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On harmonic analysis of spherical convolutions on semisimple Lie groups

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Abstract

This paper contains a non-trivial generalization of the Harish-Chandra transforms on a connected semisimple Lie group G, with finite center, into what we term spherical convolutions. Among other results we show that its integral over the collection of bounded spherical functions at the identity element $e \in G$ is a weighted Fourier transforms of the Abel transform at 0. Being a function on G, the restriction of this integral of its spherical Fourier transforms to the positive-definite spherical functions is then shown to be (the non-zero constant multiple of) a positive-definite distribution on G, which is tempered and invariant on $G = SL(2, \mathbb{R})$. These results suggest the consideration of a calculus on the Schwartz algebras of spherical functions. The Plancherel measure of the spherical convolutions is also explicitly computed.

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1 Introduction

Let G be a connected semisimple Lie group with finite center, and denote the Harish-Chandra-type Schwartz spaces of functions on G by $\mathcal{C}^p(G)$, $0 < p \leq 2$. We know that $\mathcal{C}^p(G) \subset L^p(G)$ for every such p, and if K is a maximal compact subgroup of G such that $\mathcal{C}^p(G//K)$ represents the subspace of $\mathcal{C}^p(G)$ consisting of the $K-$bi-invariant functions, Trombi and Varadarajan [11] have shown that the spherical Fourier transform $f \mapsto \widehat{f}$ is a linear topological isomorphism of $\mathcal{C}^p(G//K)$ onto the spaces $\bar{\mathcal{Z}}(\mathfrak{F}^\epsilon)$, $\epsilon = (2/p) - 1$, consisting of rapidly decreasing functions on certain sets $\mathfrak{F}^\epsilon$ of elementary spherical functions. It then follows that every positive-definite distribution on $C_c^\infty(G)$ can be uniquely extended to $\mathcal{C}^1(G//K)$.

Using these, and improving on the results of Godement [6] on the *Bochner theorem*, Barker [3] has shown that every positive-definite distribution, T, on G extends uniquely to a continuous linear functional on $\mathcal{C}^1(G)$ and that

$$T[f] = \int_{\mathcal{P}} \widehat{f} d\mu$$

for a uniquely defined *Borel measure*, μ. Here $f \in \mathcal{C}^1(G//K)$ and $\mathcal{P}$ is the space of positive-definite spherical functions on G. This is his *spherical Bochner theorem* which has been extended to all $\mathcal{C}^p(G//K)$, $1 \leq p \leq 2$, with the requirement that $supp(\mu) \subset \mathfrak{F}^\epsilon$.

Now if $f \in C_c^\infty(G)$ and $\varphi_\lambda \in \mathcal{C}^p(G//K)$ we define a function on G, termed *spherical convolutions* and denoted $\mathcal{H}_{x,\lambda}f$, as

$$\mathcal{H}_{x,\lambda}f = (f * \varphi_\lambda)(x).$$

We show, among other properties, that the map $\lambda \mapsto \mathcal{H}_{x,\lambda}f$ is well-defined on $\mathcal{P}$, *Weyl group* invariant, and that the integral over $\mathcal{P}$ of its spherical Fourier transform is a non-zero constant multiple of $T[\varphi_\lambda]$ for every $f \in \mathcal{C}^p(G//K)$ whenever $supp(\mu) \subset \mathfrak{F}^\epsilon$. This gives an expansion formula for this integral when $\varphi_\lambda \in \mathcal{C}_\tau^2(SL(2,\mathbb{R}))$, where τ is a double representation on $K = SO(2)$. When considered as the function $x \mapsto \mathcal{H}_{x,\lambda}f$ on G, the behaviours of the spherical convolutions at the identity element $x = e$ and at $\lambda = 0$ show both its generalization of the *Harish-Chandra transforms* and its relationship with the *elementary* spherical function Ξ respectively. Its membership of the Schwartz algebra $\mathcal{C}^2(G//K)$, which leads to the consideration of its spherical Fourier transforms, results to the proof of a more inclusive *Plancherel formula* for $\mathcal{C}^2(G//K)$.

Details of these results are contained in §4. after giving a preliminary on the structure theory of G in §2. and the spherical Bochner theorems in §3.

2 Preliminaries

For the connected semisimple Lie group G with finite center, we denote its Lie algebra by $\mathfrak{g}$ whose *Cartan decomposition* is given as $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$. Denote by θ the *Cartan involution* on $\mathfrak{g}$ whose collection of fixed points is $\mathfrak{t}$. We also denote by K the analytic subgroup of G with Lie algebra $\mathfrak{t}$. K is then a maximal compact subgroup of G. Choose a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ with algebraic dual $\mathfrak{a}^*$ and set $A = \exp \mathfrak{a}$. For every $\lambda \in \mathfrak{a}^*$ put

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} : [H, X] = \lambda(H)X, \forall H \in \mathfrak{a}\},$$

and call λ a restricted root of $(\mathfrak{g}, \mathfrak{a})$ whenever $\mathfrak{g}_\lambda \neq \{0\}$. Denote by $\mathfrak{a}'$ the open subset of $\mathfrak{a}$ where all restricted roots are $\neq 0$, and call its connected components the *Weyl chambers.* Let $\mathfrak{a}^+$ be one of the Weyl chambers, define the restricted root λ positive whenever it is positive on $\mathfrak{a}^+$ and denote by $\triangle^+$ the set of all restricted positive roots. Members of $\triangle^+$ which form a basis for $\triangle$ and can not be written as a linear combination of other members of $\triangle^+$ are called *simple.* We then have the *Iwasawa decomposition* $G = KAN$, where N is the analytic subgroup of G corresponding to $\mathfrak{n} = \sum_{\lambda \in \triangle^+} \mathfrak{g}_\lambda$, and the *polar decomposition* $G = K \cdot cl(A^+) \cdot K$, with $A^+ = \exp \mathfrak{a}^+$, and $cl(A^+)$ denoting the closure of A^+.

If we set $M = \{k \in K : Ad(k)H = H,\ H \in \mathfrak{a}\}$ and $M' = \{k \in K : Ad(k)\mathfrak{a} \subset \mathfrak{a}\}$ and call them the *centralizer* and *normalizer* of $\mathfrak{a}$ in K, respectively, then (see [7, p. 284]); (i) M and M' are compact and have the same Lie algebra and (ii) the factor $\mathfrak{w} = M'/M$ is a finite group called the *Weyl group.* $\mathfrak{w}$ acts on $\mathfrak{a}^*_{\mathbb{C}}$ as a group of linear transformations by the requirement

$$(s\lambda)(H) = \lambda(s^{-1}H),$$

$H \in \mathfrak{a}$, $s \in \mathfrak{w}$, $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, the complexification of $\mathfrak{a}^*$. We then have the *Bruhat* *decomposition*

$$G = \bigsqcup_{s \in \mathfrak{w}} B m_s B$$

where $B = MAN$ is a closed subgroup of G and $m_s \in M'$ is the representative of s (i.e., $s = m_s M$). The Weyl group invariant members of a space shall be denoted by the superscript $^{\mathfrak{w}}$.

