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Bounded solutions to the differential equation of planetary motion under general relativity

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Abstract

In this brief note, the nonlinear differential equation describing planetary motion under general relativity is studied using polar coordinates and the phase space (r, r'). Conditions are then given when the relativistic equation yield bounded solutions by looking at the equilibrium points of first integrals of the equation.

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1 Introduction

In this note, a straightforward account will be given of the well-known nonlinear differential equation for planetary motion under general relativity. For the derivation of the relativistic model see [1, pp. 270-276] while for a thorough discussion of the Newtonian or classical model of planetary motion see [2, pp. 471-

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496]. Planetary motion is an integral part of celestial mechanics. For an excellent introduction to this subject see [3].

2 Main results

The relativistic equation is given by

$$u''(\theta) + u(\theta) - c_1 u(\theta)^2 = c_2 \tag{1.1}$$

where $= {}^{1}/_{r}$, r being the radius from the given object to a foci and c_1 and c_2 are positive constants.

First, multiply (1.1) by $2u'$ and integrate for 0 to θ obtaining

$$u'(\theta)^2 + u(\theta)^2 - \frac{2c_1 u(\theta)^3}{3} = 2c_2 u(\theta) - 2c_2 u(0) + u'(0)^2 + u(0)^2 - 2c_1 u(0)^3/3 \tag{1.2}$$

Next, using the fact that $u = 1/r$ and $u' = -r/r^2$ and then multiplying equation (1.2) by r^4 transforms equation (1.2) into

$$r'(\theta)^2 + r(\theta)^2 - \frac{2c_1 r(\theta)}{3} = 2c_2 r(\theta)^3 - 2c_2 r(\theta)^4 u(0) + kr(\theta)^4 \tag{1.3}$$

where $k = u'(0)^2 + u(0)^2 - 2c_1 u(0)^3/3$.

If $k - 2c_2 u(0) < 0$, then should r → ∞ the LHS of (1.3) approaches ∞ while the RHS approaches -∞ which is impossible. In other words, the solutions must remain bounded as t→∞ given these conditions. Should $k - 2c_2 u(0) \geq 0$, then the solutions may be unbounded.

We could study boundedness in another way by looking at the phase space(r, r'). We start by finding the equilibrium points of equation (1.3), i.e., the points $(0, r_i)$ where $r_i \geq 0$. When $r'(\theta) = 0$, then from (1.3) after rearranging terms we have

$$2u(0)c_2r^4 - kr^4 - 2c_2r^3 + r^2 - 2c_1r/3 = 0. \tag{1.4}$$

Equation (1.4) now can be rewritten as

$$f(r) = r[(2u(0)c_2 - k)r^3 - 2c_2r^2 + r - 2c_1/3] = 0 \tag{1.5}$$

In other words, equation (1.3) may be transformed into

$$r'(\theta)^2 + f(r(\theta)) = 0 \tag{1.6}$$

Next, we need to discuss the zeros of quartic polynomial (1.6) in the phase space(r, r'). They are $r_0 = 0$ and the zeros of the cubic polynomial
$g(r) = (2u(0)c_2 - k)r^3 - 2c_2r^2 + r - 2c_1/3$.

As long as $2c_2u(0) - k > 0$, we can invoke Descartes rule of signs (see [4, p. 211]) to conclude that $g(r)$ has at least one positive real root r_1 and possibly two more positive real roots r_2 and r_3 (with the possibility that r_2=r_3 or r_1=r_2) since there are three sign changes occurring in the cubic polynomial $g(r)$. Moreover, since the signs alternate in (1.5), $f(r)$ has no negative roots. Should there only be one positive real root, then the other two roots must be imaginary. Furthermore, equation (1.6) implies that the bounded solutions occur only when $f(r) \leq 0$ which occurs when the cubic polynomial $g(r) \leq 0$. Should there be only one positive root r_1, then bounded non-zero solutions must exist over the interval (0,r_1). When r_2 and r_3 are two additional distinct zeroes of $g(r)$ then bounded solutions also exist over the interval [r_2,r_3]. Since $g(r)$ is positive over the interval(r_1,r_2), no bounded solutions can exist there. When r_2=$r_3$we have a local minimum at the equilibrium point (r_2,0) so non-constant bounded solutions exist only on (0,r_1) since $f(r) > 0$ for r >r_1. On the other hand, should r_1=r_2, then the non-constant bounded solutions exist on the entire interval (0,r_3). In this case, we have a local maximum for r = r_1 so $f(r) \leq 0$ on the entire interval [0,r_3] and $f(r) > 0$ for $r > r_3$ which again contains no bounded solution by our previous remarks.

3 Remark

In the case of a double root, one can easily calculate its value since it is a root of both $g(r)$ and $f(r)$ and a critical point as well. Consequently, we have $g'(r) = 0$ there. Therefore, we have

$$g'(r) = 3(2u(0)c_2 - k)r^2 - 4c_2 r + 1 \ . \tag{1.7}$$

Solving for r yields

$$r = \frac{4c_2 \pm \left((4c_2)^2 - 12(2u(0)c_2 - k)\right)^{1/2}}{12u(0)c_2 - 6k} \tag{1.8}$$

The correct root can be chosen by inspection. The sign of the second derivative of (1.7), i.e.

$$g''(r) = (12u(0))c_2 - 6k)r - 4c_2 \tag{1.9}$$

determines whether r is a local maximum or minimum of g(r).

4 Conclusion

By using standard methods from differential equations the above analysis clearly gives a straightforward and qualitative analysis of planetary motion under general relativity which plays an essential role in celestial mechanics.

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