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Extended Adomian's polynomials for solving non-linear fractional differential equations

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Abstract

In this paper, we implement the improved Adomian's polynomials introduced by Abassy in 2010 with an extension to solve numerically the non-linear initial value problems of fractional differential equations (FDEs). This proposed extension is called extended Adomian's polynomials. An adaption of the convergence analysis which was introduced by Hosseini and Nasabzadeh in 2006 is formulated to be used to prove that these polynomials accelerate the convergence rate of the series solution comparing with the standard Adomian's polynomials. Also, we use the so called improved Adomian decomposition method (IADM) as a special case of the proposed method where the fractional derivative $\alpha = 1$. A comparison is made between IADM and ADM for some examples to illustrate the efficiency of the proposed treatment for non-linear initial value problems of FDEs.

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1 Introduction and main result

Non-linear phenomena which is appeared in many scientific fields can be modeled by fractional differential equations. There are many approximate and numerical techniques to seek with numerical solution of non-linear FDEs ([10]-[19], [24]-[27]). In some of these techniques, the linear operator with fractional derivatives was replaced approximately by a linear operator with integer derivatives so, the convergence rate was very low such as ADM which was introduced by Odibat and Momani [20].

In our analytical treatment, the modified Riemann-Liouville fractional derivative which was introduced by Jumarie [8] is used instead of Riemann-Liouville or Caputo fractional derivatives. So, we can deal with the linear operator with fractional derivatives using the properties of Jumarie fractional derivative [9] without any replacement which mean that it is still fractional so the rate of convergence is accelerated comparing with ADM. Also this treatment is included an extension in formulation of Adomian's polynomials which derived by Abassy [1] and which provides a good improvement to the speed of convergence and cancels the calculations of all the inaccurate terms which deteriorate the convergence especially for higher fractional-order differential equations. To prove the acceleration of our treatment (IADM), we adapt the convergence analysis which was introduced by Hosseini and Nasabzadeh [6] to be used for non-linear fractional differential equations. In our test examples, we introduce a comparison between the obtained numerical results with those obtained using ADM to clarify the efficiency and the applicability of the proposed treatment.

2 Basic definitions

In this section, we present some basic definitions and properties of the fractional calculus which useful in the next sections.

Definition 2.1. *Let $f : \mathbb{R} \to \mathbb{R}$, $x \to f(x)$, denotes a continuous function, then its fractional derivative of order α is defined by [8]*

$$f^{(\alpha)}(x) := \frac{1}{\Gamma(-\alpha)} \int_0^x (x-\xi)^{-\alpha-1} f(\xi) d\xi, \qquad \alpha < 0. \tag{1}$$

For positive α, one will set

$$f^{(\alpha)}(x) := (f^{(\alpha-1)}(x))', \qquad 0 < \alpha < 1, \tag{2}$$

and

$$f^{(\alpha)}(x) := (f^{(\alpha-n)}(x))^{(n)}, \qquad n \geq 1, \qquad n \leq \alpha < n+1. \tag{3}$$

If $f(x) = k$ (constant), using Eq.(2) we find that the α^{th} derivative is $\frac{kx^\alpha}{\Gamma(1-\alpha)}$ which is different from zero. To circumvent this defect some authors, (say [4]) proposed the following definition.

Definition 2.2. *Let $f : \mathbb{R} \to \mathbb{R}$, $x \to f(x)$ denotes a continuous function, then its Riemann-Liouville fractional derivative of order α, D^α is defined by [8]*

$$D^\alpha f(x) := \frac{1}{\Gamma(n+1-\alpha)} \int_0^x (x-\xi)^{n-\alpha} f^{(n+1)}(\xi) d\xi, \qquad n \leq \alpha < n+1. \tag{4}$$

This definition doesn't apply when $f(x)$ isn't differentiable and if we want to get the first derivative of $f(x)$, we must before have its second derivative. So, Jumarie alternated Riemann-Liouville fractional derivative via finite difference is defined as follows.

