has been reported in literature that the direct method of solving the above equation is more efficient in terms of speed and accuracy than the method of reduction to a system of first order ODES (see [3-6]).

Implicit linear multistep methods have better stability properties than explicit methods and are solved using predictor and corrector method. However, several authors have proposed multi-derivative multistep methods for the solution of (1). These methods were implemented in predictor-corrector mode [5, 6]. Although these methods yield good results, but it has a major setback which includes computational burden and the reducing order of accuracy of the predictors.

In this study, our interest is to develop a class of k-step linear multistep methods for the solution of general third order initial value problems.

2 Materials and Methods

We proposed a numerical method of the form

$$y_{n+k} = \alpha_0 y_n + \alpha_1 y_{n+1} + \ldots + \alpha_{k-1} y_{n+k-1} + h^3\left(\beta_0 f_n + \beta_1 f_{n+1} + \ldots + \beta_k f_{n+k}\right) \quad (3)$$

Taken from the classical K-step method of the form

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^3 \sum_{j=0}^{k} \beta_j f_{n+j}, \quad f_{n+j} = f\left(x_{n+j}, y_{n+j}, y'_{n+j}, y''_{n+j}\right) \quad (4)$$

The coefficients α_j and β_j are constants with the conditions $\alpha_k = 1, |\alpha_0| + |\beta_0| \neq 0$ and are determined to ensure the method is symmetric, consistent and zero stable. The method is implicit with $\beta_k \neq 0$. The value of the coefficients is determined from the local truncation error (lte) defined by

$$T_{n+k} = y_{n+k} + \left(\sum_{j=0}^{k-1} \alpha_j y_{n+k-1} - h^3 \sum_{j=0}^{k} \beta_j f_{n+k} \right) \quad (5)$$

$$= y_{n+k} + \left[\alpha_0 y_n + \alpha_1 y_{n+1} + \ldots + \alpha_{k-1} y_{n+k-1} - h^3\left(\beta_0 f_n + \beta_1 f_{n+1} + \ldots + \beta_k f_{n+k}\right)\right] \quad (6)$$

The accuracy of the methods depends on the real constants α_j and β_j. To obtain the values of these constants, we adopted the Taylor series expansion of y_{n+k}, $y_{n+1}, y_{n+2}, \ldots, y_{n+k-1}$ and $f_{n+1}, f_{n+2}, \ldots, f_{n+k}$ about the point $\left(x_n, y_n\right)$ to yield

$$\begin{aligned} T_{n+k} &= y_n + (kh) y_n^{(1)} + \frac{(kh)^2}{2!} y_n^{(2)} + \ldots + \frac{(kh)^p}{p!} y_n^{(p)} + O\left(h^{(p+1)}\right) \\ &+ \sum_{j=0}^{k-1} \alpha_j \left\{ y_n + (jh) y_n^{(1)} + \frac{(jh)^2}{2!} y_n^{(2)} + \ldots + \frac{(jh)^p}{p!} y_n^{(p)} + O\frac{(jh)^{p+1}}{(p+1)!} y_n^{(p+1)} \right\} \\ &- h^3 \sum_{j=0}^{k} \beta_j \left\{ y_n^{(3)} + (jh) y_n^{(4)} + \frac{(jh)^2}{2!} y_n^{(5)} + \ldots + \frac{(jh)^{p-3}}{(p-3)!} y_n^{(p)} + 0\frac{(jh)^{p-2}}{(p-2)!} y_n^{(p+1)} \right\} \end{aligned} \quad (7)$$

Collecting terms in equal powers of h to obtain

$$T_{n+k} = \left(1+\sum_{j=0}^{k-1}\alpha_j\right)y_n + \left(k+\sum_{j=0}^{k-1} j\alpha_j\right)hy_n^{(1)} + \left(\frac{k^2}{2!}+\sum_{j=0}^{k-1}\frac{(j)^2}{2!}\alpha_j\right)h^2y_n^{(2)}$$

$$+\left(\frac{k^3}{3!}+\sum_{j=0}^{k-1}\frac{(j)^3}{3!}\alpha_j - \sum_{j=0}^{k}\beta_j\right)h^3y_n^{(3)}$$

