Theoretical Mathematics & Applications, vol. 6, no. 3, 2016, 49-65
ISSN: 1792- 9687 (print), 1792-9709 (online)
Scienpress Ltd, 2016

On the nature of the logistic function as a nonlinear discrete dynamical system

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Abstract

In an attempt to discover the effect of recurrence on the topological dynamics, a nonlinear function whose regime of periodicity amounts to recurrence was considered, thus the logistic function. This research seeks to study the logistic function as to how it really behaves. In the field of dynamics most especially this function in terms of discrete form has been studied. Logistic equation as a model based on population growth was initially originated by the famous Pierre-Francios Verhulst. It is a continuous form written as $\frac{dx}{dt} = r(x - x^2)$, which depend on time. It can be restructured from the continuous form into a discrete form known as logistic function, written as; $x_{n+1} = rx_n(1 - x_n)$, with $n = 0,1,2,3 \dots \dots ,x_n$ is

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Article Info: *Received*: March 5, 2016. Revised : April 30, 2016.
Published online : July 30, 2016.

the state at the discrete time $n\, and\ r$ are the control parameters which works within a given interval. It is a very simple example of nonlinear systems in dynamics. Its true nature or behaviour in changing from one regime to another regime is solely dependent on the adjustment or variation of the control parameter r. Therefore this research is also about the transitions of this function. For instance, for some parameter values of r the logistic map display periodic behavior (period-1 orbits "fixed point", period-2 orbits and period-n orbits), and for others, it displays chaotic behavior.

Keywords: logistic function; nonlinear dynamical system; fixed points; periodic points; bifurcation; chaos; orbits

1 Introduction

The logistic equation or the logistic map as a nonlinear dynamic system has a class of different behaviors. This function is a polynomial mapping of a degree 2 which is a nonlinear equation that behaves in series. The biologist Robert May in 1976 came out of with this paper through his seminal paper, which first created by Pierre-Francois Verhulst as a discrete-time demographic model analogous to the logistic equation. And it was written mathematically as T (x) = kx(1- x), where $x = [0{,}1]$ and k is the value of interest which is a parameter ($k > 0$).

The concepts of chaos can studied without any burden through simplicity of the logistic map. The logistic map provides a rich example as to how to explore periodic regions and complex chaotic behavior. For the interest of this research, where the logistic map changes in behavior due to the parameter introduced into the map and allowed to vary continuously in a way that changes in the logistic function can be noticed

2 Useful definitions and theorem for our work

Definition 1: orbits (fixed and periodic orbits)

let f: $R \rightarrow R$ The point x_0 is a *periodic point* of period n for f if $f^n(x_0) = x_0$ where the point x_0 is *fixed point* for f if $f(x_0) = x_0$ but $f^i(x_0) \neq x_0$ for $0 < i < n$. The sequence $\{x_0, f(x_0), f^2(x_0) \ldots f^n(x_0) \ldots\}$ is called the orbit of x_0 under f.

Definition 2: Types of periodic points

1. periodic point p is *attracting* if $|(f^n)'(p)| < 1$
2. periodic point p is *repelling* if $|(f^n)'(p)| > 1$
3. point p is *neutral* if $|(f^n)'(p)| = 1$

Note: the prime denote differentiation with respect to p

Theorem 1: if f is a continuous function/map f: $R \rightarrow R$ and if there exists period-3 periodic point in f, then all periods exist in f leading to chaos.

Definition 3: Let f: $R \rightarrow R$, if $\delta > 0$ is a constant such that $\varepsilon > 0$, then there is x satisfying $|x - x_0| < \delta$ such that $|f^n(x) - f^n(x_0)| \geq \varepsilon$, where n is an integer. Where the point x_0 is called sensitive point, x is the initial condition.

