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Perturbation Least-Squares Chebyshev method for solving fractional order integro-differential equations

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Abstract

A numerical scheme based on the perturbation Least-Squares Chebyshev procedure for solving fractional order integro - differential equations is presented in this paper. An approximate solution taken together with the Least - Squares method are utilized to reduce the fractional integro-differential equations to system of algebraic equations, which are solved for the unknown constants associated with the approximate solution. Three numerical examples are considered to demonstrate the accuracy and effectiveness of the method. The results obtained are in good agreement with existing results in literature to a reasonable extent and converge to the exact solutions of the chosen problems when such existed in closed form .

Keywords: Perturbation; curve fitting; fractional integro-differential and Least-squares

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1 Introduction

Fractional Calculus entails Fractional Differential equations and Fractional Integro-Differential Equations. Fractional Integro-Differential Equations FIDEs arose in many phenomena in Applied Sciences and Engineering. It is a known fact that many occurrences found in Physics, Chemistry, Biology, Mathematics, Acoustics, Biotechnology and so on are more accurately modeled with FIDEs. Still, many of these models do not have analytic or exact solutions. So researchers have lately proposed many numerical and analytic approaches for proffering solutions for this class of problems. [1], used the Adomian Decomposition Method to solve fractional Integro-differntial equations. Homotopy Perturbation and Homotopy Analysis methods were applied to solve initial value problems of fractional order by [2]. [3] employed Variational Iteration Method and Homotopy perturbation Method for finding the numerical solutions of fourth-order fractional integro- differential equations. B-spline Wavelets was applied to solve FIDEs by [4]. [5] gave an application of Chebyshev wavelets Method for the solutions of class of nonlinear fractional integro-differential equations in large interval. FIDEs was also solved with the Laplace Decomposition Method by [6]. [7] applied Least squares method and Shifted Chebyshev Polynomial for the numerical solutions of Fractional Integro-Differential Equations.

In this paper, we are presenting a perturbation Least squares Chebyshev method for the solution of Fractional Integro- differential equations of the type:

$$D^{\alpha}u(t) = f(t) + \int_0^1 k(t,s)u(s)ds, \qquad 0 \leq t \leq 1 \tag{1}$$

with initial condition $y(0) = y_0$

where $k(t,s)$, $f(t)$ are given smooth functions and u(t) is the unknown function to be determined.

2 Definitions of relevant terms of calculus of fractional order

In this section, we give brief definitions and properties of fractional derivatives relevant to the presentation in the next sections.

Definition 2.1. *A real function $f(t)$, $t \in N$ is said to be in space C_μ ,$\mu \in R$ if there exist a real number $\rho > \mu$, such that*

$$f(t) = t^\rho f_1(x) \tag{2}$$

where $f_1(t) = c(0, \infty)$. If $\beta \leq \mu$, then $c_\mu \in c_\beta$.

Definition 2.2. *The Riemann-Liouville integral operator of order $\alpha > 0$ of a function, $f \in c_u$,$u \geq -1$ is defined as [8]:*

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \alpha > 0, t > 0 \tag{3}$$

Listed here are some of the properties of Riemann-Liouville fraction integration. For $f \in c_u$, $u \geq -1$, $a, b \geq 0$,$c > -1$:

$$J^a J^b f(t) = J^{a+b} f(t) \tag{4}$$

$$J^a J^b f(t) = J^b J^a f(t) \tag{5}$$

$$J^a t^c = \frac{\Gamma(c+1)}{\Gamma(a+c+1)} t^{a+c} \tag{6}$$

Definition 2.3. *The fractional derivative of f(t) in the Caputo sense is defined as [9].*

$$D_*^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau) d\tau \tag{7}$$

for $m - 1 < \alpha \leq m$, $m \in N$,$t > 0$; where $\alpha \geq 0$ is the order of the derivative. Stated here, are basic properties of Caputo derivatives; if k, k_1, k_2 are constants, then;

$$D_*^\alpha f(k) = 0 \tag{8}$$

$$D_*^\alpha f(t^n) = 0, if, n \in N_0, n < \lceil \alpha \rfloor \tag{9}$$

$$D_*^\alpha f(t^n) = \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} if, n \in N_0, n \geq \lceil \alpha \rfloor \tag{10}$$

