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Dynamic Economic Model for Optimal Investment and Insurance

Dan Zhu[1]

Abstract

In this paper, we consider a multi-period investment-insurance problem. Assume the transition matrix of the market stage is certain, we obtain a multi-period dynamic programming model. The target of decision is maximizing excess return. We can obtain an overall optimal investment-insurance strategy of the multi-period dynamic programming model.

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1 Introduction

Insurance of deductibles is a form of insurance that the two sides agreed in

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[1]Hunan University of Finance and Economics, Changsha, 410205, China.
E-mail: dzhu2000123@sina.com

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advance that the insured to bear part of the loss. When the amount of loss within the deductible, the insurer is not liable, and when the loss exceeds the amount of the deductible, the insurer pays the balance between the amount of loss and deductibles. It first appeared in the Western developed countries' insurance industry. The insurance of deductibles has two advantages compared with the traditional insurance products. First, it can reduce the amount of claims of small claims cases to reduce the operating costs of the insurance companies; the second is that it can reduce the premium expenses of the insured so that social resources can e saved. In China, the absolute deductible has been applied to the car, property, health and other types of insurance provisions.

Generally speaking, the premiums of this insurance is inversely with deductibles. The higher the deductible is, the lower the premium will be, but at the same time, when the loss occurs, the insured can obtain lesser compensation. Thus in the deductible average insurance, the insured are faced with the problem of how to choose a deductible. A proper decision on the size of deductible can not only help to achieve a rational spreading of risks between the insurance company and the insured, but can also meet the requirement of a low premium, thus the overall claim costs can be reduced. The academic world's research on the issue of deductible pricing is usually the objective function to establish the interest of the insurer or the insured. Arrow et al studied the method of deductible pricing under different assumptions [1, 2]. Schlesinger (1997) and Gollier (2000) studied the case when the utility function of the insured has the characteristics of mean constant contraction and showed the optimal deductibles must be non-negative. They also explained that the deductible must be 0 if the claims costs are fixed and the amount of compensation has nothing to do with the deductible [3, 4].

Some domestic scholars also carried out some research on this issue, the main results are as first follows: Dan Zhu et al, studied the case of the optimal deductible in single-period market by using the theory of expectations - variance utility [5]. Xiao-xia Li obtained the equation that the optimal deductible should satisfy when the insured person is risk averse by zero utility principle [6]. Wen Cai discussed whether there is an optimal value of the deductible that will make the insurance company annual profit utility maximization in the two forms of deductibles and deductibles relative [7]. But most of the above studies are confined to purely insurance market, but in fact, the applicant is in a complex capital market. In addition to insurance, he also faces many other investment options. Based on the above studies, we add other risk factors asset investment and consider a multi-period investment-insurance problem. Using dynamic programming principle and modern utility theory, we discuss the optimal investment amount and optimal deductible of risk assets in multi-period investment and insurance market under the discrete case. Assume the transition matrix of the market stage is certain, we obtain a multi-period dynamic programming model. The target of decision is maximizing excess return. We can obtain an overall optimal investment-insurance strategy of the multi-period dynamic programming model.

2 Market model and basic assumptions

2.1 In this paper, we consider there is a finite number of discrete point k, $k \in \{0,1,2,\cdots,n\} = T$. Investment and Insurance are carried out at discrete points k.

2.2 Let $(\Omega, F, \{F_k\}_{k\geq 0}^n, P)$ be a probability space with domain streams, where $\Omega = (\omega_1, \omega_2, \cdots, \omega_m)$ is a market state set which is limited, $F = 2^\Omega$ is the subset of Ω, P is the probability of market, domain streams $\{F_k\}_{k\geq 0}^n$ is satisfied

$$\{\Phi,\ \Omega\} \equiv F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n \subseteq F .$$

2.2.1 In the market of investment and insurance, the state of market ξ changed in discrete interval $[k, k+1]$, $k \in \{0,1,\cdots,n-1\}$, and this change is only related with current state and the state that will be transferred to , but is unrelated with the previous state. It means the state of market $\{\xi(t,\omega)\}$ is a process of Markov.