3 Main results

3.1 The logistic map/function

The logistic map is defined by; $x_{n+1} = rx_n(1 - x_n)$, where $n = 0,1,2,3 \ldots \ldots$ Let $x_{n+1} = f(x_n)$, then $f(x_n) = rx_n(1 - x_n)$, $x_n \in [0,1]$ and $r \in [1,4]$. By setting the parameter $r = 1$, the logistic equation becomes; $f(x_n) = x_n(1 - x_n) = x_n - x_n^2$

3.2 The roots and the maximum values of x in the function

The roots of x in the function $f(x_n) = rx - rx^2$

If $f(x_n) = 0$, then r$x - rx^2 = 0$

$rx\,(1-x) = 0$

Implies, $rx = 0$ and $1 - x = 0$

Then, $x = 0$ and $x = 1$

The maximum and minimum point of $x\ in\ f(x_n) = rx - rx^2$

$df = r - 2rx$

$df = 0$ implies that, $r - 2rx =$ o

$r(1 - 2x) = 0$

$r = 0$

$1 - 2x = 0$

$x = \frac{1}{2}$

The second derivate gives room for the conclusion on the sign, thus $d^2f(\frac{1}{2}) = -2r$. Hence from the two main features of the logistic function it is clear that it passes through x at $x = 0$ and $x = 1$ and maximum at $x = \frac{1}{2}$ since it is concave down.

Note: df and d^2f are the first and second derivatives of the above function.

3.3 Solutions/locations of logistic function

The solution of the logistic function occurs when a diagonal line $y = x$ is been introduced and there is an intersection between the diagonal line and the function as indicated on the diagram below with dash red line. Thus, $y = f(x_n)$

$x = rx - rx^2$, then $rx^2 - rx + x = 0$

$x\,(rx - r + 1) = 0$,

$x = 0$ and $rx - r - 1 = 0$

Hence, $x = \frac{r-1}{r}$ or $x = 0$ are the two main solutions or locations of $f(x_n)$.

3.4 Determination of a fixed point of the logistic function for period-1 orbit/point

A period-1 orbits do occur when a function goes through series of iterations with an initial value popping-out as the same value (mensah, 2016).

From the solutions of the logistic function, that is; $x_0 = 0$ and $x_0 = \frac{r-1}{r}$ which is based on the intersection of the diagonal line and the function.

Then, for $f(x_n) = rx_n - rx_n{}^2 = rx_n(1 - x_n)$

At intersection $x_0 = 0$, $f(0) = r(0)\ (1\text{-}0)$

$f(0) = 0$

At intersection $x_0 = \frac{r-1}{r}$, $f\left(\frac{r-1}{r}\right) = r\left(\frac{r-1}{r}\right)\left(1 - \left(\frac{r-1}{r}\right)\right)$

$$= (r-1)\left(1 - \frac{r-1}{r}\right)$$

$$= (r-1)\left(\frac{r-r+1}{r}\right)$$

$$= (r-1)\frac{1}{r}$$

$$= \frac{r-1}{r}$$

$$f\left(\frac{r-1}{r}\right) = \frac{r-1}{r}$$

Clearly, both deduction points out that, the point of intersection $x_0 = 0$ and $x_0 = \frac{r-1}{r}$ are the fixed/period-1 points of the logistic map for the period-1 orbit/point and also serve as solutions.

Example 1: let $f(x_n) = rx_n(1 - x_n)$ and$r = 2.7$. Show that $x_0 = \frac{17}{27}$ is the fixed point of the function.

Illustration: if $x_0 = \frac{17}{27}$, then,$f\left(\frac{17}{27}\right) = 2.7\left(\frac{17}{27}\right)\left[1 - \frac{17}{27}\right] = \frac{17}{27}$

Hence $x_0 = \frac{17}{27}$ as the initial point is the fixed point of the function, since it gives back the same point after several iterating, therefor serving as the fixed point for the function.