Definition 2.3. *(Modified Riemann-Liouville fractional derivative [8]) Let $f : \mathbb{R} \to \mathbb{R}$, $x \to f(x)$ denotes a continuous function, $f(x)$ is not a constant, then its fractional derivative of order α is defined by*

$$f^{(\alpha)}(x) := \frac{1}{\Gamma(-\alpha)} \int_0^x (x-\xi)^{-\alpha-1} (f(\xi) - f(0)) d\xi, \qquad \alpha < 0. \tag{5}$$

For positive α, one will set

$$f^{(\alpha)}(x) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-\xi)^{-\alpha} (f(\xi) - f(0)) d\xi, \qquad 0 < \alpha < 1, \tag{6}$$

and

$$f^{(\alpha)}(x) := (f^{(n)}(x))^{(\alpha-n)}, \quad n \geq 1, \quad n \leq \alpha < n+1. \tag{7}$$

Definition 2.4. *The integral with respect to $(dx)^\alpha$ is defined as the solution of the fractional differential equation*

$$dy \cong f(x)(dx)^\alpha, \qquad x \geq 0, \qquad y(0) = 0, \qquad 0 < \alpha \leq 1. \tag{8}$$

Let $f(x)$ denote a continuous function, then the solution of Eq.(8) is defined as [8]

$$y = \int_0^x f(\xi)(d\xi)^\alpha = \alpha \int_0^x (x-\xi)^{\alpha-1} f(\xi) d\xi, \qquad 0 < \alpha \leq 1. \tag{9}$$

Proposition 1:
Assume that $f : \mathbb{R} \to \mathbb{R}$, has a fractional derivative of order αk, for any positive integer k and any α, $0 < \alpha \leq 1$, then $f(x)$ is expanded in the series form

$$f(x) := \sum_{k=0}^{\infty} \frac{f^{(\alpha k)}(0)}{\Gamma(1+\alpha k)} x^{\alpha k}. \tag{10}$$

Proposition 2: (Jumarie fractional derivative via fractional difference [9])
Let $f : \mathbb{R} \to \mathbb{R}$, $x \to f(x)$, denotes a continuous (but not necessarily differentiable) function, and let the partition $h > 0$ in the interval $[0, 1]$.
Define the forward operator FW(h) in the form

$$FW(h).f(x) := f(x+h),$$

then the fractional difference of order α, $(0 < \alpha \leq 1)$, of $f(x)$ is defined by the expression

$$\triangle^\alpha.f(x) := (FW-1)^\alpha.f(x) = \sum_{k=0}^{\infty} (-1)^k \binom{n}{k} f(x + (\alpha - k)h),$$

and its fractional derivative of order α is

$$f^{(\alpha)}(x) = \lim_{h \to 0} \frac{\triangle^\alpha.f(x)}{h^\alpha}.$$

Remark: The α^{th} derivative of a constant using this definition is zero.
For more details about the fractional calculus see [21].

3 The improved Adomian decomposition method

Consider the non-linear initial value problem of fractional partial differential equation in the following general form

$$L^{s\alpha}u(x,t) = Ru(x,t) + Nu(x,t) + g(x,t), \qquad 0 < \alpha \leq 1, \tag{11}$$

under the initial conditions

$$\frac{\partial^{k\alpha}u(x,0)}{\partial t^{k\alpha}} = f_k(x), \qquad k = 0, 1, ..., s-1,$$

where $L^{s\alpha} = \frac{\partial^{s\alpha}}{\partial t^{s\alpha}}$, $s = 1, 2, ...$, is the highest fractional partial derivative with respect to t in terms of Jumarie fractional derivative, R is a linear operator, Nu is the non-linear term and $g(x,t)$ is the source function.

Define the inverse operator $L^{-s\alpha}$ in terms of Jumarie derivative in the following form

$$L^{-s\alpha}(.) = \frac{1}{\Gamma^s(\alpha+1)} \int_0^{(t_s)} \int_0^{(t_{s-1})} ... \int_0^{(t_1)} (.)(d\tau_1)^\alpha ...(d\tau_{s-1})^\alpha (d\tau_s)^\alpha. \tag{12}$$

Applying the inverse operator $L^{-s\alpha}$ to both sides of (11) gives

$$u(x,t) = \sum_{k=0}^{s-1} \frac{t^{\alpha k}}{\Gamma(1+\alpha k)} u^{(\alpha k)}(x,0) + L^{-s\alpha}\left(g(x,t)\right) + L^{-s\alpha}\Big(Ru(x,t) + Nu(x,t)\Big). \tag{13}$$

Where the first part from the right hand side of formula (13) is obtained from the solution of the homogenous fractional differential equation $L^{s\alpha}u(x,t) = 0$ using the Maclurin series of fractional order introduced by Jumarie [8].