Some of the most important functions on G are the *spherical functions* which we now discuss as follows. A non-zero continuous function φ on G shall be called a *(zonal) spherical function* whenever $\varphi(e) = 1$, $\varphi \in C(G//K) := \{g \in C(G)\colon g(k_1 x k_2) = g(x),\, k_1, k_2 \in K,\, x \in G\}$ and $f * \varphi = (f * \varphi)(e) \cdot \varphi$ for every $f \in C_c(G//K)$, where $(f * g)(x) := \int_G f(y) g(y^{-1}x) dy$. This leads to the existence of a homomorphism $\lambda : C_c(G//K) \to \mathbb{C}$ given as $\lambda(f) = (f * \varphi)(e)$. This definition is equivalent to the satisfaction of the functional relation

$$\int_K \varphi(xky) dk = \varphi(x)\varphi(y), \quad x, y \in G.$$

It has been shown by Harish-Chandra [8] that spherical functions on G can be parametrized by members of $\mathfrak{a}^*_{\mathbb{C}}$. Indeed every spherical function on G is of the form

$$\varphi_\lambda(x) = \int_K e^{(i\lambda - p)H(xk)} dk, \ \lambda \in \mathfrak{a}^*_{\mathbb{C}},$$

$\rho = \frac{1}{2}\sum_{\lambda \in \triangle^+} m_\lambda \cdot \lambda$, where $m_\lambda = dim(\mathfrak{g}_\lambda)$, and that $\varphi_\lambda = \varphi_\mu$ iff $\lambda = s\mu$ for some $s \in \mathfrak{w}$. Some of the well-known properties of spherical functions are $\varphi_{-\lambda}(x^{-1}) = \varphi_\lambda(x)$, $\varphi_{-\lambda}(x) = \bar{\varphi}_{\bar{\lambda}}(x)$, $\mid \varphi_\lambda(x) \mid \leq \varphi_{\Re\lambda}(x)$, $\mid \varphi_\lambda(x) \mid \leq \varphi_{i\Im\lambda}(x)$, $\varphi_{-i\rho}(x) = 1$, $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, while $\mid \varphi_\lambda(x) \mid \leq \varphi_0(x)$, $\lambda \in i\mathfrak{a}^*$, $x \in G$. Also if Ω is the *Casimir operator* on G then

$$\Omega\varphi_\lambda = -(\langle\lambda, \lambda\rangle + \langle\rho, \rho\rangle)\varphi_\lambda,$$

where $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ and $\langle\lambda, \mu\rangle := tr(adH_\lambda\ adH_\mu)$ for elements $H_\lambda,\ H_\mu \in \mathfrak{a}$. The elements $H_\lambda,\ H_\mu \in \mathfrak{a}$ are uniquely defined by the requirement that $\lambda(H) = tr(adH\ adH_\lambda)$ and $\mu(H) = tr(adH\ adH_\mu)$ for every $H \in \mathfrak{a}$ ([7, Theorem 4.2]). Clearly $\Omega\varphi_0 = 0$.

Due to a hint dropped by Dixmier [5] (cf. [10]) in his discussion of some functional calculus, it is necessary to recall the notion of a *'positive-definite'* function and then discuss the situation for positive-definite spherical functions.

We call a continuous function $f : G \rightarrow \mathbb{C}$ (algebraically) positive-definite whenever, for all $x_1, \ldots, x_m$ in G and all $\alpha_1, \ldots, \alpha_m$ in $\mathbb{C}$, we have

$$\sum_{i,j=1}^{m} \alpha_i \bar{\alpha}_j f(x_i^{-1} x_j) \geq 0.$$

It can be shown ($cf.$ [7.]) that $f(e) \geq 0$ and $|f(x)| \leq f(e)$ for every $x \in G$ implying that the space $\mathcal{P}$ of all positive-definite spherical functions on G is a subset of the space $\mathfrak{F}^1$ of all bounded spherical functions on G.

We know, by the Helgason-Johnson theorem ([9.]), that

$$\mathfrak{F}^1 = \mathfrak{a}^* + iC_\rho$$

where C_ρ is the convex hull of $\{s\rho : s \in \mathfrak{w}\}$ in $\mathfrak{a}^*$. Defining the *involution* f^* of f as $f^*(x) = \overline{f(x^{-1})}$, it follows that $f = f^*$ for every $f \in \mathcal{P}$, and if $\varphi_\lambda \in \mathcal{P}$, then λ and $\bar{\lambda}$ are Weyl group conjugate, leading to a realization of $\mathcal{P}$ as a subset of $\mathfrak{w} \setminus \mathfrak{a}^*_{\mathbb{C}}$. $\mathcal{P}$ becomes a locally compact Hausdorff space when endowed with the *weak* $^*-$*topology* as a subset of $L^\infty(G)$.

3 The Spherical Bochner Theorem and its Extension

Let

$$\varphi_0(x) := \int_K \exp(-\rho(H(xk)))dk$$

be denoted as $\Xi(x)$ and define $\sigma : G \rightarrow \mathbb{C}$ as

$$\sigma(x) = \|X\|$$

for every $x = k \exp X \in G, \quad k \in K, \ X \in \mathfrak{a}$, where $\| \cdot \|$ is a norm on the finite-dimensional space $\mathfrak{a}$. These two functions are spherical functions on G and there exist numbers c, d such that

$$1 \leq \Xi(a) e^{\rho(\log a)} \leq c(1 + \sigma(a))^d.$$

Also there exists $r_0 > 0$ such that $c_0 =: \int_G \Xi(x)^2 (1 + \sigma(x))^{r_0} dx < \infty$, [13, p. 231]. For each $0 \leq p \leq 2$ define $\mathcal{C}^p(G)$ to be the set consisting of functions f

in $C^{\infty}(G)$ for which

$$\|f\|_{g_1,g_2;m} := \sup_{G} |f(g_1;x;g_2)|\Xi(x)^{-2/p}(1+\sigma(x))^m < \infty$$

where $g_1, g_2 \in \mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$, the *universal enveloping algebra* of $\mathfrak{g}_{\mathbb{C}}$, $m \in \mathbb{Z}^+, x \in G$, $f(x;g_2) := \frac{d}{dt}\big|_{t=0} f(x \cdot (\exp tg_2))$ and $f(g_1;x) := \frac{d}{dt}\big|_{t=0} f((\exp tg_1) \cdot x)$. We call $\mathcal{C}^p(G)$ the Schwartz space on G for each $0 < p \leq 2$ and note that $\mathcal{C}^2(G)$ is the well-known (see, [1]) Harish-Chandra space of rapidly decreasing functions on G. The inclusions

$$C_c^{\infty}(G) \subset \mathcal{C}^p(G) \subset L^p(G)$$

hold and with dense images. It also follows that $\mathcal{C}^p(G) \subseteq \mathcal{C}^q(G)$ whenever $0 \leq p \leq q \leq 2$. Each $\mathcal{C}^p(G)$ is closed under *involution* and the *convolution*, $*$. Indeed $\mathcal{C}^p(G)$ is a Fréchet algebra [12, p. 69]. We endow $\mathcal{C}^p(G//K)$ with the relative topology as a subset of $\mathcal{C}^p(G)$.

