Theoretical Mathematics & Applications, vol. 6, no. 2, 2016, 139-151
ISSN: 1792-9687(print), 1792-9709(online)
Scienpress Ltd, 2016

Stability Properties With Cone –Perturbing Liapunov Function method

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Abstract

$\phi_0 - L_P -$ equistability, integrally $\phi_0 -$ equistability, eventually ϕ_0 – equistability, eventually equistability of a system of differential equations are studied, perturbing Laipunov function. Our methods are cone valued perturbing Liapunov function method and comparison methods. Some results of these concepts are given.

Mathematics Subject Classification: 34D20

Keyword: $\phi_0 - L_P -$ equistable, integrally ϕ_0– equistable, eventually ϕ_0 – equistable and eventually equistable, cone valued perturbing Liapunov function.

1 Introduction

Stability concepts of differential equations has been interested important from many authors, Lakshmikantham and Leela [4] discussed some different concepts of stability of system of ordinary differential equations namely, eventually stability, integrally stability, totally stability, L_P stability, partially stability, strongly stability, practically stability of the zero solution of systems of ordinary differential equations, Liapunov function method [6] that extend to perturbing Liapunov functional method in [3] play essential role to determine stability

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Article Info: *Received:* November 2, 2015. *Revised:* December 12, 2015.
Published online : April 5, 2016.

properties.
Akpan et, al [1] discussed new concept namely, ϕ_0– equitable of the zero solution of systems of ordinary differential equations using cone -valued Liapunov function method. Soliman [7] extent perturbing Liapunov function to so-called cone-perturbing Liapunov function method that lies between perturbing Liapunov function and perturbing Liapunov function.
In [2], and [3] El-Shiekh et.al discussed and improved some concepts stability of [4] and discussed new concepts mix between φ_0–equitable and the previous kinds of stability [3-5],[8-11]
In this paper, we discuss and improve the concept of L_P – equistability of the system of ordinary differential equations with cone perturbing Liapunov function method and comparison technique. Furthermore, we prove that some results of $\varphi_0 - L_P$ – equitability of the zero solution of the nonlinear system of function differential equations with cone -valued Liapunov function method. Also we discuss some results of $\varphi_0 - L_P -$ equitability of the zero solution of ordinary differential equations using a cone - perturbing Liapunov function method.
Let R^n be Euclidean n –dimensional real space with any convenient norm $\| \quad \|$, and scalar product $(.,.) \leq \|.\|\|.\|$. Let for some $\rho > 0$

$$S_\rho = \{x \in R^n, \|x\| < \rho\}.$$

Consider the nonlinear system of ordinary differential equations

$$x' = f(t,x), \qquad x(t_0) = x_o, \tag{1.1}$$

where $f \in C\big[J \times S_\rho, R^n\big], J = [0,\infty)$ and $C[J \times S_\rho, R^n]$ denotes the space of continuous mappings $J \times S_\rho$ into R^n.
Consider the differential equation

$$\mathrm{u}' = \mathrm{g}(\mathrm{t},\mathrm{u}) \qquad \mathrm{u}(\mathrm{t}_0) = \mathrm{u}_0 \tag{1.2}$$

where $g \in C[J \times R^n, R^n], E$ be an open $(t,u) -$ set in R^{n+1}.

The following definitions [1] will be needed in the sequel.
Definition 1.1 A proper subset K of R^n is called a cone if

$$(i)\lambda K \subset K, \qquad \lambda \geq 0. \ \ (ii) K + K \subset K,$$
$$(iii)\overline{K} = K, (iv)K^0 \neq \emptyset, \quad (v)K \cap (-K) = \{0\}.$$

where K and K^0 denotes the closure and interior of K respectively, and ∂K denote the boundary of K.
Definition 1.2.The set $K^* = \{\phi \in R^n , (\phi, x) \geq 0 , x \in K\}$ is called the adjoint cone if it satisfies the properties of the definition 1.1.

$$x \in \partial K \text{ if } (\phi, x) = 0 \text{ for some } \phi \in K_0^* \ , K_0 = K/\{0\}.$$

Definition 1.3. A function $g: D \to K , D \subset R^n$ is called quasimonotone relative to the cone $K \ if \ x, y \in D , y - x \in \partial K$ then there exists $\phi_0 \in K_0^*$ such that

$$(\phi_0, y - x) = 0 \text{ and } (\phi_0, g(y) - g(x)) > 0.$$

Definition 1.4. A function $a(.)$ is said to belong to the class $\mathcal{K} \ if \ a \in [R^+, R^+]$, $a(0) = 0$ and $a(r)$ is strictly monotone increasing in r .

2 On $\phi_0 - L_P -$ equistability

Perturbing Liapunov function method was introduced in [2] to discuss ϕ_0 – equitability properties for ordinary differential equations. In this section, we will discuss $\phi_0 - L_P -$ equistability of the zero solution of the non linear system of ordinary differential equations using cone valued perturbing Liapunov functions method.

The following definitions will be needed in the sequel and related with [2].