$\lceil \alpha \rfloor$ is a function called the smallest integer greater than or equal to α and α an element of N_0 is the integer order derivative. $N_0 = (0, 1, 2, \cdots)$

$$D_*^\alpha J^\alpha f(t) = f(t) \tag{11}$$

$$J^{\alpha} D_*^{\alpha} f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{t^k}{k!} \quad ,t > 0 \tag{12}$$

$$D_*^{\alpha}(k_1 f(t) + k_2 f(t)) = k_1 D_*^{\alpha} f(t) + k_2 D_*^{\alpha} f(t) \tag{13}$$

3 Basic properties of Chebyshev Polynomials

The Chebyshev polynomials of the first kind and of degree k are defined on the interval $[-1, 1]$ [10] as;

$$T_k(t) = cos^{-1}(kcos(t)) \tag{14}$$

$$T_0(t) = 1, \quad T_1(t) = t, \quad T_2(t) = 2t^2 - 1. \tag{15}$$

and the recurrence relation is given as

$$T_{k+1}(t) = 2tT_k(t) - T_{k-1}(t), \qquad k = 2, 3 \cdots \tag{16}$$

Snyder(1966) also stated shifted Chebyshev polynomials of degree n on the closed interval [0,1] as;

$$T_n^*(t) = T_n\,(2t - 1) \tag{17}$$

The recurrence formula on the closed form interval $[0, 1]$ is ;

$$T_{n+1}^*(t) = 2(2t - 1)T_n^*(t) - T_{n-1}^*(t); \qquad n \geq 1 \tag{18}$$

Also, few terms are listed thus:

$$T_0^*(t) = 1, T_1^*(t) = 2t - 1, T_2^*(t) = 8t^2 - 8t + 1 \tag{19}$$

3.1 Perturbation Least - Squares Chebyshev Polynomials Method

The Least - Squares curve fitting is a procedure for fitting a unique curve through giving set of data points. The method of curve fitting is discussed in detail by [11]. The Least Squares - Chebyshev Polynomials method as applied to fractional integro differential equations is hereby presented in this subsection. In this approach, we employ an approximate solution of the form:

$$u_k(t) = \sum_{k=0}^{m} c_k T_k^*(t) + \sum_{k=0}^{n} \tau_k T_k^*(t) \tag{20}$$

where $u_k(t)$ denotes the approximate solution of the given problem in equation (1), c_k and τ_k are unknown constants defined for $k = 0, 1, 2, ...m$ and $k = 0, 1, 2, ...n$ also $T_k(t)$ is the shifted Chebyshev Polynomials defined in equation (14) with the recurrence relation given in equation (18). Now, we substituted equation (20) into (1) to get;

$$\begin{aligned} D^\alpha \left(\sum_{k=0}^{m} c_k T_k^*(t) + \sum_{k=0}^{n} \tau_k T_k^*(t) \right) &= f(t) \\ + \int_0^1 k(t,s) \left[\sum_{k=0}^{m} c_k T_k^*(t) + \sum_{k=0}^{n} \tau_k T_k^*(t) \right] dt \end{aligned} \tag{21}$$

We computed the error in equation (21) above and is denoted by $E(t,\, c_0,\, c_1,\, \cdots,\, c_k)$ and given as:

$$E(t, c_0, c_1, \cdots, c_k) = D^\alpha \left(\sum_{k=0}^{m} c_k T_k^*(t) + \sum_{k=0}^{n} \tau_k T_k^*(t) \right) - f(t)$$

$$-k(t,s) \left[\sum_{k=0}^{m} c_k T_k^*(t) + \sum_{k=0}^{n} \tau_k T_k^*(t) \right] dt \tag{22}$$