2.2.2 The probability of change from state i_k (at time k) to state i_{k+1} (at time $k+1$) can be known by previous data, and it is not affected by investment strategy of individual investors. Let innovation the change probability matrix of market state

$$A = \left(p_{i_k i_{k+1}}\right) = \begin{pmatrix} p_{\omega_1\omega_1} & p_{\omega_1\omega_2} & \cdots & p_{\omega_1\omega_m} \\ p_{\omega_2\omega_1} & p_{\omega_2\omega_2} & \cdots & p_{\omega_2\omega_m} \\ \cdots & \cdots & \cdots & \cdots \\ p_{\omega_m\omega_1} & p_{\omega_m\omega_2} & \cdots & p_{\omega_m\omega_m} \end{pmatrix}$$

is known before.

2.3 There is a risk-free asset and a risky asset in the market whose rice is determined by market at any moment. In the process of $[k,k+1]$, when the state is changed from i_k to i_{k+1}, the rate of risk-free asset and a risky asset are $r_{i_k i_{k+1}}$, $\tilde{r}_{i_k i_{k+1}}$, where

2.3.1 $r_{i_k i_{k+1}} \geq 0$, $r_{i_k i_{k+1}} \geq 0$

2.3.2 $\tilde{r}_{i_k i_{k+1}} = \tilde{r}_{1 i_k i_{k+1}} - \tilde{r}_{2 i_k i_{k+1}}$, the $\tilde{r}_{1 i_k i_{k+1}}$ is the normal risk rate of investment during $[k,k+1]$, $\tilde{r}_{2 i_k i_{k+1}}$ is the loss-risk interest rate of net endogenous during $[k,k+1]$.

2.3.3 An assumption of the state rate matrix of risk-free asset and a risky asset

$$B \equiv \left(r_{i_k i_{k+1}}\right) = \begin{pmatrix} r_{\omega_1\omega_1} & r_{\omega_1\omega_2} & \cdots & r_{\omega_1\omega_m} \\ r_{\omega_2\omega_1} & r_{\omega_2\omega_2} & \cdots & r_{\omega_2\omega_m} \\ \cdots & \cdots & \cdots & \cdots \\ r_{\omega_m\omega_1} & r_{\omega_m\omega_2} & \cdots & r_{\omega_m\omega_m} \end{pmatrix}$$

and

$$C_j \equiv \left(\tilde{r}_{j i_k i_{k+1}}\right) = \begin{pmatrix} \tilde{r}_{j\omega_1\omega_1} & \tilde{r}_{j\omega_1\omega_2} & \cdots & \tilde{r}_{j\omega_1\omega_m} \\ \tilde{r}_{j\omega_2\omega_1} & \tilde{r}_{j\omega_2\omega_2} & \cdots & \tilde{r}_{j\omega_2\omega_m} \\ \cdots & \cdots & \cdots & \cdots \\ \tilde{r}_{j\omega_m\omega_1} & \tilde{r}_{j\omega_m\omega_2} & \cdots & \tilde{r}_{j\omega_m\omega_m} \end{pmatrix}, \ j = 1, 2$$

can be known from previous data.

In fact, assumptions2.3.2 and 2.3.3 mean that on the premise the state i_k at time k is known, the rate $r_{i_k \bullet}$, $\tilde{r}_{j i_k \bullet}$ are random variables whose distribution function are known.