Definition2.1. The zero solution of the system (1.1) is said to be $\phi_0 -$ equistable, if for $\epsilon > 0$, $t_0 \in J$ there exists a positive function $\delta(t_0, \epsilon) > 0$ that is continuous in t_0 such that for $t \geq t_0$.

$$(\Phi_0, x_0) \leq \delta, \text{ implies } (\phi_0, x(t, t_0, x_0)) < \epsilon.$$

where $x(t, t_0, x_0)$ is the maximal solution of the system (1.1).

In case of uniformly ϕ_o-equistable , the δ is independent of t_o.

Definition2.2. The zero solution of the system (1.1) is said to be $\phi_0 - L_p -$ equistable and $P > 0$, if it is $\phi_0 -$ equistable and for each $\epsilon > 0$, $t_0 \in J$ there exists a positive function $\delta_0 = \delta_0(t_0, \epsilon) > 0$ continuous in t_0 such that the inequality

$$(\phi_0, x_0) \leq \delta_0, \quad \text{implies } (\phi_0, \int_{t_o}^{\infty} \|x(s, t_0, x_0)\|^P ds) < \epsilon.$$

In case of uniformly $\phi_0 - L_P -$ equistable, the δ_0 is independent of t_0.

Let for some $\rho > 0 \phi_0 - L_p$–equistability of (1.1), integrally ϕ_0- equistability

$$S_\rho^* = \{x \in R^n, (\phi_0, x) < \rho, \phi_0 \in K_0^*\}.$$

We define for $V \in C[J \times S_\rho^*, K]$, the function $D^+V(t, x)$by

$$D^+V(t, x) = \lim_{h \to 0} \sup \frac{1}{h} (V(t + h, x + hf(t, x)) - V(t, x)).$$

The following result will discuss the concept of $\phi_0 - L_p -$ equistability of (1.1) using comparison principle method.

Theorem 2.1. Suppose that there exist two functions $g_1 \in C[J \times R, R]$ and $g_2 \in C[J \times R, R]$ with $g_1(t, 0) = g_2(t, 0) = 0$ are monotone non decreasing functions, and there exist two Liapunov functions where $V_1(t, 0) = V_{2\eta}(t, 0) = 0$, and $S_\rho^* = \{x \in R^n;\ (\phi_0, x) < \eta\ , \phi_0 \in K_0^*\}$ and ${S_\rho^*}^C$denotes the complement of S_ρ^*, satisfying the following conditions:

$(H_1)V_1(t, x)$ is locally Lipschitzian in x and

$$D^+(\phi_0, V_1(t, x)) \leq g_1\big(t, V_1(t, x)\big) \quad \text{for} \quad (t, x) \in J \times S_\rho^*.$$

$(H_2)V_{2\eta}(t, x)$ is locally Lipschitzian in x and

$$b(\phi_0, x) \leq (\phi_0, V_{2\eta}(t, x)) \leq a(\phi_0, x) \tag{2.1}$$

$$(\phi_0, \int_{t_o}^{t} \|x(s,t_0,x_0)\|^P ds)$$
$$\leq (\phi_0, V_{2\eta}(t, x(t_0,x_0)) \leq a_1(\phi_0, \int_{t_o}^{t} \|x(s,t_0,x_0)\|^P ds), \quad (2.2)$$

where $a, a_1, b, b_1 \in \mathcal{K}$ for $(t, x_t) \in J \times S_\rho^* \cap {S_\rho^*}^C$.

$(H_3) D^+(\phi_0, V_1(t,x)) + D^+(\phi_0, V_{2\eta}(t,x)) \leq g_2(t, V_1(t,x) + V_{2\eta}(t,x))$ for $(t,x) \in J \times S_\rho^* \cap {S_\rho^*}^C$.

(H_4)If the zero solution of the equation

$$u' = g_1(t,u), u(t_0) = u_0. \quad (2.3)$$

is $\phi_0 -$ equistable, and the zero solution of the equation

$$\omega' = g_2(t,\omega), \ \ \omega(t_0) = \omega_0 \quad (2.4)$$

is uniformly $\phi_0 -$ equistable. Then the zero solution of the system (1.1) is $\phi_0 - L_P -$ equistable.

Proof. Since the zero solution of (2.4) is uniformly $\phi_0 -$equistable , given $0 < \epsilon < \rho$ and $b_1(\epsilon) > 0$ there exists $\delta_0 = \delta_0(\epsilon) > 0$ such that $t \geq t_0$

$$(\phi_0, \omega_0) \leq \delta_0, \text{implies } (\phi_0, r_2(t,t_0,\omega_0)) < b_1(\epsilon). \quad (2.5)$$

where $r_2(t,t_0,\omega_0)$ is the maximal solution of the system (2.4).

From the condition (H_2), there exists $\delta_2 = \delta_2(\epsilon) > 0$ such that

$$a(\delta_2) \leq \frac{\delta_0}{2} \quad (2.6)$$

From our assumption that the zero solution of the system (2.3) is $\phi_0 -$ equistable, given $\frac{\delta_0}{2}$ and $t_0 \in R_+$, there exists $\delta^* = \delta^*(t_0,\epsilon) > 0$ such that

$$(\phi_0, u_0) \leq \delta^*, \ \text{implies} \ (\phi_0, r_1(t,t_0,u_0)) < \frac{\delta_0}{2}, \quad for\ t \geq t_0 \quad (2.7)$$

where $r_1(t,t_0,u_0)$ is the maximal solution of the system (2.3).