ADM defines the solution $u(x,t)$ as an infinite series in the form ([2], [3], [28])

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{14}$$

where the components $u_n(x,t)$ can be obtained in recursive form.
Also, the non-linear term $N(u)$ can be decomposed by an infinite series of polynomials given by

$$N(u) = \sum_{n=0}^{\infty} A_n, \tag{15}$$

where the components A_n can be obtained using the following formula

$$A_n = \frac{1}{(ns)!}\Big[\frac{d^{ns}}{d\lambda^{ns}} N(\sum_{i=0}^{\infty} \lambda^i h_i t^{\alpha i})\Big]_{\lambda=0} + \frac{1}{(ns+1)!}\Big[\frac{d^{(ns+1)}}{d\lambda^{(ns+1)}} N(\sum_{i=0}^{\infty} \lambda^i h_i t^{\alpha i})\Big]_{\lambda=0} + ... \\ + \frac{1}{(ns+s-1)!}\Big[\frac{d^{(ns+s-1)}}{d\lambda^{(ns+s-1)}} N(\sum_{i=0}^{\infty} \lambda^i h_i t^{\alpha i})\Big]_{\lambda=0}, \qquad n = 0, 1, ..., \tag{16}$$

where $h_n^{\prime}s$ are the coefficients of $t^{n\alpha}$ in the components $u_n(x,t)$.

The formula (16) is a generalization of the formula which introduced by Abassy [1] and obtained when we put $(\alpha = 1)$ in Eq.(16) which is used to improve the accuracy of the solutions. Also there are many approaches which was introduced to improve the accuracy of Adomian decomposition method ([7], [22]). The formula (16) is used in our treatment instead of the formula of standard Adomian's polynomials

$$A_n = \frac{1}{n!}\Big[\frac{d^n}{d\lambda^n} N(\sum_{i=0}^{\infty} \lambda^i u_i)\Big]_{\lambda=0}, \qquad n = 0, 1, \dots . \tag{17}$$

The two formulas are similar to each other when $(s = 1)$ but the formula (16) accelerates the speed of convergence than the formula (17) and cancels the inaccurate terms when $s = 2, 3, ...$ which appear when using the Adomian's polynomials (17) which deteriorate the convergence of the series solution. Substituting by (14) and (15) into Eq.(13) gives

$$\sum_{n=0}^{\infty} u_n(x,t) = \sum_{k=0}^{s-1} \frac{t^{\alpha k}}{\Gamma(1+\alpha k)} u^{(\alpha k)}(x,0) + L^{-s\alpha}(g(x,t)) + L^{-s\alpha}\Big(R\sum_{n=0}^{\infty} u_n + \sum_{n=0}^{\infty} A_n\Big). \tag{18}$$

Substitute by the initial conditions, we can obtain the components $u_n(x,t)$ of the solution by the following formula

$$u_0(x,t) = f_0(x) + \frac{f_1(x)}{\Gamma(1+\alpha)} t^{\alpha} + ... + \frac{f_{s-1}(x)}{\Gamma(1+(s-1)\alpha)} t^{(s-1)\alpha} + L^{-s\alpha}(g(x,t)), \\ u_{n+1}(x,t) = L^{-s\alpha}(R u_n + A_n), \qquad n \geq 0. \tag{19}$$

4 Convergence analysis of IADM and ADM

Hosseini and Nasabzadeh introduced a simple method to determine the rate of convergence of Adomian decomposition method [6]. In this section, we adapt it to seek here.

Theorem 4.1. *Let N be an operator from a Hilbert space into itself and $u(x,t)$ be the exact solution of Eq.(11), then, the approximate solution which is obtained by (19) converges to $u(x,t)$ if $0 \leq \gamma < 1$, and satisfies the following condition [6]*

$$\|u_{k+1}\| \leq \gamma \|u_k\|, \qquad k = 0, 1, \ldots . \tag{20}$$

Remark: If U_i and $\hat{U}_i$ are obtained by ADM and IADM, respectively, then the rate of convergence of $\sum_{i=0}^{\infty} \hat{u}_i$ to the exact solution is higher than $\sum_{i=0}^{\infty} u_i$ if $\hat{\gamma}_i \leq \gamma_i$ and both of them are less than one.