3.5 The nature of the fixed point of the logistic function

We now show that whether the nature of the fixed/period-1 points of the logistic function can be classified under definition 2. That is;

1. Attracting fixed point
2. Repelling fixed point
3. Neutral

Illustration: We determine if the fixed points of the logistic function are attracting and repelling fixed points under period-1 orbit base on the definition above.
Let the logistic function $f(x_n) = rx_n(1 - x_n)$
Then, we take the derivative and evaluate the absolute value of the derived function, at x_0 = 0 and x_0 = $\frac{r-1}{r}$
That is, f' (x) = $r - 2rx$
At the fixed point x_0 = 0

$\boldsymbol{f}'(x) = r - 2rx,$

1. Attracting, $|f'(0)| < 1$

$|r - 2r(0)| < 1$

$-1 < r - 2r(0) < 1$

$-1 < r < 1$

2. Repelling, $|f'(0)| > 1$

$|r - 2r(0)| > 1$, then

$r - 2r(0) > 1$ or $r - 2r(0) < 1$

$r > 1$ or $r < 1$

Clearly, $r \in [0, 1)$ is inside the domain of $r \in [1, 4]$, hence $x_0 = 0$ is attracting and stable at -1 $< r <$ 1 for period-1 orbit of the logistic function but repelling at $r > 1$ or $r < 1$ since $r \in (1, 4]$

At the fixed point $x_0 = \frac{r-1}{r}$

$f'(x) = r - 2rx$

1. Attracting, $\left|f'\left(\frac{r-1}{r}\right)\right| < 1$

$|-r+2| < 1$

$-1 < -r + 2 < 1$

$-3 < -r < -1$ this implies $1 < r < 3$

2. Repelling, $\left|f'\left(\frac{r-1}{r}\right)\right| > 1$

$|-r+2| > 1$, then $-r + 2 >$ or $-r + 2 < -1$

Therefore, $r < 1$ or $r > 3$

Note: the fixed point $x_0 = 0$ and $x_0 = \frac{r-1}{r}$ is a neutral fixed points at $r = 1$ and $r = 3$. ***Very trivial***

Example 2: Algebraic illustration of the attracting fixed point of the logistic function, when $r = 2.3 < 3$ then the fixed point will be 0.57

Algebraically; taking $x = 0.10$ as an initial point and the control parameter $r = 2.3 < 3$. Then for $f(x_n) = 2.3x_n - 2.3x_n^2$ at $x_0 = 0.10$, $x_1 = f(x_0) = 0.207$, $x_2 = f^2(x_1) = 0.378$, $x_3 = f^3(x_2) = 0.541$, $x_4 = f^4(x_3) = 0.571$, $x_5 = f^5(x_4) = 0.563$, $x_6 = f^6(x_5) = 0.570$, $x_7 = f^7(x_6) = 0.570$

Example 3: Illustration of the repelling fixed point through algebraic when $r = 3.5 > 3$ with a fixed point $x_0 = 0.71$

Algebraically, taking $x = 0.10$ as an initial point and the control parameter $r = 3.5 > 3$. Then for $f(x_n) = 3.5x - 3.5x^2$ at $x_0 = 0.10$ $x_1 = f(x_0) = 0.315$, $x_2 = f^2(x_1) =$

0.755, $x_3 = f^3(x_2) = 0.647$, $x_4 = f^4(x_3) = 0.799$, $x_5 = f^5(x_4) = 0.561$, $x_6 = f^6(x_5) = 0.862$, $x_7 = f^7(x_6) = 0.417$

3.6 The periodic orbits (period-2) and the bifurcation diagram of the logistic map

From the logistic function $f(x_n) = rx_n - rx_n^2$, r as a parameter of interest can lie within 0 and 3 i.e. $0<r<3$. Taken $r= 3$ or beyond and iterate the function base on this interval it gives birth to an Orbit that alternate between two values (twice the period). The second iterate of the logistic map with the fixed point gives the period−2 Orbits (Mensah, 2016).