From the conditions (H_1), (2.1) , (H_3), (H_4) and applying Theorem (2) of [6], it follows the zero solution of the system (1.1) is ϕ_0 −equistable.

To show that there exists $\delta_0 = \delta_0(t_0,\epsilon) > 0$, such that

$$(\phi_0, x_0) \leq \delta_0, \ \text{implies} \ (\phi_0, \int_{t_o}^{\infty} \|x(s,t_0,x_0)\|^P ds) < \epsilon.$$

Suppose this is false, then there exists $t_1 > t_2 > t_0$. such that for $(\phi_0, \psi) \leq \delta_0$.

$$(\phi_0, \int_{t_o}^{t_1} \|x(s,t_0,x_0)\|^P ds) = \delta_2 \ , (\phi_0, \int_{t_o}^{t_2} \|x(s,t_0,x_0)\|^P ds) = \epsilon \quad (2.8)$$

$$\delta_2 \leq (\phi_0, \int_{t_o}^{t} \|x(s,t_0,x_0)\|^P ds) \leq \epsilon \ \text{ for } t \in [t_1,t_2].$$

Let $\delta_2 = \eta$ and setting $m(t,x) = V_1(t,x) + V_{2\eta}(t,x)$for $t \in [t_1,t_2]$.

From the condition (H_3), we obtain

$$D^+(\phi_0, m(t,x)) \leq g_2(t, m(t,x)).$$

We can choose $m(t_1, x(t_1)) = V_1(t_1, x(t_1)) + V_{2\eta}(t_1, x(t_1)) = \omega_0$.

Applying Theorem (8.1.1) of [5], we get

$$(\phi_0, m(t,x)) \leq (\phi_0, r_2(t, t_1, m(t_1, x(t_1)))) \text{ for } t \in [t_1, t_2]. \quad (2.9)$$

Choosing $u_0 = V_1(t_0, x_0)$. From the condition (H_1) and applying the comparison Theorem, we get

$$(\phi_0, V_1(t,x)) \leq (\phi_0, r_1(t, t_0, u_0))$$

Let $t = t_1$and from (2.7),we get

$$(\phi_0, V_1(t_1, x(t_1)) \leq (\phi_0, r_1(t_1, t_0, u_0)) < \frac{\delta_0}{2}.$$

From the condition(H_2), (2.6) and(2.8), we obtain

$$(\phi_0, V_{2\eta}(t_1, x(t_1)) \leq a_1(\phi_0, \int_{t_o}^{t_1} \|x(s, t_0, x_0)\|^P ds) \leq a_1(\delta_2) \leq \frac{\delta_0}{2}..$$

So we get $(\phi_0, \omega_0) = (\phi_0, V_1(t_1, x(t_1)) + V_{2\eta}(t_1, x_{t_1} x(t_1)) \leq \delta_0$.

Then from (2.5) and (2.9), we get

$$(\phi_0, m(t, x_t)) \leq (\phi_0, r_2(t, t_1, \omega(t_1)) < b_1(\epsilon). \quad (2.10)$$

From the condition (H_2), (2.8) and (2.10) at $t = t_2$

$$b_1(\epsilon) = b_1(\phi_0, \int_{t_o}^{t_2} \|x(s, t_0, x_0)\|^P ds) \leq (\phi_0, V_{2\eta}(t_2, x(t_2))) < (\phi_0, m(t_2, x(t_2)))$$
$$\leq b_1(\epsilon).$$

This is a contradiction, therefore it must be

$$(\phi_0, \int_{t_o}^{\infty} \|x(s, t_0, x_0)\|^P ds) < \epsilon \text{ provided that } (\phi_0, x_0) \leq \delta_0.$$

Then the zero solution of the system (1.1) is $\phi_0 - L_P -$ equistable.

3 On Integrally ϕ_0-equistable

In this section, we discuss the concept of Integrally ϕ_o - equistability of the zero solution of non linear system of ordinary diffrential equations using cone valued perturbing liapunow functions method and comparison principle method. Consider the non linear system of differential equation(1.1) and the perturbed system

$$x' = f(t,x) + R(t,x), x(t_0) = x_0 \quad (3.1)$$

where $f, R \in C[J \times S_\rho^*, R^n], J = [0, \infty]$ and $C[J \times S_\rho^*, R^n]$ denotes the space of continuous mapping $J \times S_\rho^*$into R^n. Consider the scalar differetail equation (2.3), (2.4) and the perturbing equations

$$u' = g_1(t.u) + \varphi_1(t), u(t_0) = u_0 \quad (3.2)$$
$$\omega' = g_2(t.\omega) + \varphi_2(t), \omega(t_0) = \omega_0 \quad (3.3)$$

where $g_1, g_2 \in C[J \times R, R]$, $\varphi_1, \varphi_2 \in C[J, R]$ respectively.