2.4 Supposing that the investor insure deductible insurance for $\tilde{r}_{2 i_k \bullet}$ during $\left[k, k+1\right]$ according to the principle that the expected value of compensation expenseis equal to the premium income, and taking into account the administrative costs of insurance companies, the insurance contract provides that:

2.4.1 the Compensation rates is

$$I^{k+1}(\tilde{r}_{2 i_k \bullet}, c^{k+1}) = \begin{cases} 0, & \tilde{r}_{2 i_k \bullet} \le c^{k+1} \\ \tilde{r}_{2 i_k \bullet} - c^{k+1}, & \tilde{r}_{2 i_k \bullet} > c^{k+1} \end{cases}$$

2.4.2 the Cent premium rate is

$$H^{k+1}(\tilde{r}_{2 i_k \bullet}, c^{k+1}) = (1+\lambda) E I^{k+1}(\tilde{r}_{2 i_k \bullet}, c^{k+1}) + b$$

The $\lambda \ge 0, b \ge 0$ are Load factor [8], $c^{k+1} > 0$ is deductible-factor in period $\left[k, k+1\right]$, it is time functions and can be determined by investor himself.

2.5 Suppose that in every invest period$\left[k, k+1\right]$, the premium expend happens at time k and the compensation happens at time $k+1$.

mark $I \equiv (I^1, I^2, \cdots, I^n)$, $H \equiv (H^1, H^2, \cdots, H^n)$.

Definition 2.1 $M = \left\{T, \left(\Omega, F, \{F_k\}_{k\ge 0}^{n}, P\right), (A, B, C_j), (I, H)\right\}$ *is a multi-period market of investment and insurance.*

Now, we suppose that the initial assets of the investor is and he enters the market to invest for n times at the time $k=0$. As a matter of convenience, we further assume that:

2.6 During the all period, there is no capital injected and no capital extracted except initial assets. In every period $[k,k+1]$, the investor allocates $z_0^{k+1}(\cdot)$ in risk-free asset, and $z_1^{k+1}(\cdot)$ in risky asset, and signs a contract with the insurance company . The deductible factor c^{k+1} in period $[k,k+1]$ is determined, so the insurance premium that he spend is $z_1^{k+1}(\cdot)\times H^{k+1}$.

Definition 2.2 *Define* $Z(k+1)\equiv(z_0^{k+1}(\cdot),z_1^{k+1}(\cdot),z_1^{k+1}(\cdot)\times H^{k+1})$ *is a strategy of investment and insurance in* $[k,k+1]$.

Obviously, the strategy- process $\{Z(\cdot)\}$ is Predictable Process of $\{F_k\}_{k\geq 0}$ − from assumptions 2.6, we can see it is self-financing.

Definition 2.3 *The strategy-process* $\{Z(\cdot)\}$ *is self-financing. If* $\forall k\in\{1,2,\cdots,n-1\}$, *we can reach a conclusion that*

$$z_0^k(\cdot)\times(1+r_{i_k i_{k+1}})+z_1^k(\cdot)\times[1+\tilde{r}_{1i_k i_{k+1}}-\tilde{r}_{2i_k i_{k+1}}]+z_1^k(\cdot)I^{k+1}(\tilde{r}_{2i_k i_{k+1}},c^{k+1})$$
$$=z_0^{k+1}(\cdot)+z_1^{k+1}(\cdot)+z_1^{k+1}(\cdot)H^{k+1}(\tilde{r}_{2i_k\bullet},c^{k+1})$$

3 Dynamic planning model of the optimal strategy of investment and insurance

Suppose that the utility function of the investor is NMU, which is a concave function, and it is satisfied:

$$U'(x)\geq 0,\quad U''(x)\leq 0,\quad x\in[0,+\infty) \tag{3.1}$$

To the inter-temporal strategy, we restrict that in period $[k,k+1]$ the utility function is

$$U_{k+1}((E(V),Var(V)) = \beta^k U(E(V),Var(V)) \tag{3.2}$$

(Obviously, $U_1 = U$,), and $0 \le \beta \le 1$ is discount-factor of utility function [8].