For any measurable function f on G we define the *spherical Fourier transform* $\widehat{f}$ as

$$\widehat{f}(\lambda) = \int_G f(x)\varphi_{-\lambda}(x)dx,$$

$\lambda \in \mathfrak{a}^*_{\mathbb{C}}$. It is known (see [3]) that for $f, g \in L^1(G)$ we have:

(*i.*) $(f * g)^{\wedge} = \widehat{f} \cdot \widehat{g}$ on $\mathfrak{F}^1$ whenever f (or g) is right - (or left-) K-invariant;

(*ii.*) $(f^*)^{\wedge}(\varphi) = \overline{\widehat{f}(\varphi^*)}, \varphi \in \mathfrak{F}^1$; hence $(f^*)^{\wedge} = \overline{\widehat{f}}$ on $\mathcal{P}$: and, if we define $f^{\#}(g) := \int_{K \times K} f(k_1 x k_2) dk_1 dk_2, x \in G$, then

(*iii.*) $(f^{\#})^{\wedge} = \widehat{f}$ on $\mathfrak{F}^1$.

In order to know the image of the spherical Fourier transform when restricted to $\mathcal{C}^p(G//K)$ we need the following spaces that are central to the statement of the well-known result of Trombi and Varadarajan [11] (Theorem 3.1 below).

Let C_ρ be the closed convex hull of the (finite) set $\{s\rho : s \in \mathfrak{w}\}$ in $\mathfrak{a}^*$, i.e.,

$$C_\rho = \left\{ \sum_{i=1}^{n} \lambda_i(s_i\rho) : \lambda_i \geq 0, \ \sum_{i=1}^{n} \lambda_i = 1, \ \ s_i \in \mathfrak{w} \right\}$$

where we recall that, for every $H \in \mathfrak{a}$,

$$(s\rho)(H) = \frac{1}{2} \sum_{\lambda \in \triangle^+} m_\lambda \cdot \lambda(s^{-1}H).$$

Now for each $\epsilon > 0$ set $\mathfrak{F}^\epsilon = \mathfrak{a}^* + i\epsilon C_\rho$. Each $\mathfrak{F}^\epsilon$ is convex in $\mathfrak{a}^*_{\mathbb{C}}$ and

$$int(\mathfrak{F}^\epsilon) = \bigcup_{0<\epsilon'<\epsilon} \mathfrak{F}^{\epsilon'}$$

[11, Lemma 3.2.2]. Let us define $\mathcal{Z}(\mathfrak{F}^0) = \mathcal{S}(\mathfrak{a}^*)$ and, for each $\epsilon > 0$, let $\mathcal{Z}(\mathfrak{F}^\epsilon)$ be the space of all $\mathbb{C}$-valued functions Φ such that $(i.)$ Φ is defined and holomorphic on $int(\mathfrak{F}^\epsilon)$, and $(ii.)$ for each holomorphic differential operator D with polynomial coefficients we have $\sup_{int(\mathfrak{F}^\epsilon)} |D\Phi| < \infty$. The space $\mathcal{Z}(\mathfrak{F}^\epsilon)$ is converted to a Fr$\acute{e}$chet algebra by equipping it with the topology generated by the collection, $\|\cdot\|_{\mathcal{Z}(\mathfrak{F}^\epsilon)}$, of seminorms given by $\|\Phi\|_{\mathcal{Z}(\mathfrak{F}^\epsilon)} := \sup_{int(\mathfrak{F}^\epsilon)} |D\Phi|$. It is known that $D\Phi$ above extends to a continuous function on all of $\mathfrak{F}^\epsilon$ [11, p. 278-279]. An appropriate subalgebra of $\mathcal{Z}(\mathfrak{F}^\epsilon)$ for our purpose is the closed subalgebra $\bar{\mathcal{Z}}(\mathfrak{F}^\epsilon)$ consisting of $\mathfrak{w}$-invariant elements of $\mathcal{Z}(\mathfrak{F}^\epsilon)$, $\epsilon \geq 0$. The following well-known result affords us the opportunity of defining a distribution on $\mathcal{C}^p(G//K)$.

3.1 Theorem (Trombi-Varadarajan [11]). *Let $0 < p \leq 2$ and set $\epsilon = (2/p) - 1$. Then the spherical Fourier transform $f \mapsto \widehat{f}$ is a linear topological algebra isomorphism of $\mathcal{C}^p(G//K)$ onto $\bar{\mathcal{Z}}(\mathfrak{F}^\epsilon)$.* $\square$

In order to use the above theorem to state the results of Barker [3], we require the following notions.

3.1 Definitions.

$(i.)$ *A distribution T on G (i.e., $T \in C_c^\infty(G)'$) is said to be (integrally) positive-definite (written as $T \gg 0$) whenever*

$$T[f * f^*] \geq 0,$$

for $f \in C_c^\infty(G)$.

$(ii.)$ *A distribution T on G is called K-bi-invariant whenever $T^\# = T$ where*

$$T^\#[f] := T[f^{L(k_1)R(k_2)}],$$

for $f \in C_c^\infty(G)$.

$(iii.)$ *A measure μ defined on $\mathcal{P}$ is said to be of polynomial growth if there exists a holomorphic polynomial Q on $\mathfrak{a}^*_{\mathbb{C}}$ such that $\int_{\mathcal{P}} (d\mu/|Q|) < \infty$.*

$(iv.)$ *The support, $supp(\mu)$, of a regular Borel measure μ is the smallest closed set A such that $\mu(B) = 0$ for all Borel sets B disjoint from A.*

The following is the first of the main results of [3].

3.2 Theorem (The spherical Bochner theorem). *Suppose $T \in C_c^\infty(G)'$ and $T \gg 0$. Then T extends uniquely to an element in $(\mathcal{C}^1(G))'$ and there exists a unique positive regular Borel measure μ of polynomial growth on $\mathcal{P}$ such that*

$$T[f] = \int_{\mathcal{P}} \widehat{f} d\mu, \quad f \in \mathcal{C}^1(G//K).$$

The correspondence between T and μ is bijective when restricted to $K-$bi-invariant distributions, in which case the formula holds for all $f \in \mathcal{C}^1(G)$. $\square$

The second of the main results of [3] is a consequence of the Trombi-Varadarajan theorem (Theorem 3.1 above) and is stated as follows.