The following definitions [4] will be needed in the sequal.

Definition 3.1. The zero solution of the system (1.1) is said to be integrally

ϕ_0-equistable if for every $\alpha \geq 0$ and $t_0 \in J$, there exists a positive function $\beta = \beta(t_0, \alpha)$ which in continuous in t_0, for each α and $\beta \in K$, such that for $\phi_0 \in K_0^*$ every solution $x(t, t_0, x_0)$ of pertubing differential equation (3.1), the inequality

$$(\phi_0, x(t, t_0, x_0)) < \beta, \qquad t \geq t_0$$

holds, provided that $(\phi_0, x_0) \leq \alpha$, and every T$> 0$,

$$(\phi_0, \int_{t_0}^{t_0+T} sup_{\|x\|<\beta} \|R(s,x)\| ds) \leq \alpha.$$

Definition 3.2. The zero solution of (3.2) is said to be integrally ϕ_0-equistable if , for every $\alpha_1 \geq 0$ and $t_0 \in J$, there exists a positive function $\beta_1 = \beta_1(t_0, \alpha)$ which in continuous in t_0, for each α_1 and $\beta_1 \in \mathcal{K}$, such that for $\phi_0 \in K_0^*$ every solution $u(t, t_0, u_0)$ of perturbing differential equation (2.3), the inequality

$$(\phi_0, u(t, t_0, u_0)) < \beta_1, \qquad t \geq t_0$$

holds, provided that $(\phi_0, u_0) \leq \alpha_1$, and for every T> 0,

$$(\phi_0, \int_{t_0}^{t_0+T} \varphi_1(s) ds) \leq \alpha.$$

In the case of uniformly integrally ϕ_0-equistable , $\beta_1$is independent of t_0.
We define for a cone valued Liapunov function $V(t,x) \in C[J \times S_\rho^*, K]$ is Lipschitzian in x. The function

$$D^+V(t,x)_{3.1} = \lim_{h \to 0} sup \frac{1}{h}(V(t+h, x + h\big(f(t,x) + R(t,x)\big)) - V(t,x)).$$

The following result is related with that of [5].

Theorem 3.1. Let the function $g_2(t, \omega)$ be nonincreasig in ω for each $t \in R^+$, and the assumptions $(H_1), (H_2) - (2.1)$ and (H_3) be satisfied.
If the zero solution of (2.3) is integrally ϕ_0-equistable, and the zero solution of (2.4) is uniformly integrally ϕ_0-equistable.
Then the zero solution of (1.1) is integrally ϕ_0-equistable.
Proof . Since the zero solution of (2.4) is integrally $\phi_0 -$ equistable, given $\alpha_1 \geq 0$ and $t_0 \geq 0$ there exists $\beta_0 = \beta_0(t_0, \alpha_1)$ such that $t \geq t_0$ such that for any $\phi_0 \in K_0^*$ and for any solution $\mathrm{u}(\mathrm{t}, t_0, u_0)$ of the perturbed system (3.2) satisfies the inequality

$$\big(\phi_0, u(t, t_0, u_0)\big) < \beta_0 \tag{3.4}$$

holds provided that $(\phi_0, u_0) \leq \alpha_1$, and for every T$> 0$,

$$(\phi_0, \int_{t_0}^{t_0+T} \varphi_1(s) ds) \leq \alpha_1.$$

From our assumption that the zero solution of the system (2.3) is uniformly integrally ϕ_o - equistable given $\alpha_2 \geq 0$, there exists $\beta_1 = \beta_1(\alpha_2)$ such that every solution $\omega(\mathrm{t}, t_0, \omega_0)$ of the perturbed equation (3.3) satisfies the inequality

$$\big(\phi_0, \omega(t, t_0, \omega_0)\big) < \beta_1 \tag{3.5}$$

holds provided that $(\phi_0, \omega_0) \leq \alpha_2$ and for every T> 0,

$$(\phi_0, \int_{t_0}^{t_0+T} \varphi_2(s)ds) \le \alpha_2.$$

Suppose that there exists $\alpha > 0$ such that

$$\alpha_2 = a(\alpha) + \beta_0 \tag{3.6}$$

since $b(u) \to \infty \; as \; u \to \infty$ then we can find $\beta(t_0, \alpha)$ such that

$$b(\beta) > \beta_1(\alpha_2) \tag{3.7}$$

To prove that the zero solution of (1.1) is integrally $\phi_0 -$ equistable, it must be for every $\alpha \ge 0$ and $t_0 \in J$, there exists a positive function $\beta = \beta(t_0, \alpha)$ which in continuous in t_0, for each α and $\beta \in \mathcal{K}$, such that for $\phi_0 \in K_0^*$ every solution $x(t, t_0, x_0)$ of pertubing differential equation (3.1), the inequality

$$(\phi_0, x(t, t_0, x_0)) < \beta, \qquad t \ge t_0$$

holds , provided that $(\phi_0, x_0) \le \alpha$ and every T> 0,

$$(\phi_0, \int_{t_0}^{t_0+T} sup_{\|x\|<\beta} \|R(s,x)\| ds) \le \alpha.$$

Suppose this is false, then there exists $t_2 > t_1 > t_0$ such that

$$\begin{gathered} \left(\phi_0, x(t_1, t_0, x_0)\right) = \alpha \;, (\phi_0, x(t_2, t_0, x_0)) = \beta \\ \alpha \le \left(\phi_0, x(t, t_0, x_0)\right) \le \beta \; for \; t \in [t_1, t_2] \end{gathered} \tag{3.8}$$

Let $\delta_2 = \alpha$, and setting $m(t,x) = V_1(t,x) + V_{2\eta}(t,x)$ for $t \in [t_1, t_2]$.