$$+\left(\frac{k^4}{4!}+\sum_{j=0}^{k-1}\frac{(j)^4}{4!}\alpha_j - \sum_{j=0}^{k} j\beta_j\right)h^4y_n^{(4)}$$

$$+\left(\frac{k^5}{5!}+\sum_{j=0}^{k-1}\frac{(j)^5}{5!}\alpha_j - \sum_{j=0}^{k}\frac{(j)^2}{2!}\beta_j\right)h^5y_n^{(5)} + \ldots \tag{8}$$

$$+\left(\frac{k^p}{p!}+\sum_{j=0}^{k-1}\frac{(j)^p}{p!}\alpha_j - \sum_{j=0}^{k}\frac{(j)^{(p-3)}}{(p-3!}\beta_j\right)h^p y_n^{(p)} + 0(h^{p+1})$$

By imposing an accuracy of order p on T_{n+k} to obtain the $C_i = 0, 0 \le i \le p$ and setting $k = 3(1)5$ in (7) above, we obtain a system of algebraic equation in the form

$$AX = B \tag{9}$$

For various step- number. This has helped us to determine the coefficients of the methods as displayed in Table 0.

Table 0: Coefficients and order of the methods

K	α_0	α_1	α_2	α_3	α_4	α_5	β_0	β_1	β_2	β_3	β_4	β_5
3	1	-3	3	1			0	$\frac{1}{2}$	$\frac{1}{2}$	0		
4	-1	-2	0	2	1		$\frac{1}{120}$	$\frac{56}{120}$	$\frac{126}{120}$	$\frac{56}{120}$	$\frac{1}{120}$	
5	1	$\frac{29}{31}$	$\frac{68}{31}$	$\frac{-68}{31}$	$\frac{-29}{31}$	1	$\frac{21}{2480}$	$\frac{1177}{2480}$	$\frac{3842}{2480}$	$\frac{3842}{2480}$	$\frac{1177}{2480}$	$\frac{21}{2480}$

Thus using the information in Table 1 for $k=3(1)5$, we have the following symmetric schemes

$$y_{n+3}=3y_{n+2}-3y_{n+1}+y_n+\frac{h^3}{2}\left(f_{n+2}+f_{n+1}\right) \tag{10}$$

$p=5,\quad C_{p+2}=4.16667\times10^{-3}$

$$y_{n+4}=2y_{n+3}-2y_{n+1}-y_n+\frac{h^3}{120}\left(f_{n+4}+56f_{n+3}+126f_{n+2}+56f_{n+1}+f_n\right) \tag{11}$$

$p=7,\ C_{p+2}=3.3069\times10^{-5}$

$$y_{n+5}=\frac{29}{31}y_{n+4}+\frac{68}{31}y_{n+3}-\frac{68}{31}y_{n+2}-\frac{29}{31}y_{n+1}+y_n+\frac{h^3}{2480}\begin{pmatrix}21f_{n+5}+1177f_{n+4}+3842f_{n+3}\\+3842f_{n+2}+1177f_{n+1}+21f_n\end{pmatrix} \tag{12}$$

$p=9,\ C_{p+2}=5.403\times10^{-5}$.

3 Analysis of the basic properties of the methods

We wish to examine the basic properties of the methods in terms of the order of accuracy and error constant, symmetry, consistency, zero stability and region of absolute stability.

3.1 Order of accuracy and error constant

The error in approximation is actually the difference between the exact solution $y\left(x_{n+j}\right)$ at $x=x_{n+j}$ and solution value determined from a numerical method. This is called the local truncation error (LTE)

As stated in [13], the lcocal truncation error of the general k-step method is given by

$$T_{n+k}=\sum_{j=0}^{k}\left\{\alpha_j y\left(x+jh\right)-h^n\beta_j y^{(n)}\left(x+jh\right)\right\} \tag{13}$$

Where $\alpha_k=1$, α_0 and β_0 are not both zero and $y(x)\in\left[a,b\right]$ with $a,b\in R$.