3.3 Theorem (The extension theorem). *Suppose T is a positive-definite distribution with spherical Bochner measure μ. Then $T \in (\mathcal{C}^p(G//K))'$ iff supp $(\mu) \subset \mathfrak{F}^\epsilon$ where $1 \leq p \leq 2$ and $\epsilon = (2/p) - 1$. In such a case*

$$T[f] = \int_{\mathcal{P}} \widehat{f} d\mu, \quad f \in \mathcal{C}^p(G//K). \ \square$$

4 Spherical Convolutions

We start by defining the central notion of this research work.

4.1 Definition. *Let f be any measurable function on G. The spherical convolution of f is the measurable function, $\mathcal{H}_{x,\lambda}f$, on $G \times \mathfrak{a}_{\mathbb{C}}^*$ given by the map*

$$(x, \lambda) \mapsto \mathcal{H}_{x,\lambda}f := (f * \varphi_\lambda)(x),$$

where $x \in G, \lambda \in \mathfrak{a}_{\mathbb{C}}^*$.

We shall refer to the map $\lambda \mapsto \mathcal{H}_{x,\lambda}f$ as the *spherical convolution of f at $x \in G$*. The importance of Definition 4.1 is seen from the next Lemma (especially the realization of the spherical Fourier transforms in item $(ii.)$).

4.1 Lemma. *Let f, f_1 and f_2 be measurable functions on G, whose identity element is denoted as e. Then*

$(i.)$ $\mathcal{H}_{x,\lambda}(f_1 \pm cf_2) = \mathcal{H}_{x,\lambda}f_1 \pm c\mathcal{H}_{x,\lambda}f_2, \quad x \in G, \ c \in \mathbb{C}, \ \lambda \in \mathfrak{a}_{\mathbb{C}}^*;$

$(ii.)$ $\mathcal{H}_{e,\lambda}f = \widehat{f}(\lambda), \quad \lambda \in \mathfrak{a}_{\mathbb{C}}^*;$

$(iii.)$ $\mathcal{H}_{x,0}1 = \int_G \Xi(y^{-1}x)dy, \;\; x \in G.$

$(iv.)$ $\overline{\mathcal{H}_{x,-\lambda}f} = \mathcal{H}_{x,\overline{\lambda}}\overline{f}, \;\; x \in G, \; \lambda \in \mathfrak{a}_{\mathbb{C}}^*;$

Proof. Items $(i.)$ and $(iv.)$ are clear. We recall that, for any measurable function f on G, the spherical Fourier transform, $\widehat{f}$, of f is given as $\widehat{f}(\lambda) = \int_G f(x)\varphi_{-\lambda}(x)dx$, $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Since $\varphi_{-\lambda}(x) = \varphi_\lambda(x^{-1})$, for every $\lambda \in \mathfrak{a}_{\mathbb{C}}^*, \; x \in G$, this may be written as

$$\widehat{f}(\lambda) = \int_G f(x)\varphi_\lambda(x^{-1})dx = (f * \varphi_\lambda)(e) = \mathcal{H}_{e,\lambda}f.$$

This proves $(ii.)$. Item $(iii.)$ follows if we recall that $\varphi_0(x) = \Xi(x)$. $\square$

Item $(iv.)$ of Lemma 4.1 gives the *functional equation* for spherical convolutions. This Lemma (especially in item $(ii.)$) explains that the harmonic analysis of G has so far been explored only with the spherical convolution at e. The implication of considering only

$$(e, \lambda) \mapsto \mathcal{H}_{e,\lambda}f =: \widehat{f}(\lambda)$$

is that the direct contribution of the non-identity members of G to its harmonic analysis are suppressed and may never be suspected or known in the context of $\widehat{f}(\lambda)$. Indeed a great deal of properties of the spherical convolutions and their contributions to harmonic analysis on G would not be available if, instead of considering the entirety of the map $(x, \lambda) \mapsto \mathcal{H}_{x,\lambda}f$, we restrict ourselves to either $\lambda \mapsto \mathcal{H}_{e,\lambda}f = \widehat{f}(\lambda)$ or $x \mapsto \mathcal{H}_{x,0}1$ or any other special case of the spherical convolutions as has been done till now.

We shall therefore show the importance of including spherical convolutions in the harmonic analysis of G by giving its bounds, $\mathfrak{w}-$group transformation and differential equation. These are contained in the following Theorem while a Plancherel formula for the functions $x \mapsto \mathcal{H}_{x,\lambda}f$ on G is proved after a study of its spherical Fourier transforms.

4.1 Theorem. *Consider a measurable function f on G, $x \in G$ and let $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Then*

$(i.)$ $\mid \mathcal{H}_{x,\lambda}f \mid \leq \parallel f \parallel_1,$ *with* $f \in L^1(G)$;

$(ii.)$ $\mid \mathcal{H}_{x,\lambda}f \mid \leq \mathcal{H}_{x,\Re\lambda}f$, $\mid \mathcal{H}_{x,\lambda}f \mid \leq \mathcal{H}_{x,i\Im\lambda}f$ *and* $\mid \mathcal{H}_{x,\lambda}f \mid \leq \mathcal{H}_{x,0}f$, *for* $f \geq 0$;

$(iii.)$ $\mathcal{H}_{x,s\lambda}f = \mathcal{H}_{x,\lambda}f,$ *for every* $s \in \mathfrak{w}$;

(iv.) $\Omega\mathcal{H}_{x,\lambda}f = -(\langle\lambda,\lambda\rangle + \langle\rho,\rho\rangle)\cdot\mathcal{H}_{x,\lambda}f,$ *for* $f \in \mathcal{C}^p(G),\ 0 < p \leq 2;$

(v.) $\mathcal{H}_{x,-i\rho}f = \int_G f(y)dy.$

Proof. We employ the properties of spherical functions given in §2 to establish $(i.), (ii.), (iii.)$ and (v). The proof of $(iv.)$ follows if we recall that

$$\Omega(f * \varphi_\lambda) = f * \Omega\varphi_\lambda. \ \square$$

The equation established in Theorem 4.1 $(iv.)$, or any other such equation for $q \in \mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$, shows that the spherical convolutions inherit the differential equations satisfied by φ_λ. This resemblance suggests the choice of the name adopted in Definition 4.1. It therefore has the following series expansion.