Since $V_1(t,x)$ and $V_{2\eta}(t,x)$ are Lipschitizian in x for constants M and K respectively. Then

$$\begin{gathered} D^+(\phi_0, V_1(t,x))_{3.1} + D^+(\phi_0, V_{2\eta}(t,x))_{3.1} \le \\ D^+(\phi_0, V_1(t,x))_{1.1} + D^+(\phi_0, V_{2\eta}(t,x))_{1.1} + N(\phi_0, R(t,x)). \end{gathered}$$

where $N = M + K$. From the condition (H_3), we obtain

$$D^+(\phi_0, m(t,x)) \le g_2(t, m(t,x)) + N(\phi_0, R(t,x)).$$

We can choose $m(t_1, x(t_1)) = V_1(t_1, x(t_1)) + V_{2\eta}(t_1, x(t_1)) = \omega_0$.

Applying Theorem (8.1.1) of [5], we get

$$(\phi_0, m(t,x)) \le (\phi_0, r_2\left(t, t_1, m(t_1, x(t_1))\right)) \quad \text{for} \quad t \in [t_1, t_2] \tag{3.9}$$

where $r_2\left(t, t_1, m(t_1, x(t_1))\right)$ is the maximal solution of the perturbed system (3.3), where $\varphi_2(t) = NR(t,x)$. To prove that $(\phi_0, r_2(t, t_1, \omega_0)) < \beta_1(\alpha_2)$, it must be shown that

$$(\phi_0, \omega_0) \le \alpha_2 \;, \;\; (\phi_0, \textstyle\int_{t_0}^{t_0+T} \varphi_2(s)ds) \le \alpha_2$$

Choosing $u_0 = V_1(t_0, x_0)$, since $V_1(t,x)$ is a Lipschitizian in x for a constant $M > 0$,then

$$\begin{gathered} \|\phi_0\| \|V_1(t_0, x_0)\| \le M \|\phi_0\| \|x_0\| \\ (\phi_0, u_0) = \left(\phi_0, V_1(t_0, x_0)\right) \le M(\phi_0, x_0) \le M\alpha = \alpha_1 \end{gathered} \tag{3.10}$$

Also we get

$$D^+(\phi_0, V_1(t,x))_{3.1} \le D^+(\phi_0, V_1(t,x))_{1.1} + M(\phi_0, R(t,x)).$$

From the condition (H_1) we get

$$(\phi_0, V_1(t,x)) \le (\phi_0, r_1(t, t_0, u_0)) \text{ for } t \in [t_1, t_2].$$

where $r_1(t, t_0, u_0)$ is the maximal solution of (3.2) and define $\varphi_1(t) = MR(t,x)$.

Integrating it,we get

$$\int_{t_0}^{t_0+T} \varphi_1(s)\, ds = \int_{t_0}^{t_0+T} M\|R(s,x)\|ds \leq \ M\int_{t_0}^{t_0+T} sup_{\|x\|<\beta}\|R(s,x)\|ds$$

which leads to

$$\left(\phi_0, \int_{t_0}^{t_0+T} \varphi_1(s)\, ds\right) \leq \ M\left(\phi_0, \int_{t_0}^{t_0+T} sup_{\|x\|<\beta}\|R(s,x)\|ds\right) \leq M = \alpha_1 \quad (3.11)$$

From (3.4),(3.10) and (3.11) at $t = t_1$,we get

$$(\phi_0, V_1(t_1, x(t_1)) \leq (\phi_0, r_1(t_1, t_0, u_0)) < \beta_0.$$

From the condition (2.1) and (3.7), we obtain

$$(\phi_0, V_{2\eta}(t_1, x(t_1)) \leq a_1(\ \phi_0, x(t_1)) \leq a(\alpha).$$

From (3.6), we get

$$(\phi_0, \omega_0) = (\phi_0, V_1(t_1, x(t_1)) + V_{2\eta}(t_1, x_{t_1}x(t_1)) \leq \alpha_2 \quad (3.12)$$

Since $\varphi_2(t) = NR(t,x)$, then integrating both sides

$$\int_{t_0}^{t_0+T} \varphi_2(s)ds = \int_{t_0}^{t_0+T} N\|R(t,s)\|ds \leq N\int_{t_0}^{t_0+T} sup_{\|x\|<\beta}\|R(t,x)\|ds$$

which leads to

$$\left(\phi_0, \int_{t_0}^{t_0+T} \varphi_2(s)\, ds\right) \leq \ \mathrm{N}\left(\phi_0, \int_{t_0}^{t_0+T} sup_{\|x\|<\beta}\|R(s,x)\|ds\right) \leq N = \alpha_2 \quad (3.13)$$