5 Illustrative examples

In this section, we introduce four examples of non-linear fractional differential equations two of them are ODEs and the others are PDEs. We find the truncated series solutions for these examples using our treatment (IADM) and compare them with the solutions obtained using ADM and plot the curves of these solutions at different values of α. Also, we study the convergence of our treatment solutions comparing to solutions of ADM using the generalization of Hosseini and Nasabzadeh study on the convergence of Adomian decomposition method.

Example 1:

Consider the initial value problem for higher fractional-order differential equation [23]

$$u^{(2\alpha)}(t) = u^2(t) + 1, \quad t \geq 0, \quad 0 < \alpha \leq 1, \tag{21}$$

subject to the initial conditions

$$u(0) = 0, \quad u^{(\alpha)}(0) = 1.$$

In order to obtain the numerical solutions for Eq.(21) using our proposed treatment, we follow the following steps:

1: Rewrite Eq.(21) in the following operator form

$$L^{2\alpha}u(t) = Ru(t) + Nu(t) + g(t), \tag{22}$$

where $L^{2\alpha} = \frac{d^{2\alpha}}{dt^{2\alpha}}$, $Ru(t) = 0$, $Nu(t) = u^2$, and $g(t) = 1$.

2: Apply the inverse operator $L^{-2\alpha}$ which is defined by

$$L^{-2\alpha}(.) = \frac{1}{\Gamma^2(1+\alpha)} \int_0^t \int_0^t (.)(d\tau)^\alpha (d\tau)^\alpha, \tag{23}$$

to both sides of Eq.(22) gives

$$u(t) = \sum_{k=0}^{1} \frac{u^{(\alpha k)}(0)}{\Gamma(1+\alpha k)} t^{\alpha k} + L^{-2\alpha}(1) + L^{-2\alpha}(Nu(t)). \tag{24}$$

Substituting by Eqs.(14) and (15) in Eq.(24) gives

$$\sum_{n=0}^{\infty} u_n(t) = u(0) + \frac{u^{(\alpha)}(0)}{\Gamma(1+\alpha)} t^\alpha + \frac{1}{\Gamma^2(1+\alpha)} \int_0^t \int_0^t (1)(d\tau)^\alpha (d\tau)^\alpha + L^{-2\alpha}\Big(\sum_{n=0}^{\infty} A_n\Big), \tag{25}$$

substituting by initial conditions, then, the components $u_n(t)$ of the solution $u(t)$ can be written as

$$u_0(t) = \frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}, \qquad u_{n+1}(t) = L^{-2\alpha}(A_n), \qquad n \geq 0, \tag{26}$$

where A_n can be obtained by the formula (16) at $s = 2$.

3: In order to obtain the components $u_n(t)$ of the solution $u(t)$ using the iteration formula (26), follow the following

$$u_0(t) = \frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} = h_0 + h_1 t^\alpha + h_2 t^{2\alpha},$$

where,

$$h_0 = 0, \quad h_1 = \frac{1}{\Gamma(1+\alpha)} \quad \text{and} \quad h_2 = \frac{1}{\Gamma(1+2\alpha)},$$

$$A_0 = h_0^2 + 2h_0h_1\, t^\alpha + h_1^2\, t^{2\alpha} + 2h_0h_2\, t^{2\alpha}.$$

The first component $u_1(t)$ can be obtained using the formula (26) as follows

$$u_1(t) \;=\; L^{-2\alpha}(A_0) = \frac{\Gamma(1+2\alpha)}{\Gamma(\alpha)\Gamma(1+\alpha)}\, {}_2F_1[1, 1+3\alpha, 2+4\alpha, 1]\; t^{4\alpha}.$$