Simplifying equation (22) gives;

$$\begin{aligned} E(t, c_0, c_1, \cdots, c_k) = &\sum_{k=0}^{m} c_k D^\alpha \left(T_k^*(t) \right) + \sum_{k=0}^{n} D^\alpha \tau_k T_k^*(t) - f(t) \\ &- \int_0^1 k(t,s) \left[\sum_{k=0}^{m} c_k T_k^*(t) + \sum_{k=0}^{n} \tau_k T_k^*(t) \right] dt \end{aligned} \tag{23}$$

Let $w(s)$ be the positive weight function defined in $(0, 1)$ and because it is defined in this interval, $w(s)$=1. Consequently, we have:

$$S(c_0, c_1, \cdots, c_k) = \int_0^1 E\left[(s, c_0, c_1, \cdots, c_k)\right]^2 w(s) ds \tag{24}$$

$$S(c_0, c_1, \cdots, c_k) = \int_0^1 \left\{ \sum_{k=0}^{m} c_k D^\alpha T_k^*(t) + \sum_{k=0}^{n} D^\alpha \tau_k T_k^*(t) - f(t) \right. \\ \left. - \int_0^1 k(t,s) \left[\sum_{k=0}^{m} c_k T_k^*(t) + \sum_{k=0}^{n} \tau_k T_k^*(t) \right] dt \right\}^2 ds \tag{25}$$

The values of $c_0, c_1, \cdots, c_k$ give the coefficients of the approximate solution of equation (1). To get these values using least squares method, we need to find the minimum value of $S(c_0, c_1, \cdots, c_k)$. This is done by finding partial derivatives of $S(c_0, c_1, \cdots, c_k)$ and equating the results to zero. Consequently the results in $(m+1)$ system of equations are expressed in matrix form as:

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1m} & \tau_{11} & \tau_{12} & \cdots & \tau_{1m} \\ A_{21} & A_{22} & A_{23} & \cdots & A_{2m} & \tau_{21} & \tau_{22} & \cdots & \tau_{2m} \\ \vdots & \vdots & \vdots & & & & & & \\ \vdots & \vdots & \vdots & & & & & & \\ \vdots & \vdots & \vdots & & & & & & \\ \vdots & \vdots & \vdots & & & & & & \\ \vdots & \vdots & \vdots & & & & & & \\ A_{m1} & A_{m2} & A_{m3} & \cdots & A_{mn} & \tau_{m1} & \tau_{m2} & \cdots & \tau_{mn} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_m \\ \tau_1 \\ \tau_2 \\ \vdots \\ \tau_n \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_m \\ \vdots \\ \vdots \\ \vdots \\ B_{m+1} \end{pmatrix}$$

The matrix is then solved with the Gaussian elimination method to get the unknown constants.

4 Numerical Experiments

The method discussed above is hereby demonstrated with the following numerical examples. The examples are Fractional integro-differential equations of fractional order.

Example 1. Consider the fractional order Integro- Differential equation.

$$D^{\frac{1}{2}} u(t) = \frac{\left(\frac{8}{3} t^{\frac{3}{2}} - 2t^{\frac{1}{2}}\right)}{\sqrt{\pi}} + \frac{t}{12} + \int_0^1 tsu(s)ds, \ 0 \le t, \ s \le 1 \tag{26}$$

subject to $u(0) = 0$. This problem has an exact solution of $t^2 - t$ [7]. We take $m = 5, \ n = 1$ and use the Perturbation Least-Squares Chebyshev Method in equation (20). Also, we make use of six terms of the shifted Chebyshev Polynomials for $k = 5$.