Then from (3.5), (3.12) and (3.13), we get

$$(\phi_0, m(t,x)) \leq (\phi_0, r_2(t, t_1, \omega(t_1)) < \beta_1(\alpha_2). \quad (3.14)$$

From the condition (2.1), (3.7) and (3.14) at $t = t_2$,we have

$$b(\beta) = b\big(\phi_0, x(t_2)\big) \leq (\phi_0, V_{2\eta}(t_2, x(t_2)) < (\phi_0, m(t_2, x(t_2))$$
$$\leq \beta_1(\alpha_2) < b(\beta).$$

That is a contradiction, therefore it must be

$$(\phi_0, x(t, t_0\,, x_0)) < \beta, t\ \geq t_0$$

Then the zero solution of the system (1.1) is integrally $\phi_0 -$ equistable.

4 Eventually equistable

In this section, we discuss the notion of eventually-equistable of the zero solution of non linear system (1.1) using perturbing liapunow functions method and comparison principle method.

The following definition will be needed in the sequel and related with that [3].

Definition 4.1. The zero solution of the system (1.1) is said to eventually uniformly equistable if for $\epsilon > 0$, there exists a positive function $\delta(\epsilon) > 0 \quad and \quad \tau = \tau(\epsilon)$ such that the inequality

$$\|x_0\| \leq \delta, \ \text{ implies}\|\ x(t, t_0, x_0)\| < \epsilon, \ \ t \geq t_0 \geq \tau(\epsilon)$$

where $x(t, t_0, x_0)$ is any solution of the system (1.1).

Theorem 4.1. Suppose that there exist two functions $g_1, g_2 \in C[J \times R^+, R]$ with $g_1(t,0) = g_2(t,0) = 0$, and two Liapunov functions

$$V_1(t,x) \in C[J \times S_\rho, R^n] \text{ and } V_{2\eta}(t,x) \in C[J \times S_\rho \cap S_\eta^C, R^n]$$

where $V_1(t,0) = V_{2\eta}(t,0) = 0$, and $S_\eta = \{x \in R^n; \|x\| < \eta\}$ and S_η^C denotes the complement of S_η, satisfying the following conditions

$(h_1)V_1(t,x)$ is locally Lipschitzian in x and

$$D^+V_1(t,x) \le g_1(t, V_1(t,x)) \text{ for } (t,x_t) \in J \times S_\rho.$$

$(h_2)V_{2\eta}(t,x)$ is locally Lipschitzian in x, and

$$b(\|x\|) \le V_{2\eta}(t,x) \le a(\|x\|)$$

for $0 < r < \|x\| < \rho$ and $t \ge \theta(r)$, where $\theta(r)$ is a continuous monotone decreasing in r , $for\ 0 < r < \rho$ where $a, b \in \mathcal{K}$. For $(t,x) \in J \times S_\rho \cap S_\eta^C$.

$(h_3)D^+V_1(t,x) + D^+V_{2\eta}(t,x) \le g_2(t, V_1(t,x) + V_{2\eta}(t,x))$ for $(t,x)) \in J \times S_\rho \cap S_\eta^C$.

(h_4) If the zero solution of (2.3) is uniformly equistable, and the zero solution of (2.4) is eventually uniformly equistable, then the zero solution of the system (1.1) is uniformly eventually equistable.

Proof. Since the zero solution of (2.4) is eventually uniformly equistable, given $b(\epsilon) > 0$ there exists $\tau_1 = \tau_1(\epsilon) > 0$ and $\delta_0 = \delta_0(\epsilon) > 0$ such that

$$\omega_0 \le \delta_0, \text{ implies } \omega(t, t_0, \omega_0) < b(\epsilon),\ \ t \ge t_0 \ge \tau_1(\epsilon) \tag{4.1}$$

where$\omega(t, t_0, \omega_0)$ is any solution of the system (2.4).

Since $a(u) \to \infty$ as $u \to \infty$ for $a \in \mathcal{K}$, it is possible to choose $\delta_1 = \delta_1(\epsilon) > 0$ such that

$$a(\delta_1) \le \frac{\delta_0}{2} \tag{4.2}$$

From our assumption that the zero solution of the system (2.3) is uniformly equistable. Given $\frac{\delta_0}{2}$, there exists $\delta^* = \delta^*(\epsilon) > 0$ such that

$$u_0 \le \delta^*,\ \ \text{implies } u(t, t_0, u_0) < \frac{\delta_0}{2} \tag{4.3}$$

where$u(t, t_0, u_0)$ is any solution of the system (2.3).

Choosing $u_0 = V_1(t_0, x_0)$, since $V_1(t,x)$ is a Lipschitizian function for a contant M. Then there exists $\delta_2 = \delta_2(\epsilon) > 0$ such that

$$\|x_0\| \le \delta_2,\ \ \text{implies } V_1(t_0, x_0) \le M\ \|x_0\| \le M\delta_2 \le \delta^*$$

$\max[\tau_1(\epsilon), \tau_2(\epsilon)]$.