$y(x)$ is the theoretical solution assumed to have continuous derivatives of sufficiently higher order. By expanding $y\left(x+jh\right)$ and $y^{(3)}\left(x+jh\right)$, $j=0(1)k$ and comparing terms in equal power of h, we have that

$$\begin{aligned}T_{n+k}&=C_0y\left(x_n\right)+C_1hy'\left(x_n\right)+C_2h^2y''\left(x_n\right)+...+C_{p+1}h^{p+1}y^{p+1}\left(x_n\right)\\&+C_{p+2}h^{p+2}y^{p+2}\left(x_n\right)\end{aligned} \tag{14}$$

Definition 3.1 : A linear multistep method (9-11) is said to be of order p if in (13) $C_0=C_1=C_2=...=C_p=C_{p+1}=0,C_{p+2}\neq 0$, C_{p+2} is error constant.

Hence the methods (10-12) are of orders $\left(5,7,9\right)$ and the error constants are $\left[4.16667\times10^{-3},3.3069\times10^{-5},5.403\times10^{-5}\right]$.

3.2 Region of Absolute stability for the method

The region of absolute stability is the set of points in the $\lambda h-$plane for which the method is absolutely stable.

Definition 3.2 An interval $\left(\alpha,\beta\right)$ of the real line is said to be an interval of absolute stability if the method is absolutely stable for all $\bar{h}\in\left(\alpha,\beta\right)$ [1]

When **k = 3**

Using the boundary locus method [1]

$$\bar{h}(r)=\frac{\rho(r)}{\sigma(r)}=\frac{2\left(r^3-3r^2+3r-1\right)}{r^2+r}\text{ , where, }\bar{h}=\lambda h^3$$

By setting $r=e^{i\theta}$, where $e^{i\theta}=\cos\theta+i\sin\theta$, we have

$$\bar{h}(\theta)=\frac{2\left[(\cos 3\theta-3\cos 2\theta-1)+i(\sin 3\theta-3\sin 2\theta+3\sin\theta\right]}{(\cos 2\theta+\cos\theta)+i(\sin 2\theta+\sin\theta)} \quad (15)$$

This is simplified to the form $x(\theta)+iy(\theta)$

The region of absolute stability curve for k=3 is given as

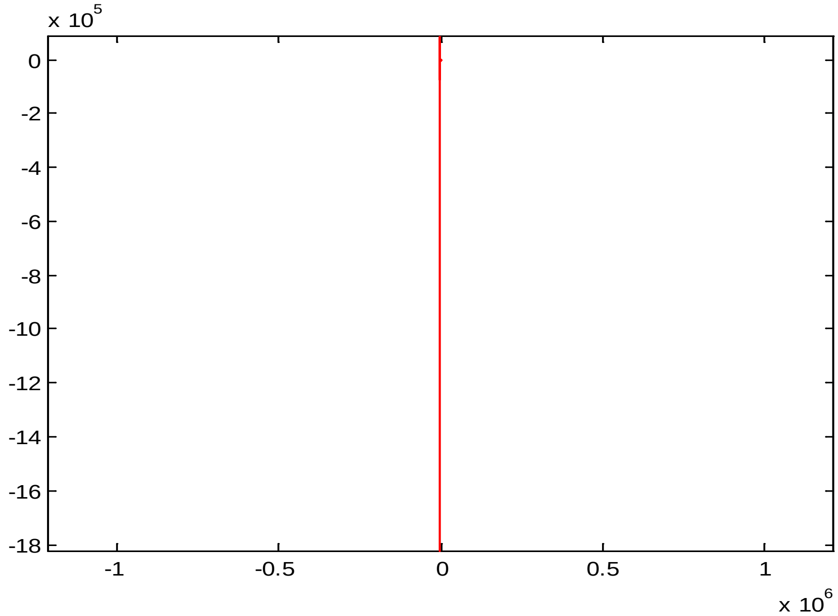

Figure 1: Region of absolute stability curve for k=3

When **k=4**

$$\rho(r)=(r^4-2r^3+2r-1),\sigma(r)=\frac{1}{120}(r^4+56r^3+126r^2+56r+1)$$

$$h(r)=\frac{\rho(r)}{\sigma(r)}=\frac{120\left(r^4-2r^3+2r-1\right)}{r^4+56r^3+126r^2+56r+1}$$

By putting $r=e^{i\theta}=\cos\theta+i\sin\theta$

$$h(\theta)=\frac{120\left[(\cos 4\theta+i\sin 4\theta)-2(\cos 3\theta+i\sin 3\theta)+2(\cos\theta+i\sin\theta)-1\right]}{\left[(\cos 4\theta+i\sin 4\theta)+56(\cos 3\theta+i\sin 3\theta)+126(\cos 2\theta+\sin 2\theta)+56(\cos\theta+i\sin\theta)+1\right]} \quad (16)$$

After much algebraic simplification of (15) we obtained the equation of the form

$$x(\theta)+iy(\theta)$$

The region of absolute stability curve for the method k=4 is given below.