4.1 Corollary. *The spherical convolutions, $\mathcal{H}_{h,\lambda}f$ admit the series expansion*

$$\mathcal{H}_{h,\lambda}f = \sum_{s\in\mathfrak{w}} c(s\lambda)\left(e^{(s\lambda-\rho)(\log h)} + \sum_{\mu\in L^+} a_\mu(s\lambda)e^{(s\lambda-\rho-\mu)(\log h)}\right),$$

regardless of the functions $f \in \mathcal{C}^p(G),\ 0 < p \leq 2,$ where $h \in A^+,\ \lambda \in {}^*\mathfrak{F}' := \{\nu \in {}^*\mathfrak{F} : \nu \text{ is regular}\},\ L^+ = L \setminus \{0\},$ with

$$L := \left\{\sum_{1\leq i\leq r} m_i\alpha_i : m_1, \ldots, m_r \ \text{ are integers } \geq 0\right\},$$

for the simple roots $\alpha_i, 1 \leq i \leq r$, some subset ${}^*\mathfrak{F}$ of $\mathfrak{a}_{\mathbb{C}}^*$ and coefficient functions $a_\mu(\lambda)$ which may be generated from the recursive relation

$$(\langle\mu,\mu\rangle - 2\langle\mu,\lambda\rangle)\, a_\mu(\lambda)$$

$$= -2 \sum_{\substack{\alpha > 0, k \geq 1 \\ \mu - 2k\alpha \in L}} n(\alpha)\langle\lambda - \mu + 2k\alpha - \rho, \alpha\rangle a_{\mu-2k\alpha}(\lambda),\ n(\alpha) := dim(\mathfrak{g}_\alpha). \ \square$$

We shall now consider the map $\lambda \mapsto \mathcal{H}_{e,\lambda}f$ for its differentiability and/or integrability with respect to λ in some specified subset, Y, of $\mathfrak{a}_{\mathbb{C}}^*$. Indeed, $\mathcal{H}_{e,\lambda}f \in C_c(Y)$, for every $f \in C_c(G)$ and any subset, Y, of $\mathfrak{a}_{\mathbb{C}}^*$. This makes its integral, $\int_Y \mathcal{H}_{e,\lambda}f d\mu(\lambda)$, with respect to some normalised measure, μ, on Y, worthy of an indepth study. To this end we define the map $f \mapsto f\{\varphi_\lambda\}$ on G at the identity element, e, as

$$f\{\varphi_\lambda\}(e) = \int_Y \mathcal{H}_{e,\lambda} f d\mu(\lambda), \;\; f \in C_c(G), \; \varphi_\lambda \in \mathcal{C}^p(G).$$

Before considering the generality of $f\{\varphi_\lambda\}(x)$, for every $x \in G$, we state our first major result on $f\{\varphi_\lambda\}(e)$ which gives an important application of its integral for $Y = \mathfrak{F}^1$.

Define the map $a \mapsto \beta_{\mathfrak{F}^1}(a)$ as $\beta_{\mathfrak{F}^1}(a) = \int_{\mathfrak{F}^1} e^{\nu(\log a)} d\mu(\nu), \; a \in A$ and the $\beta_{\mathfrak{F}^1}-$*weighted Fourier transforms,* $\widetilde{f}$, of $f \in C_c(A)$ at $\lambda \in \mathfrak{F}^1$ as

$$\widetilde{f}(\lambda) = \int_A f(a) e^{\lambda(\log a)} d\eta(a),$$

where $d\eta(a) = \beta_{\mathfrak{F}^1}(a)da$. Observe that the above weighted Fourier transforms $\widetilde{f}(\lambda)$ reduces to the classical Fourier transforms $\widehat{f}(\lambda) = \int_A f(a)e^{\lambda(\log a)}da$ when $\beta_{\mathfrak{F}^1}(a) = 1, \; \forall \; a \in A$. We shall however use this (weighted) transforms only at the identity element $0 \in \mathfrak{F}^1$ of the vector space $\mathfrak{F}^1$ (*i.e.*, $\widetilde{f}(0) = \int_A f(a)d\eta(a) = \int_A f(a)\beta_{\mathfrak{F}^1}(a)da$), in the next Theorem and this, as could be seen below, may not be un-connected with the fact that

$$f\{\varphi_\lambda\}(e) = \int_{\mathfrak{F}^1} (\mathcal{H}_{x,\lambda} f)_{|x=e} d\mu(\lambda)$$

is itself an evaluation at the identity element of G.

4.2 Theorem. *Let* $dx = e^{2\rho(\log a)} dk \; da \; dn$, *where* dk, da *and* dn *are Haar measures on* K, A, *and* N, *respectively, with* dk *normalised. For every* $f \in C_c(G//K)$, *let* $\mathcal{A}(f)$ *denote the Abel transform of* f *defined on* A *as* $\mathcal{A}(f)(a) = e^{\rho(\log a)} \int_N f(an)dn$. *Then*

$$f\{\varphi_\lambda\}(e) = \widetilde{(\mathcal{A}f)}(0), \quad \lambda \in \mathfrak{F}^1.$$

Proof. We only need to prove that

$$f\{\varphi_\lambda\}(e) = \int_A \mathcal{A}f(a)\beta_{\mathfrak{F}^1}(a)da, \;\; \lambda \in \mathfrak{F}^1.$$

Indeed, using the Harish-Chandra parametrisation of φ_λ, we have

$$\begin{aligned}
f\{\varphi_\lambda\}(e) &= \int_{\mathfrak{F}^1} (f * \varphi_\lambda)(e) d\mu(\lambda) \\
&= \int_{\mathfrak{F}^1} \widehat{f}(\lambda) d\mu(\lambda) = \int_{\mathfrak{F}^1} \int_G f(x)\varphi_{-\lambda}(x) dx d\mu(\lambda) \\
&= \int_{\mathfrak{F}^1} \int_G \int_K f(xk) e^{(\lambda-\rho)(H(xk))} dx dk d\mu(\lambda) \\
&= \int_{\mathfrak{F}^1} \int_G f(y) e^{(\lambda-\rho)(H(y))} dy d\mu(\lambda) \\
&= \int_{\mathfrak{F}^1} \int_{AN} f(an) e^{(\lambda+\rho)(\log a)} da dn d\mu(\lambda) \\
&= \int_{\mathfrak{F}^1} \int_A (\mathcal{A}f)(a) e^{\lambda(\log a)} da d\mu(\lambda),
\end{aligned}$$

which implies our result, using Fubini's theorem. $\square$

The last result shows the importance of $f\{\varphi_\lambda\}(e)$ in the harmonic analysis of G and prepares the ground for the consideration of results of *Paley-Wiener type.* It will soon be clear that it is sufficient to take the measure μ as the Borel measure on $\mathcal{P}$ in defining the map $a \mapsto \beta_{\mathcal{P}}(a)$. Our motivation in this direction is to consider the general map

$$\lambda \mapsto \mathcal{H}_{x,\lambda} f,$$

not only for the identity element $x = e \in G$, but for other values of G as well. This leads to the definition of $f\{\varphi_\lambda\}(x)$, $x \in G$, as

$$f\{\varphi_\lambda\}(x) = \int_Y \mathcal{H}_{x,\lambda} f d\mu(\lambda),$$

$f \in C_c(G)$, $\varphi_\lambda \in \mathcal{C}^p(G)$, whenever the integral is absolutely convergent. We already have two candidates for the position of Y, namely $\mathfrak{F}^1$ and $\mathcal{P}$. Among other results, it would be important to evaluate the above measure, μ, on these candidates. In the mean time we study some of the properties of $x \mapsto f\{\varphi_\lambda\}(x)$.