$$u_5(t) = \sum_{k=0}^{5} c_k T_k^*(t) + \sum_{k=0}^{1} \tau_k T_k^*(t) \tag{27}$$

$$\begin{aligned} u_5(t) =& c_0T_0^*(t) + c_1T_1^*(t) + c_2T_2^*(t) + c_3T_3^*(t) + c_4T_4^*(t) \\ &+ c_5T_5^*(t) + \tau_0T_0^* + \tau_1T_1^*(t) \end{aligned} \tag{28}$$

Substituting (28) into (26) and simplifying further we got the following equations:

$$\begin{aligned} 1.166666667a_0 + 0.4951966673a_1 - 0.6241542824a_2 \\ +0.5583454737a_3 - 0.4160499541a_4 + 0.4322016693a_5 \\ +0.6063077784\tau_1 = -0.2238526193 \end{aligned} \tag{29}$$

$$\begin{aligned} 0.8249855834a_1 + 0.4951966673a_0 + 1.159046793a_2 \\ -.8272909647a_3 + .8444844662a_4 - .7634975441a_5 \\ +0.9407015114\tau_1 = 0.08298126588 \end{aligned} \tag{30}$$

$$\begin{aligned} 6.476952936a_2 - .6241542824a_0 + 1.159046793a_1 \\ -0.2395141584a_3 + 0.4786277972a_4 - 0.5156700155a_5 \\ +1.286917482\tau_1 = 0.8876384042 \end{aligned} \tag{31}$$

$$\begin{aligned} 9.994458600a_3 + 0.9061289423a_5 - 0.8729937326\tau_1 \\ +.5583454737a_0 - 0.8272909647a_1 - 0.2395141584a_2 \\ -0.8030659854a_4 = -0.09973243609 \end{aligned} \tag{32}$$

$$\begin{aligned} 13.63643757a_4 - 1.195029765a_5 + 0.8967522289\tau_1 \\ -0.4160499541a_0 + 0.8444844662a_1 + .4786277972a_2 \\ -.8030659854a_3 = 0.1118348224 \end{aligned} \tag{33}$$

$$\begin{aligned} 17.05283883a_5 + .9061289423a_3 - 1.195029765a_4 \\ -0.80644432488\tau_1 + 0.4322016693a_0 - 0.7634975441a_1 \\ -0.5156700155a_2 - = 0.1184832081 \end{aligned} \tag{34}$$

$$\begin{aligned} 1.074935954\tau_1 - 0.8729937326a_3 + 0.8967522289a_4 \\ -0.8064432488a_5 + 0.6063077784a_0 + 0.9407015114a_1 \\ +1.286917482a_2 = 0.08507621290 \end{aligned} \tag{35}$$

These equations were solved for unknowns to get following values;

$$\begin{aligned} a_0 &= -\,0.1247251552,\ a_1 = 0.004448370365,\ a_2 = 0.1250465289, \\ a_3 &= -\,0.4787098650 \times 10^{-5},\ a_4 = 0.1163226749 \times 10^{-5}, \\ a_5 &= -\,3.182034463 \times 10^{-7},\ \tau_1 = -0.4108697853 \times 10^{-2} \end{aligned} \tag{36}$$

On substitution into the approximate solution we have;

$$\begin{aligned} u_5(t) = -0.004120728171 - 0.9916147915t + 1.000915409t^2 \\ -0.0008073610644t^3 + 0.0005561934352t^4 - 0.0001629201645t^5 \end{aligned} \tag{37}$$

Example 2. Consider the fractional order Integro- Differential equation.

$$D^{\frac{5}{6}}u(t) = f(t) + \int_0^1 te^x u(x)dx,\ 0 \le t,\ x \le 1 \tag{38}$$

subject to $u(0) = 0$, where

$$f(t) = -\frac{3}{91}\frac{t^{\frac{1}{6}}\Gamma(\frac{5}{6})(-91 + 216t^2)}{\pi} + (5 - 2e)t$$

with the exact solution $u(t) = t - t^3$ (Mohammed, 2014).
Similarly, substituting (37) into(26) and simplifying further we got the following approximate solution:

$$\begin{aligned} u_5(t) = -0.00006848728738 + 1.000106452t - 0.000233146t^2 \\ -0.9994447054t^3 - 0.0007128815566t^4 + 0.0003168631342t^5 \end{aligned} \tag{39}$$