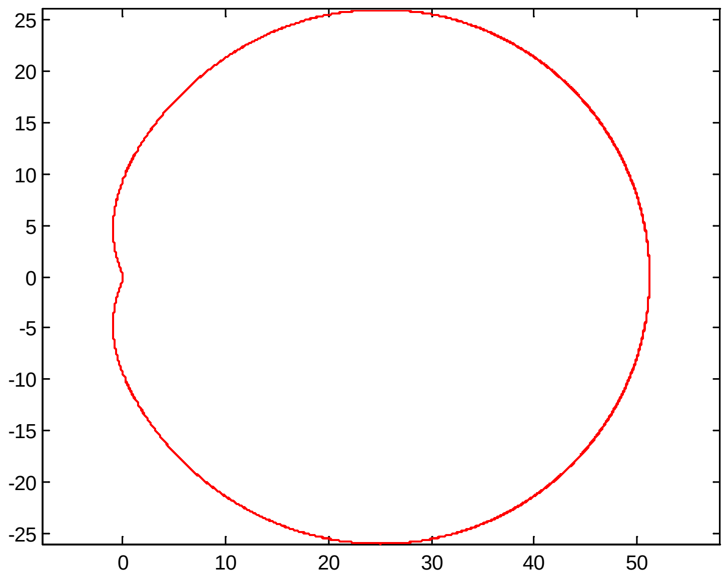

Figure.2: Region of absolute stability curve for method k=4

When **k=5**, we have

$$h(r)=\frac{r^5-\frac{29}{31}r^4-\frac{68}{31}r^3+\frac{68}{31}r^2+\frac{29}{31}r-1}{\frac{h^3}{2480}\left[21+1177r^4+3842r^3+3842r^2+1177r+21\right]} \quad (17)$$

By putting $r=e^{i\theta}=\cos\theta+i\sin\theta$ and substituting in (16), after much simplification, we have

$$\bar{h}(\theta)=\frac{2480\begin{bmatrix}(\cos 5\theta-29\cos 4\theta-68\cos 3\theta+68\cos 2\theta+29\cos\theta-1)\\ +i(\sin 5\theta-29\sin 4\theta-68\sin 3\theta+68\sin 2\theta+29\sin\theta)\end{bmatrix}}{31\begin{bmatrix}(21\cos 5\theta+1177\cos 4\theta+3542\cos 3\theta+3542\cos 2\theta+1177\cos\theta)\\ +i(21\sin 5\theta+1177\sin 4\theta+3542\sin 3\theta+3542\sin 2\theta+1177\sin\theta)\end{bmatrix}} \quad (18)$$