We rewrite the component $u_1(t)$ in the form

$$u_1(t) = h_3\, t^{3\alpha} + h_4\, t^{4\alpha},$$

where $h_3 = 0, \quad h_4 = \frac{\Gamma(1+2\alpha)}{\Gamma(\alpha)\Gamma(1+\alpha)}\, {}_2F_1[1, 1+3\alpha, 2+4\alpha, 1]$,
following the same procedure, we obtain

$$u_2(t) = \frac{3\sqrt{\pi}2^{1-2\alpha}\Gamma(3\alpha)}{\Gamma^2(\alpha)\Gamma(\frac{1}{2}+\alpha)}\, {}_2F_1[1, 1+4\alpha, 2+5\alpha, 1]\, t^{5\alpha} + \frac{2^{2-4\alpha}\pi\Gamma(4\alpha)}{\Gamma^2(\alpha)\Gamma^2(\frac{1}{2}+\alpha)}\, t^{6\alpha}.$$

So, the solution $u(t)$ can be approximated as

$$u(t) \simeq \phi_n(t) = \sum_{m=0}^{n} u_m(t). \tag{27}$$

The truncated solution $\phi_n(t)$ using IADM with $n = 2$ is given by

$$\begin{aligned}\phi_2(t) &= u_0(t) + u_1(t) + u_2(t)\\ &= \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{\Gamma(1+2\alpha)}{\Gamma(\alpha)\Gamma(1+\alpha)}\, {}_2F_1[1, 1+3\alpha, 2+4\alpha, 1] t^{4\alpha} +\\ &\quad \frac{3\sqrt{\pi}2^{1-2\alpha}\Gamma(3\alpha)}{\Gamma^2(\alpha)\Gamma(\frac{1}{2}+\alpha)}\, {}_2F_1[1, 1+4\alpha, 2+5\alpha, 1] t^{5\alpha} + \frac{2^{2-4\alpha}\pi\Gamma(4\alpha)}{\Gamma^2(\alpha)\Gamma^2(\frac{1}{2}+\alpha)} t^{6\alpha}.\end{aligned}$$

In order to obtain the numerical solutions for Eq.(21) using standard ADM, we compute the first components of the solution as follows

$$\begin{aligned}u_0(t) &= \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)},\\ u_1(t) &= \frac{1}{2\alpha^2\Gamma(2\alpha)}\Big(\alpha + \frac{4t^{2\alpha}\alpha^3\Gamma(2\alpha^2)}{\Gamma^2(1+\alpha)\Gamma(1+4\alpha)} + ...\Big) t^{2\alpha},\\ u_2(t) &= \frac{1}{\Gamma(2\alpha)}\Big(\frac{t^{2\alpha}}{2\alpha} + \frac{2^{1-4}\sqrt{\pi}\Gamma(2\alpha)\Gamma(1+5\alpha)t^{7\alpha}}{\Gamma^3(1+\alpha)+\Gamma(\frac{1}{2}+2\alpha)+\Gamma(1+7\alpha)} + ...\Big).\end{aligned}$$

Then the solution $u(t)$ which is defined by (27) can be approximated in the following form

$$\begin{aligned}\phi_2(t) &= u_0(t) + u_1(t) + u_2(t)\\ &= \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{1}{2\alpha^2\Gamma(2\alpha)}\Big(\alpha + \frac{4t^{2\alpha}\alpha^3\Gamma(2\alpha^2)}{\Gamma^2(1+\alpha)\Gamma(1+4\alpha)} + ...\Big) t^{2\alpha} +\\ &\quad \frac{1}{\Gamma(2\alpha)}\Big(\frac{t^{2\alpha}}{2\alpha} + \frac{2^{1-4}\sqrt{\pi}\Gamma(2\alpha)\Gamma(1+5\alpha)t^{7\alpha}}{\Gamma^3(1+\alpha)+\Gamma(\frac{1}{2}+2\alpha)+\Gamma(1+7\alpha)} + ...\Big).\end{aligned}$$

The behavior of the approximate solutions using our modified method (IADM) compared with ADM and approximate solution using ADM* (ADM with Jumarie derivative and Adomian's polynomials (17)) is given in figures 1.1 and 1.2 at $\alpha = 1$ and $\alpha = 0.75$, respectively.

Method	$\frac{\|\|u_1\|\|}{\|\|u_0\|\|}$	$\frac{\|\|u_2\|\|}{\|\|u_1\|\|}$	$\frac{\|\|u_3\|\|}{\|\|u_2\|\|}$
IADM	0.021963	0.389872	0.052601
ADM*	0.035102	0.543723	1.025448
ADM	0.243101	0.957275	0.052006

Table 1.1: The convergence behavior of the truncated solutions at $\alpha = 1.0$ using IADM, ADM* and ADM.