Example 3. Consider the fractional order Integro- Differential equation.

$$D^{\frac{5}{3}}u(t) = \frac{3\sqrt{3}\Gamma(\frac{2}{3})t^{\frac{1}{3}}}{\pi} - \frac{1}{5}t^2 - \frac{1}{4}t + \int_0^1 (tx + t^2x^2)u(x)dx,\ 0 \le t,\ x \le 1 \tag{40}$$

with the exact solution $u(t) = t^2$ [7].
Also, substituting (39) into(26) and simplifying further we got the following approximate solution:

$$\begin{aligned} u_5(t) = -0.000004105902790 + 0.000557383t + 1.000011307t^2 \\ +0.00005868537207t^3 + 0.000000605187348t^4 + 0.000002099039719t^5 \end{aligned} \tag{41}$$

The numerical results for these three problems are tabulated in the tables below.

Table 1: Table of Results for Example 1

t	Exact	Approx	Error
0.00	0.000000000000	-0.004120728171	4.1207e-03
0.10	-0.090000000000	-0.093273806580	3.2738e-03
0.20	-0.160000000000	-0.162412691200	2.4127e-03
0.30	-0.210000000000	-0.211540468200	1.5405e-03
0.40	-0.240000000000	-0.240659280200	6.5928e-04
0.50	-0.250000000000	-0.249770521100	2.2948e-04
0.60	-0.240000000000	-0.238875031900	1.1250e-03
0.70	-0.210000000000	-0.207973296600	2.0267e-03
0.80	-0.160000000000	-0.157065637400	2.9344e-03
0.90	-0.090000000000	-0.086152409740	3.8476e-03
1.00	0.000000000000	0.004765801506	4.7658e-03

Table 2: Table of Results for Example 2

t	Exact	Approx	Error
0.00	0.000000000000	-0.000068487287	6.8487e-05
0.10	0.099000000000	0.098940313620	5.9686e-05
0.20	0.192000000000	0.191946880500	5.3120e-05
0.30	0.273000000000	0.272952453900	4.7546e-05
0.40	0.336000000000	0.335957323900	4.2676e-05
0.50	0.375000000000	0.374961210900	3.8789e-05
0.60	0.384000000000	0.383963644800	3.6355e-05
0.70	0.357000000000	0.356964345900	3.5654e-05
0.80	0.288000000000	0.287963605100	3.6395e-05
0.90	0.171000000000	0.170962663900	3.7336e-05
1.00	0.000000000000	-0.000035904822	3.5905e-05

Table 3: Table of Results for Example 3

t	Exact	Approx	Error
0.00	0.000000000000	-0.000004105903	4.1059e-06
0.10	0.010000000000	0.010051804240	5.1804e-05
0.20	0.040000000000	0.040108294100	1.0829e-04
0.30	0.090000000000	0.090165721140	1.6572e-04
0.40	0.160000000000	0.160224449300	2.2445e-04
0.50	0.250000000000	0.250284851500	2.8485e-04
0.60	0.360000000000	0.360347312000	3.4731e-04
0.70	0.490000000000	0.490412229800	4.1223e-04
0.80	0.640000000000	0.640480019600	4.8002e-04
0.90	0.810000000000	0.810551115700	5.5112e-04
1.00	1.000000000000	1.000625973000	6.2597e-04

5 Conclusion

Perturbation Least - Squares Chebyshev method was successfully used to solve fractional order integro differential equations . The method used an approximate solution that reduced the FIDEs into a system of equations. The procedure provided realistic solutions which converged to exact solutions of the problems. This showed that the method agreed with the exact solutions when such exist to a reasonable decimals and hence confirmed that the method could handle the class of problems discussed effectively.

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