This is in the form $x(\theta)+iy(\theta)$

The region of absolute stability curve is given below

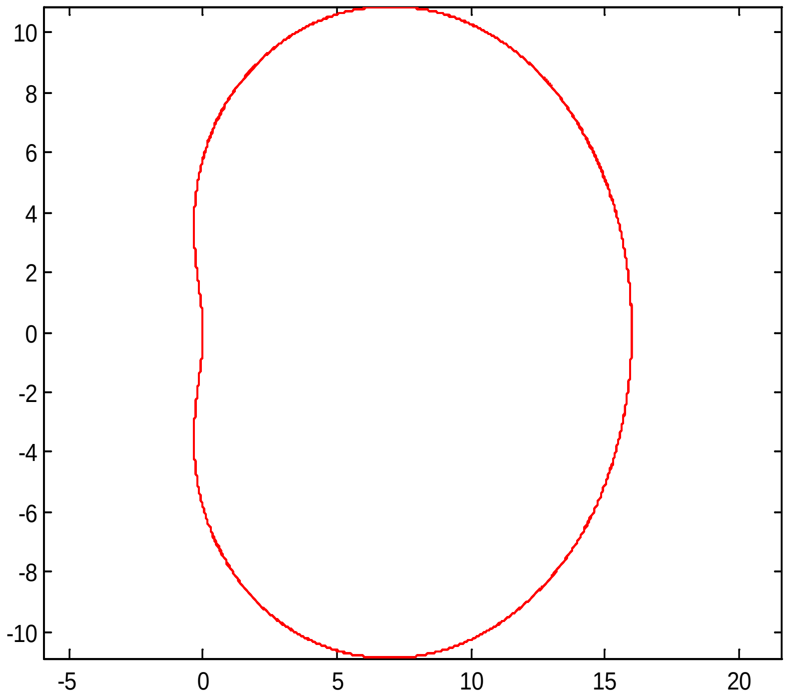

Figure 3: Region of absolute stability curve for the method k=5

3.3 Zero stability

Definition 3.3: The LMM (3) is said to be zero stable if the roots of the first characteristics polynomial lie inside or on the unit circle.

A method is stable if the cumulative effect of all errors, including the round-off errors is bounded independent of the mesh points or a numerical solution of the class (1) is said to be stable if the difference between the numerical and theoretical solution is as small as possible.The methods (9-11) were all found to be zero stable since no roots of the methods has modulus greater than one.

3.4 Consistency

According to [1, 9, 10, 12], a linear multistep method is said to be consistent if it satisfies the following conditions:

1. $p \geq 1$

2. $\sum_{j=0}^{k} \alpha_j = 0$

3. $\rho(r) = \rho'(r) = \rho''(r) = ... = \rho^{n-1}(r)$

4. $\rho^n(r) = n!\sigma(r)$

For our $n = 3$ and methods (9-11) were found to be consistent.

3.5 Convergence

The Linear multistep method (3) is said to be convergent if

$$\lim_{h \to 0} y_n = y(x_n) \ , \ 0 \leq n \leq N \tag{19}$$

provided the rounding-off errors arising from all initial conditions tend to zero.

Theorem 3.1: The LMM (3) is said to be convergent if and only if it is consistent and zero stable. Hence the methods (9-11) are convergent.

3.6 Symmetry

Following [2, 10], the methods (9-11) are symmetric if

$$\alpha_j = \alpha_{k-j}\,, \quad \beta_j = \beta_{k-j}\,, \quad j = 0(1)\frac{k}{2} \quad \text{for } k \text{ even} \tag{20}$$

$$\alpha_j = -\alpha_{k-j}\,, \quad \beta_j = -\beta_{k-j}\,, \quad j = 0(1)k \quad \text{for } k \text{ odd}$$

4 Numerical Experiments

The methods that we have derived were tested some third initial value problems of special and general nature. The results obtained compared with results obtained in some existing methods. Below are the problems used as test problems.

Problem 1:

$$y''' = x - 4y', y(0) = 0, y'(0) = 0, y''(0) = 1, h = 0.1$$

Exact solution: $y(x) = \frac{3}{16}\left(1 - \cos 2x\right) + \frac{1}{8}x^2$

Problem 2:

$$y''' + y = 0, y(0) = 1, y'(0) = -1, y''(0) = 1, 0 \le x \le 1, h = 0.1$$

Exact solution: $y(x) = e^{-x}$

Problem 3:

$$y''' = -e^x$$
$$y(0) = 1, y'(0) = -1, y''(0) = 3, 0 \le x \le 1, h = 0.1$$

Exact solution: $y(x) = 2 + 2x^2 - e^x$

Problem 4:

$$y''' + y'' + 3y' - 5y = 2 + +6x - 5x^2$$
$$y(0) = -1, y'(0) = 1, y''(0) = -3, 0 \le x \le 1, h = 0.1$$

Exact solution: $y(x) = x^2 - e^x + e^{-x}\sin(2x)$

Problem 5:

$$y''' = y'(2xy'' + y')$$
$$y(0) = 1, y'(0) = \frac{1}{2}, y''(0) = 0, h = 0.01$$