To prove theorem, it must be shown that set

$$\delta = \min(\delta_1, \delta_2) \text{ and suppose} \|x_0\| \le \delta\ , \text{ define } \tau_2(\epsilon) = \theta\big(\delta(\epsilon)\big) \text{ and let } \tau = \tau(\epsilon)$$

$$\|x_0\| \le \delta \text{ implies} \|\ x(t, t_0, x_0)\| < \epsilon\ ,\ \ t \ge t_0 \ge \tau(\epsilon)$$

Suppose that is false, then there exists $t_2 > t_1 > t_0$such that

$$\|x(t_1)\| = \delta_1, \|x(t_2)\| = \epsilon \tag{4.4}$$

$$\delta_1 \le \|x(t)\| \le \epsilon \quad \text{for } t \in [t_1, t_2].$$

Let $\delta_1 = \eta$ and setting $m(t,x) = V_1(t,x) + V_{2\eta}(t,x)$ for $t \in [t_1, t_2]$.

From the condition(h_3), we obtain

$$D^+m(t,x) \le G_2(t,m(t,x)).$$

we can choose $m(t_1,x(t_1)) = V_1(t_1,x(t_1)) + V_{2\eta}(t_1,x(t_1)) = \omega_0$.
Applying Theorem (8.1.1) of [5], we get

$$m(t,x) \le r_2(t,t_1,m(t_1,x(t_1))) \quad (4.5)$$

where$r_2(t,t_1,m(t_1,x(t_1)))$ is the maximal solution of (2.4). Choosing $u_0 = V_1(t_0,x_0)$. From the condition (h_1) and applying the comparison Theorem, we get

$$V_1(t,x) \le r_1(t,t_0,u_0) \text{for } t \in [t_0,t_1]. \quad (4.6)$$

Let $t = t_1$and from (4.3), we get

$$V_1(t_1,x(t_1)) \le r_1(t_1,t_0,u_0) < \frac{\delta_0}{2}.$$

From the condition (h_2), (4.2) and (4.4)

$$V_{2\eta}(t_1,x(t_1)) \le a(\|x(t_1)\|) \le a(\delta_1) \le \frac{\delta_0}{2}.$$

So we get

$$\omega_0 = V_1(t_1,x(t_1)) + V_{2\eta}\big(t_1,x(t_1)\big) \le \delta_0.$$

Then from (4.1) and (4.5), we get

$$m(t,x) \le r_2(t,t_1,\omega(t_1)) < b(\epsilon) \quad (4.7)$$

From (h_2),(4.4) and (4.7) at $t = t_2$

$$b(\epsilon) = b(\|x(t_2)\| \le V_{2\eta}(t_2,x(t_2)) < m(t_2,x(t_2)) \le b(\epsilon).$$

This is a contradiction, therefore it must be

$$\| x(t,t_0,x_0)\| < \epsilon , t \ge t_0 \ge \tau(\epsilon)$$

Provided that$\|x_0\| \le \delta$ .Then the zero solution of the system (1.1) is uniformly eventually equistable.

5 Eventually $\phi_0 - equistable$

In this section, we discuss the notion of eventuallyϕ_0-equistable of the zero solution of non linear system (1.1) using cone valued perturbing liapunow functionsmethod and comparison principle method. The following definition is somewhat new and related with that [3].

Definition 5.1. The zero solution of the system (1.1) is said to eventually uniformly ϕ_0-equistable if for $\epsilon > 0$,there exists a positive function $\delta(\epsilon) > 0$ *and* $\tau = \tau(\epsilon)$ such that the inequality

$$(\phi_0,x_0) \le \delta, \quad \text{implies } (\phi_0,x(t,t_0,x_0)) < \epsilon , \quad t \ge t_0 \ge \tau(\epsilon)$$

where $x(t,t_0,x_0)$ is the maximal solution of the system (1.1).

Theorem 5.1. Let the assumptions $(H_1),(H_2) - (2.1)$ and (H_3) be satisfied for $0 < r < (\phi_0,x) < \rho$ and $t \ge \theta(r)$, where $\theta(r)$ is a continuous monotone decreasing in r for $0 < r < \rho$ where $a, b \in \mathcal{K}$. If the zero solution of (2.3) is uniformly ϕ_0-equistable and the zero solution of (2.4) is uniformly eventually ϕ_0 -equistable. Then the zero solution of (1.1) is uniformly eventually ϕ_0-equistable.

Proof. Since the zero solution of (2.4) is eventually uniformly $\phi_0 -$ equistable, given $b(\epsilon) > 0$ there exists $\tau_1 = \tau_1(\epsilon) > 0$ and $\delta_0 = \delta_0(\epsilon) > 0$ such that

$$(\phi_0, \omega_0) \leq \delta_0, \text{ implies } (\phi_0, r_2(t, t_0, \omega_0)) < b(\epsilon), t \geq t_0 \geq \tau_1(\epsilon) \tag{5.1}$$

where $r_2(t, t_0, \omega_0)$ is the maximal solution of the system (2.4).