4.3 Theorem. *Let* $f \in C_c(G//K)$, $\varphi_\lambda \in \mathcal{C}^p(G//K), 1 \leq p \leq 2$, *and* $Y = \mathcal{P}$. *Then, as a function on* $\mathcal{P}$, $\lambda \mapsto \mathcal{H}_{x,\lambda} f$ *is continuous with compact support. Indeed,* $f\{\varphi_\lambda\} \in \mathcal{C}^p(G//K)$. *Moreover, we have that*

$$f\{\varphi_{s\lambda}\} = f\{\varphi_\lambda\} = f\{\varphi_{s\bar{\lambda}}\},$$

for every $s \in \mathfrak{w}$.

Proof. The first assertion holds, since $f \in C_c(G//K)$ and $C_c(G//K)$ is dense in $\mathcal{C}^p(G//K)$. The properties of φ_λ at the end of §2 imply the second assertion . $\square$

We also have that $f\{\varphi_{\lambda_1} + c\varphi_{\lambda_2}\} = f\{\varphi_{\lambda_1}\} + cf\{\varphi_{\lambda_2}\},\ c \in \mathbb{C}$, suggesting that the map $f \mapsto f\{\varphi_\lambda\}$ may be a *calculus* on $\mathcal{C}^p(G//K)$.

Let us now consider the spherical convolution map

$$(x, \lambda) \mapsto \mathcal{H}_{x,\lambda} f := (f * \varphi_\lambda)(x)$$

as the function

$$x \mapsto \mathcal{H}_{x,\lambda} f$$

on G. Since it is measurable its spherical Fourier transforms may be computed as shown in the following result which gives how to *generate* positive-definite distributions on G and which will be found useful in the proof of its Plancherel formula given later in Theorem 4.7.

4.4 Theorem. *Let* $f \in C_c(G//K),\ \varphi_\lambda \in \mathcal{C}^1(G//K)$. *Let* μ *be a spherical Bochner measure corresponding to a positive-define distribution* T *on* G. Then

(i.) $(\widehat{\mathcal{H}_{x,\lambda} f})(\nu) = \widehat{f}(\lambda) \cdot \widehat{\varphi_\lambda}(\nu),\ \ x \in G,\ \nu \in \mathfrak{a}^*_{\mathbb{C}}$.

(ii.) $\int_{\mathcal{P}} (\widehat{\mathcal{H}_{x,\lambda} f})(\nu) d\mu(\nu) = \widehat{f}(\lambda) \cdot T[\varphi_\lambda]$.

Moreover, if $T^{\#} = T$, the integral in (*ii.*) holds for all $\varphi_\lambda \in \mathcal{C}^1(G)$.

Proof. (*i.*) Employing the defining properties of a spherical function given in §2. we have, for every $x \in G,\ \nu \in \mathfrak{a}^*_{\mathbb{C}}$, that

$$(\widehat{\mathcal{H}_{x,\lambda} f})(\nu) = (f * \varphi_\lambda)(e) \cdot \widehat{\varphi_\lambda}(\nu) = \widehat{f}(\lambda) \cdot \widehat{\varphi_\lambda}(\nu).$$

(*ii.*) Now fix $\varphi_\lambda \in C_c^\infty(G//K)$, then

$$\begin{aligned} \int_{\mathcal{P}} (\widehat{\mathcal{H}_{x,\lambda} f})(\nu) d\mu(\nu) &= \int_{\mathcal{P}} \widehat{f}(\lambda) \cdot \widehat{\varphi_\lambda}(\nu) d\mu(\nu) \\ &= \widehat{f}(\lambda) \cdot \int_{\mathcal{P}} \widehat{\varphi_\lambda}(\nu) d\mu(\nu) \\ &= \widehat{f}(\lambda) \cdot T[\varphi_\lambda]. \end{aligned}$$

We apply the denseness of $C_c^\infty(G//K)$ in $\mathcal{C}^1(G//K)$ to conclude the second assertion. That (*ii.*) holds for all $\varphi_\lambda \in \mathcal{C}^1(G)$ follows from the second part of Theorem 3.2. $\square$

The extension Theorem 3.3 leads also to an extension of Theorem 4.4 given next.

4.5 Theorem (Extension Theorem). *If $supp(\mu) \subset \mathfrak{F}^{\epsilon}$, $\epsilon = (2/p) - 1$ and $1 \leq p \leq 2$, then $\int_{\mathcal{P}} (\widehat{\mathcal{H}_{x,\lambda} f})(\nu) d\mu(\nu)$ is a constant multiple of $T[\varphi_\lambda]$ for every $\varphi_\lambda \in \mathcal{C}^p(G//K)$.* $\square$

The conclusion on $\int_{\mathcal{P}} (\widehat{\mathcal{H}_{x,\lambda} f})(\nu) d\mu(\nu)$ in the last Theorem above may be generalized to the Schwartz algebra $\mathcal{C}^p_\tau(G)$ of all $\tau-$ spherical functions on G where $\tau = (\tau_1, \tau_2)$ is a *double representation* of K. This would be so immediately the Trombi-Varadarajan theorem, Theorem 3.1, is established for $\mathcal{C}^p_\tau(G)$. The case $p = 2$ has been proved and is contained in [1] for real-rank 1 Lie groups G, and in [2] for any semisimple Lie group of any rank, while the case of general p remains an open problem.

However the situation for general p and the group $G = SL(2, \mathbb{R})$, or its conjugate $SU(1,1)$, is contained in [13] from which other groups could be considered. Thus using the results of [4] on $\mathcal{C}^2_\tau(SL(2, \mathbb{R}))$ we extend the assertions of Theorem 4.5 to all the members of $\mathcal{C}^2_\tau(SL(2, \mathbb{R}))$. This leads to an expansion of $\int_{\mathcal{P}} (\widehat{\mathcal{H}_{x,\lambda} f})(\nu) d\mu(\nu)$ for φ_λ in the Schwartz algebras of all $\tau-$ spherical functions on $G = SL(2, \mathbb{R})$. This expansion brings in the involvement of the well-known global characters of the (unitary) principal and discrete series of representations of $G = SL(2, \mathbb{R})$, [13].