The convergence analysis of the approximate solution using our modified method (IADM) compared with ADM and approximate solution using ADM* is given in tables 1.1 and 1.2, respectively, in terms of Theorem 4.1 with respect to the L_2 norm which is defined as

$$||u(t)||_2 = \int_0^1 |u(t)|^2 dt.$$

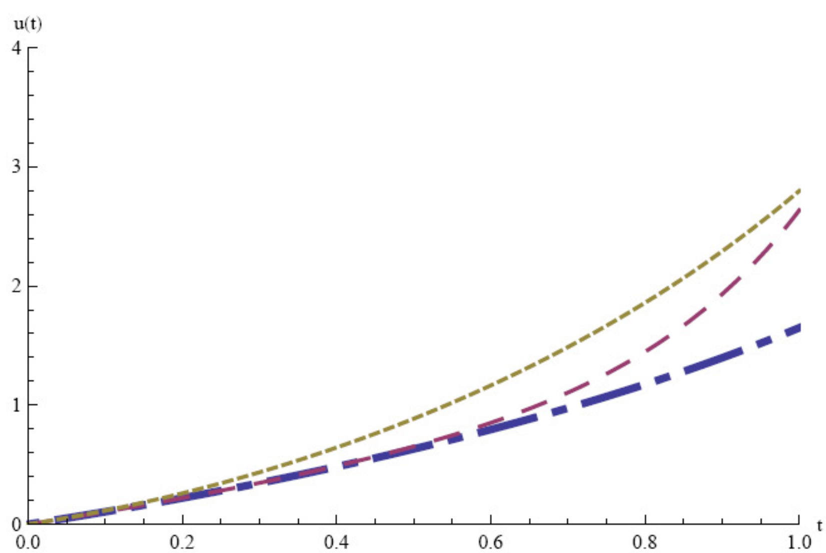

Figure 1.1: The IADM solution (dot-dashed line), ADM* (dashed line) and ADM solution (dot line) at $\alpha = 1.0$.

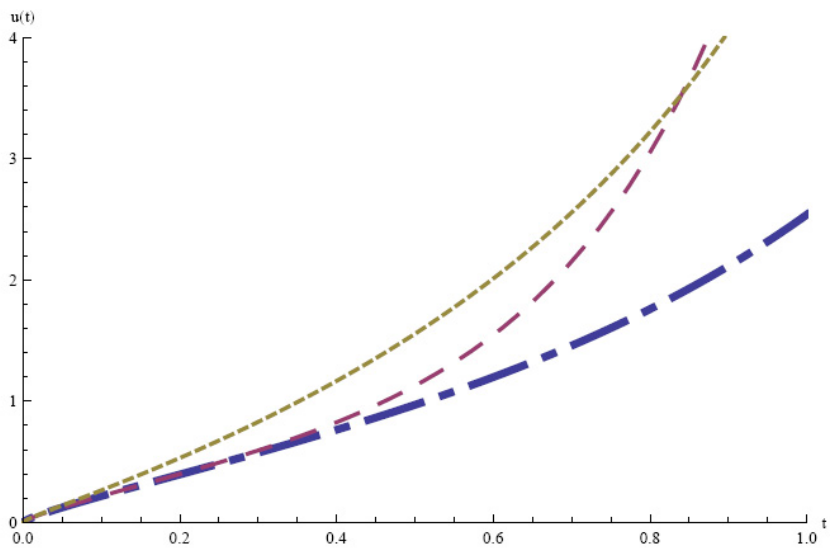

Figure 1.2: The IADM solution (dot-dashed line), ADM*(dashed line) and ADM solution (dot line) at $\alpha = 0.75$.

Method	$\frac{\|\|u_1\|\|}{\|\|u_0\|\|}$	$\frac{\|\|u_2\|\|}{\|\|u_1\|\|}$
IADM	0.064130	0.727540
ADM*	0.130161	1.123870
ADM	0.408004	0.940325

Table 1.2: The convergence behavior of the truncated solutions at $\alpha = 0.75$ using IADM, ADM* and ADM.