Since $a(u) \to \infty$ as $u \to \infty$ for $a \in \mathcal{K}$, it is possible to choose $\delta_1 = \delta_1(\epsilon) > 0$ such that

$$a(\delta_1) \leq \frac{\delta_0}{2} \tag{5.2}$$

From our assumption that the zero solution of the system (2.3) is uniformly ϕ_0 $-$equistable.Given $\frac{\delta_0}{2}$, there exists $\delta^* = \delta^*(\epsilon) > 0$ such that

$$(\phi_0, u_0) \leq \delta^*, \text{ implies } (\phi_0, r_1(t, t_0, u_0)) < \frac{\delta_0}{2} \tag{5.3}$$

where $r_1(t, t_0, u_0)$ is the maximal solution of the system (2.3).

Choosing $u_0 = V_1(t_0, x_0)$, since $V_1(t, x)$ is a Lipschitizian function for a constant M. Then there exists $\delta_2 = \delta_2(\epsilon) > 0$ such that

$$(\phi_0, x_0) \leq \delta_2 \text{ implies } (\phi_0, V_1(t_0, x_0)) \leq M (\phi_0, x_0) \leq M\delta_2 \leq \delta^*$$

Set

$$\delta = \min(\delta_1, \delta_2) \text{ and suppose} (\phi_0, x_0) \leq \delta,$$

then define $\tau_2(\epsilon) = \theta(\delta(\epsilon))$ and let $\tau(\epsilon) = \max[\tau_1(\epsilon), \tau_2(\epsilon)]$.

To prove the zeo solution of (1.1) is uniformly eventually ϕ_0-equistable, it must be shown that

$$(\phi_0, x_0) \leq \delta, \text{ implies} (\phi_0, x(t, t_0, x_0)) < \epsilon, t \geq t_0 \geq \tau(\epsilon)$$

Suppose that is false, then there exists $t_2 > t_1 > t_0$ such that

$$(\phi_0, x(t_1)) = \delta_1 , \quad (\phi_0, x(t_2) = \epsilon \tag{5.4}$$

$$\delta_1 \leq (\phi_0, x(t, t_0, x_0)) \leq \epsilon \text{ for } t \in [t_1, t_2].$$

Let $\delta_1 = \eta$, and setting $m(t, x) = V_1(t, x) + V_{2\eta}(t, x)$ for $t \in [t_1, t_2]$.

From the condition (H_3), we obtain

$$D^+(\phi_0, m(t, x)) \leq g_2(t, m(t, x)).$$

Choose $m(t_1, x(t_1)) = V_1(t_1, x(t_1)) + V_{2\eta}(t_1, x(t_1)) = \omega_0$.

Applying Theorem (8.1.1) of [5], we get A

$$(\phi_0, m(t, x)) \leq (\phi_0, r_2(t, t_1, m(t_1, x(t_1))) \tag{5.5}$$

Choosing $u_0 = V_1(t_0, x_0)$, from the condition (H_1) and applying the comparison Theorem 1.4.1 of [3] , we get

$$(\phi_0, V_1(t, x)) \leq (\phi_0, r_1(t, t_0, u_0)) \text{ for } t \in [t_0, t_1]. \tag{5.6}$$

Let $t = t_1$ and from (5.3) , we get

$$(\phi_0, V_1(t_1, x(t_1))) \leq (\phi_0, r_1(t_1, t_0, u_0)) < \frac{\delta_0}{2}.$$

From the condition (H_2), (5.2) and (5.4)

$$(\phi_0, \mathrm{V}_{2\eta}(\mathrm{t}_1, x(t_1))) \leq a(\phi_0, x(t_1)) \leq \mathrm{a}(\delta_1) \leq \frac{\delta_0}{2}$$

So we get

$$(\phi_0, \omega_0) = (\phi_0, \mathrm{V}_1(\mathrm{t}_1, x(t_1))) + (\phi_0, \mathrm{V}_{2\eta}(\mathrm{t}_1, x(t_1))) \leq \delta_0.$$

Then from (5.1) and (5.5), we get

$$(\phi_0, \mathrm{m}(\mathrm{t}, \mathrm{x})) \leq (\phi_0, \mathrm{r}_2(\mathrm{t}, \mathrm{t}_1, \omega(\mathrm{t}_1))) < b(\epsilon). \quad (5.7)$$

From (H_2),(5.4) and (5.7) at $\mathrm{t} = \mathrm{t}_2$

$$b(\epsilon) = b(\phi_0, x(t_2)) \leq (\phi_0, \mathrm{V}_{2\eta}(\mathrm{t}_2, x(t_2)) < (\phi_0, \mathrm{m}(\mathrm{t}_2, x(t_2)) \leq \mathrm{b}(\epsilon).$$

This is a contradiction, therefore it must be

$$(\phi_0, x(t, t_0, x_0)) < \epsilon, t \geq t_0 \geq \tau(\epsilon)$$

Provided that $(\phi_0, x_0) \leq \delta$. Then the zero solution of the system (1.1) is uniformly eventually $\phi_0 -$ equistable .

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