We use $V(n,0,i_0,v_0,Z(\cdot))$ to represent assets from time 0 to time *n*. The original state is i_0, the original assets is v_0, the self-financing strategy is *Z*. And we also use $\hat{V}(n,0,i_0,v_0,Z(\cdot))$ to represent the excess return in this period. From Definition 2.3, we can reach a conclusion that

$$V(n,0,i_0,v_0,Z(\cdot)) = v_0 + \sum_{k=1}^{n} z_0^k r_{i_{k-1}i_k} + \sum_{k=1}^{n} z_1^k (\tilde{r}_{1i_{k-1}i_k} - \tilde{r}_{2i_{k-1}i_k} + I^k - H^k) \tag{3.3}$$

$$\hat{V}(n,0,i_0,v_0,Z(\cdot)) = \sum_{k=1}^{n} z_0^k r_{i_{k-1}i_k} + \sum_{k=1}^{n} z_1^k (\tilde{r}_{1i_{k-1}i_k} - \tilde{r}_{2i_{k-1}i_k} + I^k - H^k) \tag{3.4}$$

and $I^k = \max(0, \tilde{r}_{2i_{k-1}i_k} - c^k)$, $H^k = (1+\lambda)EI^k + b$ are defined as before.

In this passage, the target is maximize the utility of expected excess return [9], i.e. for each v_0 and $i_0 \in \Omega$, we should search the self-financing strategy and insurance strategy progress $\overline{Z}(\cdot) =$ $\overline{z}_0(\cdot), \overline{z}_1(\cdot), \overline{z}_1(\cdot)\overline{H}(\cdot))$ in $[0,n]$ to maximize the utility sum of expected excess return. That means

$$\begin{aligned}
&\sum_{k=1}^{n} U_k (E(\overline{z}_0^k r_{i_{k-1}i_k} + \overline{z}_1^k (\tilde{r}_{1i_{k-1}i_k} - \tilde{r}_{2i_{k-1}i_k} + \overline{I}^k - \overline{H}^k))) \\
&\qquad = \max \sum_{k=1}^{n} U_k (E(z_0^k r_{i_{k-1}i_k} + z_1^k (\tilde{r}_{1i_{k-1}i_k} - \tilde{r}_{2i_{k-1}i_k} + I^k - H^k)))
\end{aligned} \tag{3.5}$$

In fact, the behavior of investor happens in every period $[k, k+1]$, and they decide the optimal investment amount of risky-asset and the optimal deductible factor according to the disposable assets v_k and the market state i_k., and then the optimal investment amount of risk-free asset $\overline{z}_0^{k+1}$ is determined too .

It is a problem of dynamic programming. In order to solve problem (3.5), we name the concept in front of the passage by the terminology of dynamic programming [10].

3.1 State variables

$v_k, r_{i_k \bullet}, \tilde{r}_{ji_k \bullet}$ are state variables.

3.2 Decision variables

$z_1^{k+1}, c^{k+1} \quad (j=1,2),\ \ k \in \{0,1,\cdots,n-1\}$ are decision variables.

3.3 The relationship between the state variables and the decision variables, and the constraint conditions

It is clear that the equation belong must satisfy this requirement:

$$\begin{aligned} v_k &= z_0^{k+1}(\cdot) + z_1^{k+1}(\cdot) + z_1^{k+1}(\cdot) H^{k+1} \\ &= z_0^{k+1}(\cdot) + z_1^{k+1}(\cdot) + z_1^{k+1}(\cdot)\left\{(1+\lambda)\left[E\max(0, \tilde{r}_{2i_k \bullet} - c^{k+1}) + b\right]\right\} \end{aligned} \tag{3.6}$$

and $c^{k+1} > 0$, the distribution of $r_{i_k \bullet}$, $r_{i_k \bullet}$ are determined by matrix A, B, C_j .