Example 4: Considering the function or map $f(x_n) = 3.2x_n - 3.2x_n^2$ for $x_n \in (0, 1)$, let $x_0 = 0.5$

By iteration of the function $f(x)$ the following sequence was obtain; at $x_0 = 0.5$, $x_1 = f(x_0) = 0.80$, $x_2 = f^2(x_1) = 0.51$, $x_3 = f^3(x_2) = 0.80$, $x_4 = f^4(x_3) = 0.51$, $x_5 = f^5(x_4) = 0.80$

Clearly, the iteration of the function $f(x_n) = 3.2x_n - 3.2x_n^2$ is a repeat of numbers that alternate between two values. Thus Orb = {0.51, 0.80} for the Orbits for the function $f(x_n) = 3.2x_n - 3.2x_n^2$ with $x_0 = 0.5$ as the initial point. This point x_0 is a period−2 points for the map

Attracting and repelling points of logistic map for period-n orbits

1. periodic point p is called ***attracting*** if $|(f^n)'(p)| < 1$
2. periodic point p is called ***repelling*** if $|(f^n)'(p)| > 1$
3. point p is called ***neutral*** if $|(f^n)'(p)| = 1$

Note: the prime denote differentiation with respect to p

It is obvious that by the definition of the periodic n point as the iteration of the fixed point p in n time, thus $f^n(p) = p$ for instance, $f(f(p)) = p$, then it is clear that the conditions for a fixed point p to be *attracting fixed point* also hold for *periodic pointp* for period n point. So it is also true for the *periodic pointp* if a fixed/period-1 point p is a *repelling fixed point* p.

3.7 Solutions/locations for the logistic function/map on its second iteration (period-2orbits/points)

For period-1 point the solution is $x = 0$ and $x = \frac{r-1}{r}$. Algebraically, we can also find the solutions for period-2 point of the logistic function.

Let $f(x_n) = rx_n - rx_n{}^2$ then, for the period-2 point that is the second iteration $f^2(x)$ of the logistic function implies; evaluating $f^2(x) = f(f(x))$,

$$f^2(x) = r\big(rx(1-x)\big)\big[1-\big(rx(1-x)\big)\big]$$
$$= r^2x(1-x)[1-rx+rx^2]$$
$$= (r^2x - r^2x^2)[1-rx+rx^2]$$
$$= r^2x[1-x-rx+2rx^2-rx^3] \text{ ………………….. } 1.1$$

But for period-2 point, $f^2(x) = x$ ……………………………1.2

Then by equating 1.1 and 1.2

$$x = r^2x[1-x-rx+2rx^2-rx^3]$$
$$0 = r^2x[1-x-rx+2rx^2-rx^3] - x$$
$$0 = x(r^2[1-x-rx+2rx^2-rx^3] - 1)$$
$$0 = -x\left(x-1+\frac{1}{r}\right)(r^2x^2-(r^2+r)x+r+1)$$

This implies $0 = -x$, $0 = \left(x-1+\frac{1}{r}\right)$ and $0 = (r^2x^2-(r^2+r)x+r+1)$

Therefore $x = 0$, $x = \frac{r-1}{r}$ and $x = \frac{\pm\sqrt{r^2-2r-3}+r+1}{2r}$ are the solutions or the fixed points for period-2 point/orbits for the logistic function

But, since our interest is in r>0 for x is real. So we setts the discriminant to be;

$$r^2 - 2r - 3 \geq 0$$

$$(r-3)(r+1) \geq 0$$

$$r \geq 3 \text{ Or } r \leq -1$$

Hence $r \geq 3$ will be our interest for this work at this section since our interest was that r>0 for x to be real.

3.8 Bifurcation diagram of the logistic function

At exactly r=3 and beyond, the behavior of the logistic map begins to change and it is as a results of the increasing nature of the control parameter r, and this bring about bifurcation (splitting), when there is a qualitative change in the long term behavior of the map as the control parameter is varied, we say that the system undergo a bifurcation.(Mensah.2016)

Now by carefully looking at the graph (bifurcation) below the first bifurcation starts at exactly r=3 that is a period-2 periodic points.