Example 2:

Consider the initial value problem for higher fractional-order differential equation

$$u^{(4\alpha)}(t) = 16u(t) + 40u^3(t) + 24u^5(t), \qquad 0 < \alpha \leq 1, \tag{28}$$

subject to the following initial conditions

$$u(0) = 0, \quad u^{(\alpha)}(0) = \frac{1}{\Gamma(1+\alpha)}, \quad u^{(2\alpha)}(0) = 0, \quad \text{and} \quad u^{(3\alpha)}(0) = \frac{2}{\Gamma(1+3\alpha)}.$$

The exact solution of Eq.(28) at $\alpha = 1$ is $u(t) = \tan(t)$.

In this example, we present the series solution of Eq.(28) using IADM, as follows:

1: Rewrite Eq.(28) in the operator form

$$L^{4\alpha}u(t) = Ru(t) + N_1u(t) + N_2u(t) + g(t), \tag{29}$$

where $L^{4\alpha} = \frac{d^{4\alpha}}{d\,t^{4\alpha}}$, $Ru(t) = 16u(t)$, $N_1u(t) = 40u^3$, $N_2u(t) = 24u^5$ and $g(t) = 0$.

2: Apply the inverse operator which is defined by

$$L^{-4\alpha}(.) = \frac{1}{\Gamma(1+4\alpha)} \int_0^t \int_0^t \int_0^t \int_0^t (.)(d\tau)^\alpha (d\tau)^\alpha (d\tau)^\alpha (d\tau)^\alpha, \tag{30}$$

to both sides of Eq.(29) gives

$$u(t) = \sum_{i=0}^{3} \frac{u^{(i\,\alpha)}(0)}{\Gamma(1+i\,\alpha)} t^{i\,\alpha} + L^{-4\alpha}\Big[Ru(t) + N_1u(t) + N_2u(t) + g(t)\Big]. \tag{31}$$

Substituting by Eqs.(14) and (15) in Eq.(31) gives

$$\sum_{n=0}^{\infty} u_n(t) = \sum_{i=0}^{3} \frac{u^{(i\,\alpha)}(0)}{\Gamma(1+i\,\alpha)} t^{i\,\alpha} + L^{-4\alpha}\Big[16\sum_{n=0}^{\infty} u_n + \sum_{n=0}^{\infty} A_n + \sum_{n=0}^{\infty} B_n\Big], \tag{32}$$

substituting by initial conditions, then, the components $u_n(t)$ of the solution $u(t)$ can be written as

$$\begin{aligned} u_0(t) &= \frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{2t^{3\alpha}}{\Gamma(1+3\alpha)}, \\ u_{n+1}(t) &= L^{-4\alpha}\Big[16u_n(t) + A_n + B_n\Big], \quad n \geq 0, \end{aligned} \tag{33}$$

where A_n and B_n can be obtained by the formula (16) at ($s = 4$).

3: To obtain the components $u_n(t)$ of the solution $u(t)$ using the iteration formula (33), follow the following

$$u_0(t) = \frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{2t^{3\alpha}}{\Gamma(1+3\alpha)} = h_0 + h_1 t^\alpha + h_2 t^{2\alpha} + h_3 t^{3\alpha},$$

$$\text{i.e.,} \quad h_0 = 0, \quad h_1 = \frac{1}{\Gamma(1+\alpha)}, \quad h_2 = 0, \quad h_3 = \frac{2}{\Gamma(1+3\alpha)},$$

and,

$A_0 = 40h_0^3 + 120t^\alpha h_0^2 h_1 + 240(h_0 h_1^2 + h_0^2 h_2)t^{2\alpha}$,

$B_0 = 24h_0^5 + 120t^\alpha h_0^4 h_1 + 120(2h_0^3 h_1^2 + h_0^4 h_2)t^{2\alpha} + 120(2h_0^2 h_1^3 + 4h_0^3 h_1 h_2 + h_0^4 h_3)t^{3\alpha}$.