Exact solution: $y(x) = 1 + \frac{1}{2} inl\left(\frac{2+x}{2-x}\right)$

Table 1: Result of test problem 1 using the third order methods of order p=5, 7 and 9

X-Value	Exact solution	New result (k=3, p=5)	New result (k=4, p=7)	New result (k=5,p = 9)
0.1	0.4987516803E-02	0.4987500148E-02	0.4987516815E-02	0.4987516803E-02
0.2	0.1980106421E-01	0.1980000058E-01	0.1980106725E-01	0.1980106420E-01
0.3	0.4399957491E-01	0.4399643528E-01	0.4399957128E-01	0.4399957317E-01
0.4	0.7686749420E-01	0.7686125420E-01	0.76867644979E-01	0.7686744887E-01
0.5	0.1174433209E+00	0.1174329233E+00	0.1174436264E+00	0.1174433151E+00
0.6	0.1645579255E+00	0.1645421445E+00	0.1645583865E+00	0.1645579086E+00
0.7	0.2168811664E+00	0.2168581157E+00	0.2168818701E+00	0.2168811499E+00
0.8	0.2729749173E+00	0.2729406708E+00	0.2729759349E+00	0.2729748812E+00
0.9	0.3313504007E+00	0.3312956599E+00	0.3313520535E+00	0.3313503618E+00
1.0	0.3905275407E+00	0.3904304488E+00	0.3905306192E+00	0.3905274468E+00

Table 2: Comparision of the errors in test problem 1 with the errors in the results obtained by Adesanya (2011) and Olabode (2007).

X	Errors in (k=3, p=5)	Errors in (k=4, p=7)	Error in (k=5, p=9)	Errors in [7] (k=4, p=7)	Error in [8] (k=4, p=7)
0.1	1.6655E-08	1.1189E-11	5.29004E-15	1.1889E-11	1.66547E-08
0.2	1.0636E-06	3.0422E-09	5.41143E-12	3.0422E-09	3.80957E-07
0.3	3.1382E-06	7.7779E-08	3.11578E-10	7.7956E-08	1.56646E-06
0.4	6.2400E-06	1.5559E-07	5.52122E-09	7.7467E-07	3.98657E-06
0.5	1.0398E-05	3.0544E-07	5.83875E-09	4.5990E-06	7.95971E-06
0.6	1.5781E-05	4.6102E-07	1.69100E-09	6.4783E-06	1.36800E-05
0.7	2.3051E-05	7.0374E-07	1.64365E-08	5.7839E-06	2.11958E-05
0.8	3.4246E-05	1.0177E-06	3.60644E-08	2.3547E-06	3.03963E-05
0.9	5.4741E-05	1.6528E-06	3.88199E-08	3.7665E-06	4.10086E-05
1.0	9.7092E-05	3.0768E-06	9.37880E-08	1.2331E-05	5.26051E-05

Table 3: Result of test problem 2 using the methods of order 5, 7and 9

X	Exact solution	New result (k=3, P= 5)	New result (k=4, P =7)	New result (k=5, P = 9)
0.1	0.9048374167	0.9048374153	0.9408374167	0.9408374167
0.2	0.8187307506	0.8187306642	0.8187307506	0.8187307506
0.3	0.7408182174	0.7408179701	0.7408182158	0.7408182174
0.4	0.6703200420	0.6703195741	0.6703200389	0.6703200420
0.5	0.6065306552	0.6065299326	0.6065306490	0.6065306552
0.6	0.5488116312	0.5488106630	0.5488116220	0.5488116311
0.7	0.4965852986	0.4965841717	0.4965852853	0.4965852985
0.8	0.4493289588	0.4493279091	0.4493289424	0.4493289586
0.9	0.4065696543	0.4065692080	0.4065696372	0.4065696542
1.0	0.3678794357	0.3678806687	0.3678794282	0.3678794355

Table 4: Comparison of the errors in the result of test problem 2 with errors in [8] and [10] .