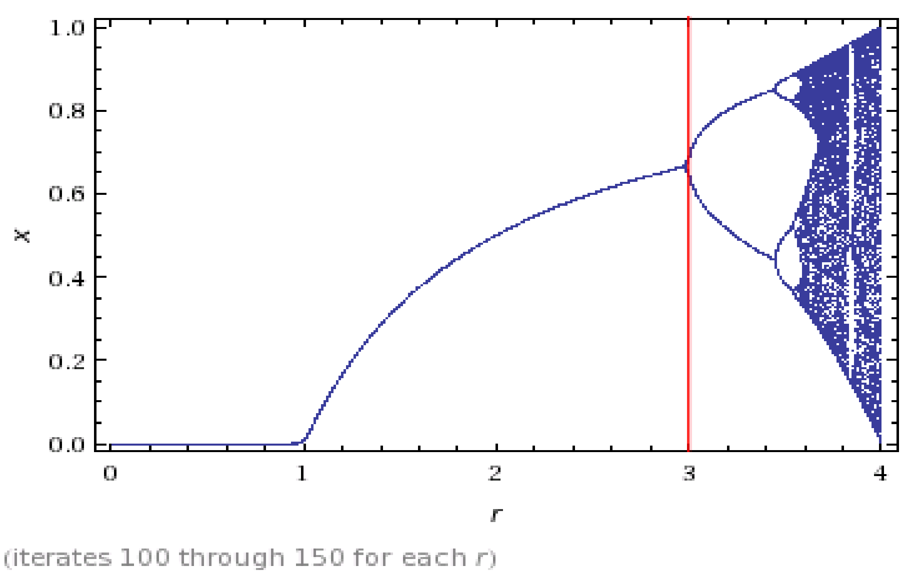

Figure 1.00: Bifurcation diagram of r and $f(x)$

We can also notice in the figure 1.00 above that there is a split (or bifurcation) which happened when r>3, this bifurcation represents the number of periods an initial value obtain when r is a certain value/ number.

From the bifurcation of the logistic function as r keeps increasing, Mitchell Feigenbaum (in 1978) worked on this process through a computer and arrived with the following table. What he discovered is defined by $lim\ n \to \infty \frac{r_{n-1}-r_{n-2}}{r_n-r_{n-1}} \approx 4.6692016$ ... ... now accepted and is called Feigenbaum Constant.

3.9 Bifurcation table of the logistic function as r keeps increasing

n	Bifurcation (2^n-cycle)	r_n	$r_{n-1} - r_{n-2}$	$\frac{r_{n-1} - r_{n-2}}{r_n - r_{n-1}}$
1	2	3	-	-
2	4	3.449490	-	-
3	8	3.544090	0.44949	4.7515
4	16	3.564407	0.09460	4.6562
5	32	3.568759	0.020317	4.6684
6	64	3.56989	0.004352	4.6692
7	128	3.56993	0.001131	4.6694

It can be seen that the distance between successive bifurcations shrinks by a constant factor. This Feigenbaum Constant can be used to predict subsequent values of r at exactly where there is a split on the bifurcation diagram.

Note: the sequence { r_n} is an infinite series called a period doubling cascade: this when the control parameter of given system is been adjusted further and further, where 2^n-cycle exist for every positive integer n. From Feigenbaum computations the location of r_n numerically appear closer and closer together through successive period doubling bifurcation. (Mensah, 2016)

When r is slightly higher than 3.54 the function alternate between 8,16,32,64 as in the Table above. Also the lengths $r_{n-1} - r_{n-2}$ of the control parameter distances/gabs producing the same values of alternation reduce speedily.

The ratio $r_{n-1} - r_{n-2} / r_n - r_{n-1}$ between the lengths $r_{n-1} - r_{n-2}$ of two successive bifurcation distances get closer to the value 4.6692016. And when r=4, chaotic behavior of the map occurs.

4 Chaotic behavior of the logistic function

The last nature/characteristic of the logistic map $f(x_n) = rx\ (1 - x)$, is the chaotic regime. To arrive at this chaotic regime, it has been shown in the various bifurcation diagrams and the Feigenbaum computations that the map moves faster or closer as r is been increase.

4.1 Proof of theorem 1: the existence of period-3 as one of the route to chaos

It is also very clear that for period-3 points, there are some indications of small open spaces which break beyond a certain point, hence periodic leading to chaos.