To establish this expansion formula we put the needed type of measures in place. A pair (μ_c, μ_d) is called a *tempered Bochner measure pair* whenever:

$(i.)$ μ_c is a non-negative Baire measure on $\mathbb{R}$ which is symmetric and of polynomial growth. That is, $d\mu_c(-\lambda) = d\mu_c(\lambda)$, for all $\lambda \in \mathbb{R}$ and

$$\int_{\mathbb{R}} d\mu_c(\lambda)/(1+ \mid \lambda \mid^r) < \infty$$

for some $r \geq 0$.

$(ii.)$ μ_d is a non-negative counting measure on $\mathbb{Z}' = Z \backslash 0$ which is of polynomial growth. That is,

$$\sum_{l \in \mathbb{Z}'} \mu_d(l)/(1+ \mid l \mid^r) < \infty$$

for some $r \geq 0$.

The following Theorem opens up the integral contained in Theorem 4.5 in the special case of $G = SL(2, \mathbb{R})$

4.6 Theorem (Expansion for $\int_{\mathcal{P}}(\widehat{\mathcal{H}_{x,\lambda}f})(\nu)d\mu(\nu)$ on $\mathcal{C}^2_\tau(SL(2,\mathbb{R}))$). *Let $f\in C_c^\infty(G)$, $\varphi_\lambda\in\mathcal{C}^2_\tau(SL(2,\mathbb{R}))$. Then, up to a non-zero constant, the positive-definite distribution $\int_{\mathcal{P}}(\widehat{\mathcal{H}_{x,\lambda}f})(\nu)d\mu(\nu)$ is given as*

$$\int_{\mathcal{P}}(\widehat{\mathcal{H}_{x,\lambda}f})(\nu)d\mu(\nu)=\widehat{f}(\lambda)\cdot lim_{n\to\infty}(\int_{-n}^{n}\Phi^\lambda[\varphi_\lambda]d\mu_c(\lambda)+\Sigma_{1\le|l|\le n}\Theta^l[\varphi_l]\mu_d(l)),$$

where Φ^λ and Θ^l are the global characters of the (unitary) principal and discrete series of representations of $G=SL(2,\mathbb{R})$ and (μ_c,μ_d) is the tempered Bochner measure pair associated to a tempered invariant positive-definite distribution on G. In particular, $\int_{\mathcal{P}}(\widehat{\mathcal{H}_{x,\lambda}f})(\nu)d\mu(\nu)$ is a tempered invariant distribution on G.

Proof. For any tempered invariant positive-definite distribution T on G there corresponds a Bochner measure pair (μ_c,μ_d) such that

$$T[f]=lim_{n\to\infty}(\int_{-n}^{n}\Phi^\lambda[f]d\mu_c(\lambda)+\Sigma_{1\le|l|\le n}\Theta^l[f]\mu_d(l)).$$

This is the main result of [4] (listed there as Theorem 9.3), which when combined with our Theorem 4.5 gives the assertion. $\square$

4.1 Remark. The expansion given above for $\int_{\mathcal{P}}(\widehat{\mathcal{H}_{x,\lambda}f})(\nu)d\mu(\nu)$ reveals the rich structure encoded in it. Indeed since the global characters above, in terms of which it is expressed (in Theorem 4.6), have well-known transformation under the action of the center, $\mathfrak{Z}$, of the universal enveloping algebra, $\mathfrak{U}(\mathfrak{g}_\mathbb{C})$, of the complexification $\mathfrak{g}_\mathbb{C}$ of the Lie algebra $\mathfrak{g}$ of G, a study of the functional and differential equations of $\int_{\mathcal{P}}(\widehat{\mathcal{H}_{x,\lambda}f})(\nu)d\mu(\nu)$ is very possible and suggests a harmonic analysis involving both the discrete and (unitary) principal series of, at least, $G=SL(2,\mathbb{R})$.

For $f\in C_c(G//K)$ and $\varphi_\lambda\in\mathcal{C}^p(G//K)$ as in Theorem 4.5, we may view the map $\lambda\mapsto f\{\varphi_\lambda\}$ as the *evaluation* of members of $C_c(G//K)$ on members of $\mathcal{C}^p(G//K)$. This means that $\lambda\mapsto f\{\varphi_\lambda\}$ is an *operational calculus* on the Schwartz algebras, $\mathcal{C}^p(G//K)$, whose spherical Fourier transform is a distribution on G. This suggests the use of the term *'distributional calculus'* for $f\mapsto f\{\varphi_\lambda\}$. A more detailed study of $f\{\varphi_\lambda\}$ may therefore be conducted by considering the invariant eigendistributions on G, most especially the global characters of the irreducible admissible representations of G.

We now consider the explicit form of the Plancherel formula for the measurable functions $x\mapsto\mathcal{H}_{x,\lambda}f$ on G. A Haar measure dx on G is said to be *admissible* if $dx=e^{2\rho(\log a)}dkdadn$ $(x=kan)$ where $\int_K dk=1$ and $\int_{\overline{N}}e^{-2\rho H(\overline{n})}d\overline{n}=1$

where $d\overline{n}$ is a Haar measure on $\overline{N} := \theta(N)$. Recall the Borel measure $d\mu(\lambda)$ from Theorem 4.4. The pair $(dx, d\mu(\lambda))$ of Haar measures on the pair $(G, \mathfrak{F}^1)$ shall be de termed *admissible* if, every f in the Schwartz space, $\mathcal{S}(A)$, of A whose Fourier transform $\widehat{f}$, already known as

$$\widehat{f}(\lambda) = \int_A f(a)e^{\lambda(\log a)}da, \quad \lambda \in \mathfrak{F}^1,$$

satisfies

$$f(a) = \int_{\mathfrak{F}^1} \widehat{f}(\lambda)e^{-\lambda(\log a)}d\mu(\lambda), \quad a \in A.$$

4.7 Theorem (Plancherel formula for spherical convolutions). *Let $(dy, d\mu(\lambda))$ be an admissible pair of Haar measures on the pair $(G, \mathfrak{F}^1)$, $x \in G$ and $f \in \mathcal{C}(G//K)$. If we define the measure $d\zeta_{x,\lambda}$ as a normalization of the spherical Bochner measure $d\mu(\lambda)$ on $\mathfrak{F}^1$ by the requirement that*

$$d\zeta_{x,\lambda}(\nu) = \frac{1}{\mid \widehat{\varphi_\lambda}(\nu) \mid^2} d\mu(\lambda),$$

then

$$\int_G \mid f(y) \mid^2 dy = \int_{\mathfrak{F}^1} \mid (\widehat{\mathcal{H}_{x,\lambda} f})(\nu) \mid^2 d\zeta_{x,\lambda}(\nu).$$

In particular the map $f \mapsto (\widehat{\mathcal{H}_{x,\lambda} f})$, for $x \in G$ and $\lambda \in \mathfrak{F}^1$, extends uniquely to a unitary isomorphism of $L^2(G//K)$ with $L^2(\mathfrak{F}^1, d\zeta_{x,\lambda}(\nu))^{\mathfrak{w}}$.