X	Errors in order (k=3, P=5)	Errors in order (k=4, P=7)	Errors in order (k=5, P=9)	Error in [10] (Predictor-corrector method) (k=3,P=5)	Errors in[8] (block method) (k=4,P=7)	Errors in [8] (block method) (k=5,P=9)
0.1	1.36929E-09	2.4525E-13	0.0000+00	1.36929E-09	1.36929E-09	2.1760E-12
0.2	8.64113E-08	6.2109E-11	2.7756E-14	3.12272E-08	3.12272E-08	1.3935E-11
0.3	2.47223E-08	1.5746E-10	1.5838E-12	1.27694E-07	1.27694E-07	3.4443E-11
0.4	2.05452E-07	3.1477E-09	2.7879E-11	3.25196E-07	3.25196E-07	6.4477E_11
0.5	7.22629E-07	6.1617E-09	2.9477E-11	6.54297E-07	6.54297E-07	1.0316E-10
0.6	9.68177E-06	9.1732E-09	8.5048E-11	1.14406E-06	1.14406E-06	1.4979E-10
0.7	1.12692E-06	1.3329E-08	8.0357E-11	1.81784E-06	1.81784E-06	2.0486E-10
0.8	1.04962E-06	1.6378E-08	1.6601E-10	2.69774E-06	2.69774E-06	2.6756E-10
0.9	4.46241E-06	1.7134E-08	1.11757E-10	3.80241E-06	3.80241E-06	6.9382E-10
1.0	1.23330E-06	7.4405E-09	1.4871E-10	5.14755E-06	5.14755E-06	1.4224E-10

Table 5: Results of test problem 3 using the methods of order p=5, 7 and 9

X	**Exact solution**	**New result (k=3, p=5)**	**New result (k=4, p=7)**	**New result (k=5, p=9)**
0.1	0.9148290809E+00	0.9148290809E+00	0.9148290809E+00	0.9148290809E+00
0.2	0.8585972406E+00	0.8585972406E+00	0.8585972406E+00	0.8585972406E+00
0.3	0.8301411918E+00	0.8301411918E+00	0.8301411918E+00	0.8301411918E+00
0.4	0.8281753030E+00	0.8281753030E+00	0.8281753030E+00	0.8281753030E+00
0.5	0.8512787319E+00	0.8512787319E+00	0.8512787319E+00	0.8512787319E+00
0.6	0.8978812048E+00	0.8978812048E+00	0.8978812048E+00	0.8978812048E+00
0.7	0.9662473007E+00	0.9662473007E+00	0.9662473007E+00	0.9662473007E+00
0.8	0.1054459083E+01	0.1054459083E+01	0.1054459083E+01	0.1054459083E+01
0.9	0.1160396904E+01	0.1160396904E+01	0.1160396904E+01	0.1160396904E+01
1.0	0.1281718191E+01	0.1281718191E+01	0.1281718191E+01	0.1281718191E+01

Table 6: Comparison of the errors in the results of test problem 3 with the errors in [8]

X	**Errors in Order (k=3, p = 5)**	**Errors in order (k=4, p=7)**	**Errors in order (k=5, p=9)**	**Errors in [8](block method) (p=9)**
0.1	1.40898E-09	2.5080E-13	0.0000+00	7.56477E-11
0.2	9.14935E-08	6.4932E-11	2.8644E-14	2.60171E-10
0.3	2.70784E-07	1.6831E-09	1.6720E-12	5.76003E-10
0.4	5.40391E-07	3.3668E-09	2.9932E-11	8.41271E-10
0.5	9.04450E-07	6.6147E-09	3.1673E-11	1.00013E-09
0.6	1.37831E-06	9.9982E-09	9.1890E-11	1.09051E-09
0.7	2.00976E-06	1.5283E-08	8.9834E-11	1.07048E09
0.8	2.92566E-06	2.1980E-08	1.9682E-10	1.49247E-09
0.9	4.42266E-06	3.4643E-08	2.1110E-10	3.15695E-09
1.0	7.12929E-06	5.9998E-08	4.9310E-10	4.45905E-09