Graphical display of the period-3 points of the logistic map (bifurcation diagram)

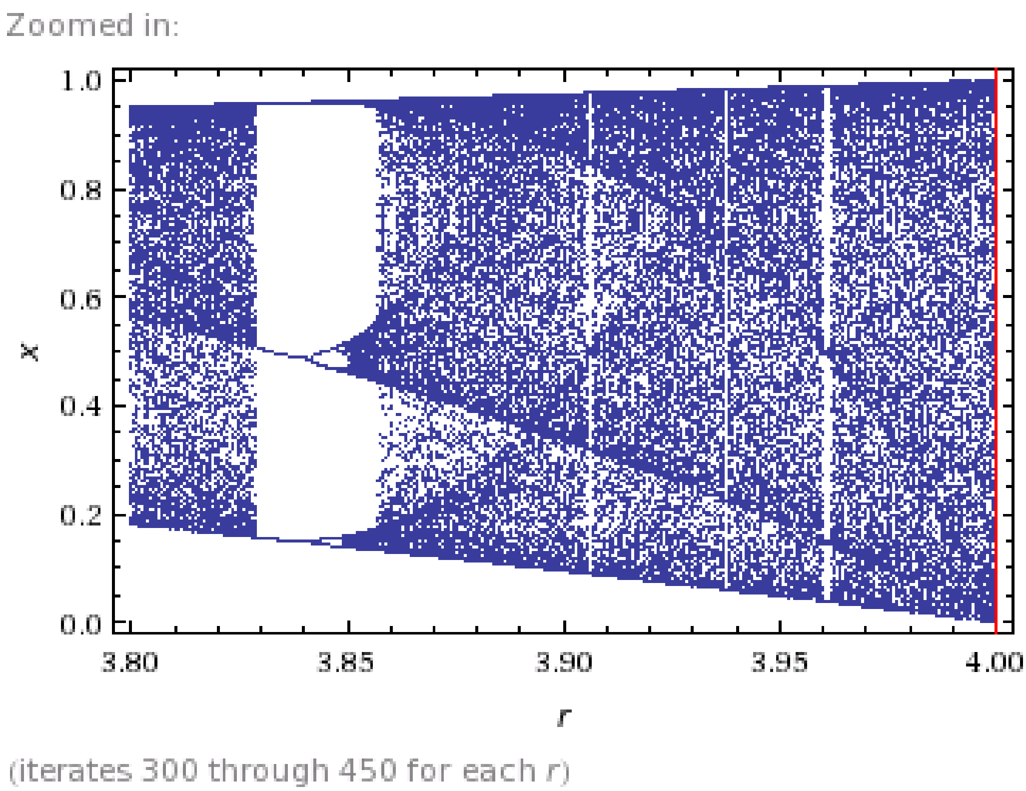

Figure 1.10: Bifurcation diagram for 3.8 <r< 4.0

4.2 Illustration: Algebraic proof of the period-3 points of the logistic function

It shows that period−3 point exists when r lies approximately between 3.83 and 3.84 as shown in Figure 1.10 above. It will be much better and easier if we use algebraic approach.

So Then, by considering the iterations of the logistic function $f(x_n) = rx_n\,(1-x_n)$ where r=3.83, implies $f(x_n) = 3.83x_n\,(1-x_n)$. Let x_0=0.5

x_1= $f(x_0)$ =0.9575, x_2= $f^2(x_1)$ =0.1559, x_3=f^3 (x_2) =0.5039, x_4=f^4 (x_3) =0.9574, x_5=f^5 (x_4) =0.1561, x_6=f^6 (x_5) =0.5044, x_7=f^7 (x_6) =0.9574, x_8=f^8 (x_7) =0.1561, x_9=f^9 (x_8) =0.5046

Upon iterating the function $f(x_n)$ = $3.83x_n$ $(1- x_n)$, the sequence we are obtaining are a repeaof numbers that alternate between three values as shown above, thus {0.96, 0.16, 0.50}

This clearly, shows that there is the existence of period-3 in logistic function. Hence period−3 point exists implying that all other periods also exist.

So the route to chaos can be seen through the existence of period-3, the doubling nature of the periodic orbits and all this route are dependents on the strength of the control parameter r.

So the question now is what happens when r=4?