3.4 State transition equation

$$\begin{aligned} v_{k+1} &\equiv V(k+1, k, i_k, v_k, Z(\cdot)) \\ &= z_0^{k+1}(\cdot)(1+r_{i_k i_{k+1}}) + z_1^{k+1}(1+\tilde{r}_{1i_k i_{k+1}} - \tilde{r}_{2i_k i_{k+1}}) + z_1^{k+1}\max(o, \tilde{r}_{2i_k i_{k+1}} - c^{k+1}) \end{aligned} \tag{3.7}$$

3.5 Index function and the optimal value function

Function

$$f_k(v_k, r_{i_k \bullet}, \tilde{r}_{ji_k \bullet}) \equiv \sum_{t=k+1}^{n} U_t(E(z_0^t r_{i_{t-1}i_t} + z_1^t(\tilde{r}_{1i_{t-1}i_t} - \tilde{r}_{2i_{t-1}i_t} + \overline{I}^t - \overline{H}^t)))$$

is the index function of the problem, it means the investor begin to invest using initial assets v_k and on the premise that market status is i_k at time k. The total stages of investing is $n-k$, the utility sum of expected excess return at time n is $f_k(v_k, r_{i_k\bullet}, \tilde{r}_{ji_k\bullet})$,

and make

$$F_k(v_k, r_{i_k\bullet}, \tilde{r}_{ji_k\bullet}) \equiv \max_{\{(z(k+1),\cdots,z(n))\}} f_k(v_k, r_{i_k\bullet}, \tilde{r}_{ji_k\bullet})$$

to be the optimal value function of the dynamic programming problem.

3.6 The basic equation of dynamic programming problems

According to the principle of optimality of problem of dynamic programming, there are two equations:

$\forall\ k \in \{0,1,\cdots,n-1\}$, we have

$$F_k(v_k, r_{i_k\bullet}, \tilde{r}_{ji_k\bullet}) \equiv$$
$$\equiv \max_{\{z(k+1)\}} \{F_{k+1}(v_{k+1}, r_{i_k\bullet}, \tilde{r}_{ji_k\bullet}) + U_{k+1}(E(z_0^{k+1} r_{i_k i_{k+1}} + z_1^{k+1}(\tilde{r}_{1i_k i_{k+1}} - \tilde{r}_{2i_k i_{k+1}} + I^{k+1} - H^{k+1})))\} \quad (3.8)$$

$$F_n(v_n, r_{i_n}, \tilde{r}_{ji_n}) \equiv 0 \quad (3.9)$$

4 The solving of the problem of dynamic programming

Lemma 4.1 If the target function of the problem of dynamic programming is a concave function to decision variables, then the first order condition of dynamic programming is necessary and sufficient condition [8].

Now, we use the reverse-recursion method [10, 11] to solve problem (3.5).

First, in equation (3.8), we suppose $k = n-1$, then

$$F_{n-1}(v_{n-1}, r_{i_{n-1}\bullet}, \tilde{r}_{ji_{n-1}\bullet}) = \max\{0 + U_n(E(z_0^n r_{i_{n-1}\bullet} + z_1^n(\tilde{r}_{1i_{n-1}\bullet} - \tilde{r}_{2i_{n-1}\bullet} + I^n - H^n)))\}$$
$$= \max \beta^{n-1} U\ (E(z_0^n r_{i_{n-1}\bullet} + z_1^n(\tilde{r}_{1i_{n-1}\bullet} - \tilde{r}_{2i_{n-1}\bullet} + I^n - H^n))) \quad (4.1)$$

From hypothesis (2.3.2) and (2.3.3), we can know the distribution of $r_{i_{n-1}\bullet}$, $\tilde{r}_{ji_{n-1}\bullet}$, $(j=1,2)$. To (4.1), we derivative respectively of z_1^n and c^n, we should have

$$\beta^{n-1}U'_{z_1^n}\,(E(z_0^n r_{i_{n-1}\bullet} + z_1^n(\tilde{r}_{1i_{n-1}\bullet} - \tilde{r}_{2i_{n-1}\bullet} + I^n - H^n))) = 0 \tag{4.2}$$

$$\beta^{n-1}U'_{c^n}\,(E(z_0^n r_{i_{n-1}\bullet} + z_1^n(\tilde{r}_{1i_{n-1}\bullet} - \tilde{r}_{2i_{n-1}\bullet} + I^n - H^n))) = 0 \tag{4.3}$$