Table 7: Result of test problem 4 using the method of orders p=5 and 7

X-value	**Exact solution**	**New result for order (k=3, P =5)**	**New result for order (k=4, p=7)**
0.1	0.9154074727E+00	0.9154074156E+00	0.9154074736E+00
0.2	0.8625739844E+00	0.8625706656E+00	0.8625740814E+00
0.3	0.8415613749E+00	0.8415516452E+00	0.8415628319E+00
0.4	0.8509665312E+00	0.8509469738E+00	0.8509694533E+00
0.5	0.8883433229E+00	0.8883086925E+00	0.8883491589E+00
0.6	0.9506049113E+00	0.9505428695E+00	0.9506151121E+00
0.7	0.1034392864E+01	0.1034270208E+01	0.1034415176E+01
0.8	0.1136403570E+01	0.1136136123E+01	0.1136461778E+01
0.9	0.1253666228E+01	0.1253061116E+01	0.1253833292E+01
1.0	0.1383770019E+01	0.1382425664E+01	0.1384234604E+01

Table 8: Comparison of the errors in the results of test problem 4 with the errors in [8]

X	**Errors in (k=3, p=5)**	**Errors in (k=4, p=7)**	**Error in [8] (block method) (k=4, P=7)**
0.1	5.70895E-08	8.4706E-10	6.40864E-07
0.2	3.31883E-06	9.7040E-08	1.51133E-05
0.3	9.72966E-06	1.4570E-06	6.36444E-05
0.4	1.95574E-05	2.9221E-06	1.67567E-04
0.5	3.46304E-05	5.8360E-06	3.56771E-04
0.6	6,20418E-05	1.0201E-05	6.410875E-04
0.7	1.22655E-04	2.2313E-05	1.071642E-03
0.8	2.67447E-04	5.8208E-05	1.682213E-03
0.9	6.05112E-04	1.6706E-04	2.520604E-03
1.0	1.34436E-03	4.6458E-04	3.644104E-03

Table 9: The result for problem 5 (a non-linear problem) using the methods of orders p =5, 7 and 9

X-value	**Exact solution**	**Result for (k=3, p = 5)**	**Result for (k=4, p=7)**	**Result for (k=5, p=9)**
0.1	0.1050041730E+01	0.1050041709E+01	0.1050041711E+01	0.1050041711E+01
0.2	0.1100335349E+01	0.1100334068E+01	0.1100334788E+01	0.1100334788E+01
0.3	0.1151140438E+01	0.1151136321E+01	0.1151136661E+01	0.1151136683E+01
0.4	0.1202732557E+01	0.1202718413E+01	0.1202719110E+01	0.1202719154E+01
0.5	0.1255412816E+01	0.1255378917E+01	0.1255380137E+01	0.1255380225E+01
0.6	0.1309519609E+01	0.1309459191E+01	0.1309461310E+01	0.1309461444E+01
0.7	0.1365443760E+01	0.1365368057E+01	0.1365372025E+01	0.1365372236E+01
0.8	0.1423648937E+01	0.1423614604E+01	0.1423622965E+01	0.1423623289E+01
0.9	0.1484700287E+01	0.1484851641E+01	0.1484870617E+01	0.1484871202E+01
1.0	0.1549306154E+01	0.1549931907E+01	0.1549975595E+01	0.1549976798E+01

Table 10: Errors in test problem 5 by using the methods of order p=5, 7 and 9. It shows that better results were obtained with increasing order

X-value	**Errors in (k=3,p=5)**	**Errors in (k=4,p=7**	**Errors in (k=5, p=9)**
0.1	2.09452E-08	1.93182E-08	1.93148E-08
0.2	6.81064E-07	5.61699E-07	5.60825E-07
0.3	4.11737E-06	3.77772E-06	3.75510E-06
0.4	1.41437E-05	1.34474E-05	1.34028E-05
0.5	3.38988E-05	3.26773E-05	3.25906E-05
0.6	6.04186E-05	5.82875E-05	5.81649E-05
0.7	7.57029E-05	7.16807E-05	7.15239E-05
0.8	3.43336E-05	2.57657E-05	2.56483E-05
0.9	1.51354E-05	1.71002E-04	1.70915E-04
1.0	6.25753E-04	6.71370E-04	6.70643E-04

5 Conclusion

In this study, we have developed a family of implicit linear multistep methods for the numerical solution of general third order ordinary differential equations. Analysis of the basic properties showed that the methods are consistent, zero-stable, convergent and absolutely stable. The results displayed in tables 1-10 shows that there is a remarkable improvement in accuracy if the order is increased. The results obtained compared favourably with some existing methods in terms of accuracy and efficiency.

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