By considering the Approximation of r within 3.83 and 3.84, there exist a periodic orbits of period-3 point for $f(x_n)$ and after r>3.84 period doubling starts, hence chaotic at r=4?

Then, by considering sensitive dependence on initial conditions as a concept for chaos and setting r=4. The logistic becomes, $f(x_n)$ = $4x_n$ (1- x_n).

4.3 Proof of definition 3: sensitive dependence on initial condition at r=4

Sensitive dependence on initial conditions as a concept for chaos on logistic function/map

Taking the logistic function $f(x_n)$ = $4x_n$ (1- x_n) and setting x_0=0.3333 as the approximation of $\frac{1}{3}$. Then, *The iteration of the logistic function with initial value* $\frac{1}{3}$ *and its approximation 0.3333*

When x_0=0.3333

x_1= $f(x_0)$ =0.8888, x_2= $f^2(x_1)$ =0.3952, x_3=f^3 (x_2) =0.9561, x_4=f^4 (x_3) =0.1680, x_5=f^5 (x_4) =0.5591, x_6=f^6 (x_5) =0.9860, x_7=f^7 (x_6) =0.0552

When x=$\frac{1}{3}$

x_1= $f(x)$ =$\frac{8}{9}$, x_2= $f^2(x_1)$ =$\frac{32}{81}$, x_3=f^3 (x_2) =$\frac{6272}{6561}$, x_4=f^4 (x_3) =0.1684, x_5=f^5 (x_4) =0.5602, x_6=f^6 (x_5) =0.9855, x_7=f^7 (x_6) =0.0572

It can be seen in that increase in iterations increases the distance between each successive number. For the chaotic regime we base our argument on the definition and the above iterations.

Then by setting δ=0.000333. We choose ε = 0.0001, it can be seen that at f^4 the difference is 0.0004 which is more than ε. And for x and x_0 to get closer let x_0 =0.3333, then 0.0000333 as the difference between x and x_0 due to the iterations and for δ=0.000333 and our fixed ε = 0.0001.

Clearly, $|x - x_0| < \delta$ implying that $|0.0000333| < 0.000333$ and at f^7, the resulting difference between the values is 0.0020 which also exceed our fixed ε = 0.0001.

Thus if $\left|f^7(x) - f^7(x_0)\right| \geq \varepsilon$ implying $|0.0020| \geq 0.0001$

Therefore it is evidently clear and easy to say that the function is sensitive to initial condition.

And since this hold for sensitive condition the function is chaotic at r=4

Finally we can accept the fact that period-3 lead to chaos since it exits by the algebraic analysis and also through the zooming of the bifurcation diagram of figure 1.10 when r lies between 3.83 and 3.84 which are less than r equal to 4. And beyond this period-3 subsequent period occurs called the period doubling cascade into chaos. Also at r=4 the function is sensitive to initial condition therefore showing chaotic behavior.

5 Conclusion

Clearly, the behavior of the logistic function into a period−1 points also known as the fixed points was as a results of the control parameter r<3 and will only results into periodic point when r =3 or beyond. It was found that the period-1 point (fixed points) of the logistic function was attracting and repelling, that is converging and diverging when r lies within 0 and 3 and r<1 and r>3 respectively. On the issue of period−2 points of the logistic map, it was shown that the bifurcation diagram (figure 1.00) gives a better and transparent solution than that of the algebraic iteration of the function.That is when it moves from the fixed points/ orbits solution x = 0, $x = \frac{r-1}{r}$ to the periodic points which also add another solutions $x = \frac{\pm\sqrt{r^2-2r-3}+r+1}{2r}$ to the previous solutions.

It can be concluded that for a logistic function, when the parameter r keeps increasing the nature or behavior moves from periodic behaviour to chaotic behaviour, making it unstable. That is an increase in r makes the solutions unstable and higher periodic oscillating occurs.

When the cycles keep on becoming unstable, period doubling gives way to a different regime hence chaos then occurs at r = 4.

So a route to chaos can be seen through the existence of period-3, the doubling nature of the periodic orbits and the sensitive dependence on its initial condition. All these routes relies on the strength of the controlparameter r.

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