Proof. Since the spherical convolutions of $f \in \mathcal{C}(G//K)$ may be considered as the functions $x \mapsto \mathcal{H}_{x,\lambda} f$ on G it follows that its spherical Fourier transforms, $\nu \mapsto (\widehat{\mathcal{H}_{x,\lambda} f})(\nu)$, is well-defined on $\mathfrak{F}^1$. Therefore

$$\begin{aligned}
\int_{\mathfrak{F}^1} \mid (\widehat{\mathcal{H}_{x,\lambda} f})(\nu) \mid^2 d\zeta_{x,\lambda}(\nu) &= \int_{\mathfrak{F}^1} \mid \widehat{f}(\lambda) \mid^2 \cdot \mid \widehat{\varphi_\lambda}(\nu) \mid^2 d\zeta_{x,\lambda}(\nu) \\
\text{(from Theorem 4.4 } (i.)) & \\
&= \int_{\mathfrak{F}^1} \mid \widehat{f}(\lambda) \mid^2 d\mu(\lambda) \\
\text{(from definition of } d\zeta_{x,\lambda}) & \\
&= \int_G \mid f(y) \mid^2 dy
\end{aligned}$$

(by the Plancherel formula for f). $\square$

The situation of Theorem 4.7 for $x = e$ is well-known, while the inverse, $(\mathcal{H}_{x,\lambda} f)^{-1}$, for fixed $f \in \mathcal{C}(G//K)$, $x \in G$ and $\lambda \in \mathfrak{F}^1$, is given as

$$(\mathcal{H}_{x,\lambda} f)^{-1}(b)(y) = \int_{\mathfrak{F}^1} b(\lambda)\varphi_\lambda(y) d\zeta_{x,\lambda}(\nu), \; b \in \mathcal{S}(\mathfrak{F}^1)^{\mathfrak{w}}, y \in G$$

and is commonly called the *exact (normalized) wave packet.* The explicit expression for the Plancherel measure, $d\zeta_{x,\lambda}(\nu)$, of the spherical convolutions in terms of elementary functions of harmonic analysis is therefore given as

$$d\zeta_{x,\lambda}(\nu) = \mid \mathfrak{w} \mid^{-1} \mid \widehat{\varphi_\lambda}(\nu) \mid^{-2} \mid c(\lambda) \mid^{-2} d\mu(\lambda),$$

for $x \in G$, $\nu, \lambda \in \mathfrak{F}^1$, where the map $\lambda \mapsto c(\lambda)$ is the Harish-Chandra $c-$function. The combination of Theorems 4.2 and 4.7 may be used to give the Paley-Wiener theorem for spherical Fourier transforms of spherical convolutions.

It is known [11, p. 298] that $\mathcal{H}_{e,\lambda}f = \widehat{f}(\lambda)$ and that, in this case, the Plancherel measure, $d\zeta_{e,\lambda}(\nu)$, of the spherical convolution, $\mathcal{H}_{x,\lambda}f$, at $x = e$ is

$$d\zeta_{e,\lambda}(\nu) = \mid \mathfrak{w} \mid^{-1} \mid c(\lambda) \mid^{-2} d\mu(\lambda).$$

A non-trivial problem is to find the relation between ν and λ for the Plancherel measure, $d\zeta_{x,\lambda}(\nu)$, of the spherical convolutions to reduce to the classical Plancherel measure, $\mid \mathfrak{w} \mid^{-1} \mid c(\lambda) \mid^{-2} d\mu(\lambda)$, on G. This is equivalent to seeking those ν in terms of λ for which $\mid \widehat{\varphi_\lambda}(\nu) \mid = 1$, where $\varphi_\lambda \in \mathcal{C}^1(G//K)$. We plan to address this problem in another paper.

The richness of our results, which may be ultimately seen in Theorem 4.7, derives from the fact that the spherical convolutions are functions on both G and $\mathfrak{F}^1$. This fact allows us to switch its domains between G and $\mathfrak{F}^1$, depending on its immediate use. In all these diverse instances of the harmonic analysis on G we still use the same defining functions for the spherical convolutions. We have however taken advantage of some known results in the harmonic analysis on G (like the Harish-Chandra series expansion inherited by $\mathcal{H}_{h,\lambda}f$ (in Corollary 4.1) and the classical Plancherel formula on G used in the proof of Theorem 4.7) in order to establish our results. Nevertheless our results could still be established from the scratch without recourse to the special case of $\mathcal{H}_{e,\lambda}f = \widehat{f}(\lambda)$.

References

[1] Arthur, J.G., *Harmonic analysis of tempered distributions on semisimple Lie groups of real rank one*, Ph.D. Dissertation, Yale University, 1970.

[2] Arthur, J.G., *Harmonic analysis of the Schwartz space of a reductive Lie group*, I, II, preprint, (1973).

[3] Barker, W.H., The spherical Bochner theorem on semisimple Lie groups, *J. Funct. Anal.,* **20**, (1975), 179-207.

[4] Barker, W.H., Tempered invariant, positive-definite distributions on $SU(1,1)/\{\pm 1\}$, *Illinois J. Maths*, **28**(1), (1984), 83-102.

[5] Dixmier, J., Opérateurs de rang fini dans les représentations unitaires, *Publ. math. de l' Inst. Hautes Études Scient.*, **6**, (1960), 13-25.

[6] Godement, R.A., A theory of spherical functions, I. *Trans. Amer. Math. Soc.*, **73**, (1952), 496-556.

[7] Helgason, S., *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press, New York, 1978.

[8] Helgason, S., *Groups and Geometric Analysis; Integral Geometry, Invariant Differential Operators, and Spherical Functions*, Academic Press, New York, 1984.

[9] Helgason, S. and Johnson, K., The bounded spherical functions on symmetric spaces, *Advances in Math*, **3**, (1969), 586-593.

[10] Kahane J.-P., Sur un théorème de Wiener-Lévy, *C.R. Acad. Sc., Paris*, **246**, (1958), 1949-1951.

[11] Trombi, P.C. and Varadarajan, V.S., Spherical transforms on semisimple Lie groups, *Ann. of Math.*, **94**, (1971), 246-303.

[12] Varadarajan, V.S., The theory of characters and the discrete series for semisimple Lie groups, in Harmonic Analysis on Homogeneous Spaces, (C.C. Moore (ed.)), *Proc. of Symposia in Pure Maths.*, **26**, (1973), 45-99.

[13] Varadarajan, V.S., *An introduction to harmonic analysis on semisimple Lie groups*, Cambridge University Press, Cambridge, 1989.