By Lemma 4.1 and the concavity of the utility function, we can solve (4.2) and (4.3), then the optimal investment amount of risky asset in period $[n-1,n]$ is

$$\bar{z}_1^n = \bar{z}_1^n(v_{n-1}, r_{i_{n-1}\bullet}, \tilde{r}_{ji_{n-1}\bullet})$$

and the optimal deductible factor is

$$\bar{c}^n = \bar{c}^n(v_{n-1}, r_{i_{n-1}\bullet}, \tilde{r}_{ji_{n-1}\bullet}),$$

and we can also get $F_{n-1}(v_{n-1}, r_{i_{n-1}\bullet}, \tilde{r}_{ji_{n-1}\bullet})$.

In equation (3.8), if $k = n-2$, using state transition equation (3.7), then

$$\begin{aligned} & F_{n-2}(v_{n-2}, r_{i_{n-2}\bullet}, \tilde{r}_{ji_{n-2}\bullet}) \\ & = \max\left\{F_{n-1}(v_{n-1}, r_{i_{n-1}\bullet}, \tilde{r}_{ji\,n-1\bullet}) + U_{n-1}(E(z_0^{n-1} r_{i_{n-2}\bullet} + z_1^{n-1}(\tilde{r}_{1i_{n-2}\bullet} - \tilde{r}_{2i_{n-2}\bullet} + I^{n-1} - H^{n-1})))\right\} \\ & = \max\{F_{n-1}(v_{n-2}, r_{i_{n-2}\bullet}, \tilde{r}_{ji_{n-2}\bullet}) + U_{n-1}(E(z_0^{n-1} r_{i_{n-2}\bullet} + z_1^{n-1}(\tilde{r}_{1i\,n-2\bullet} - \tilde{r}_{2i_{n-2}\bullet} + I^{n-1} - H^{n-1})))\} \end{aligned} \tag{4.4}$$

From assumptions (2.3.2) and (2.3.3), we can get the distribution function of $r_{i_{n-2}\bullet}$, $\tilde{r}_{ji_{n-2}\bullet}$, $(j=1,2)$, then by the same method, we derivative respectively of z_1^{n-1} and c^{n-1} in (4.4), we should have the optimal investment amount of risk asset in period $[n-2,n-1]$ is

$$\bar{z}_1^{n-1} = \bar{z}_1^{n-1}(v_{n-2}, r_{i_{n-2}\bullet}, \tilde{r}_{ji_{n-2}\bullet}),$$

and the optimal deductible factor is

$$\bar{c}^n = \bar{c}^n(v_{n-2}, r_{i_{n-2}\bullet}, \tilde{r}_{ji_{n-2}\bullet}),$$

and we can also get $F_{n-2}(v_{n-2}, r_{i_{n-2}\bullet}, \tilde{r}_{ji_{n-2}\bullet})$,…, and so on, we can obtain $F_k(v_k, r_{i_k\bullet}, \tilde{r}_{ji_k\bullet})$ and

$$\overline{Z}(k+1, v_k, r_{i_k\bullet}, \tilde{r}_{ji_k\bullet}) \equiv (\overline{z}_0^{k+1}, \overline{z}_1^{k+1}, \overline{z}_1^{k+1}\overline{H}^{k+1}), \quad k \in \{0,1,\cdots,n-1\}, \quad v_k \in F_k \tag{4.5}$$

Then we get the theorem:

Theorem 4.1 The optimal value function of problem (3.5) can be solved and the optimal strategy of investment and insurance is solved by (4.5).

5 Conclusion

Most of the solutions to dynamic programming is complicated. We usually use computer programming to get the optimal solution in practice. In this paper, we discuss a multi-period investment-insurance problem. Suppose the market transition matrix is certain, we obtain a multi-period programming model. The target of decision is maximizing excess return. We can obtain an overall optimal investment-insurance policies of the multi-period investment-insurance. We illustrate the process of finding the optimal solution. If we using computer programming, we can get the answer.

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