Therefore, using Eq.(33) the first component $u_1(t)$ is given by

$$\begin{aligned} u_1(t) &= L^{-4\alpha}\left[16u_0(t) + A_0 + B_0\right] \\ &= \frac{\alpha^4}{\Gamma^4(1+\alpha)}\left(-16\Gamma^3(\alpha)\Gamma(1+\alpha)\,{}_2F_1[1, 1+4\alpha, 2+5\alpha, 1]\right) t^{5\alpha} + \\ &\quad \frac{\alpha^4}{\Gamma^4(1+\alpha)}\left(-16\Gamma^3(\alpha)\Gamma(1+\alpha)\,{}_2F_1[1, 1+6\alpha, 2+7\alpha, 1]\right) t^{7\alpha} \\ &= h_4\, t^{4\alpha} + h_5\, t^{5\alpha} + h_6\, t^{6\alpha} + h_7\, t^{7\alpha}, \end{aligned}$$

i.e.,

$$h_4 = 0, \qquad h_5 = \frac{\alpha^4}{\Gamma^4(1+\alpha)}(-16\Gamma^3(\alpha)\Gamma(1+\alpha)\, {}_2F_1[1, 1+4\alpha, 2+5\alpha, 1]),$$
$$h_6 = 0, \qquad h_7 = \frac{\alpha^4}{\Gamma^4(1+\alpha)}(-16\Gamma^3(\alpha)\Gamma(1+\alpha)\, {}_2F_1[1, 1+6\alpha, 2+7\alpha, 1]).$$

Following the same procedure, we obtain

$$\begin{aligned} u_2(t) &= L^{-4\alpha}\Big[16u_1(t) + A_1 + B_1\Big] \\ &= \frac{\alpha^4}{\Gamma^4(1+\alpha)} \frac{8(-15\Gamma(3\alpha)\Gamma(5\alpha)\Gamma(1+11\alpha)+...)}{\alpha^4\Gamma(\alpha)\Gamma(3\alpha)\Gamma(1+9\alpha)\Gamma(1+11\alpha)}\ t^{9\alpha} + \\ &\quad \frac{\alpha^4}{\Gamma^4(1+\alpha)} \frac{(8(-70\Gamma(\alpha)\Gamma(7\alpha)\Gamma(1+9\alpha))+...)}{(\alpha^4\Gamma(\alpha)\Gamma(3\alpha)\Gamma(1+9\alpha)\Gamma(1+11\alpha))}\ t^{11\alpha}. \end{aligned}$$

Then the solution $u(t)$ using the standard IADM can be approximated as defined in (27)

$$\begin{aligned} \phi_2(t) &= u_0(t) + u_1(t) + u_2(t) \\ &= \frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{2t^{3\alpha}}{\Gamma(1+3\alpha)} - \frac{16\alpha^4\Gamma^3(\alpha)\Gamma(1+\alpha)}{\Gamma^4(1+\alpha)}\ {}_2F_1[1, 1+4\alpha, 2+5\alpha, 1]\ t^{5\alpha} + \\ &\quad \frac{-16\alpha^4\Gamma^3(\alpha)\Gamma(1+\alpha)}{\Gamma^4(1+\alpha)}\ {}_2F_1[1, 1+6\alpha, 2+7\alpha, 1]\ t^{7\alpha} + \\ &\quad \frac{\alpha^4}{\Gamma^4(1+\alpha)} \frac{(8(-15\Gamma(3\alpha)\Gamma(5\alpha)\Gamma(1+11\alpha)+...))}{\alpha^4\Gamma(\alpha)\Gamma(3\alpha)\Gamma(1+9\alpha)\Gamma(1+11\alpha)}\ t^{9\alpha} + \\ &\quad \frac{\alpha^4}{\Gamma^4(1+\alpha)} \frac{\big(8(-70\Gamma(\alpha)\Gamma(7\alpha)\Gamma(1+9\alpha)+...)\big)}{\alpha^4\Gamma(\alpha)\Gamma(3\alpha)\Gamma(1+9\alpha)\Gamma(1+11\alpha)}\ t^{11\alpha}. \end{aligned}$$

In order to obtain the numerical solutions for Eq.(28) using ADM, we compute the first components as follows

$$u_0(t) = \frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{2t^{3\alpha}}{\Gamma(1+3\alpha)},$$

$$u_1(t) = \frac{8}{\Gamma(4\alpha)}\Big(t^{5\alpha}\Gamma(4\alpha)(\frac{-2}{\Gamma(1+5\alpha)} + ... + t^{2\alpha}(\frac{-4 - \frac{5\Gamma(1+3\alpha)}{\Gamma^3(1+\alpha)}}{\Gamma(1+7\alpha)} + ...)\Big),$$

$$u_2(t) = \frac{64}{\Gamma(4\alpha)}\Big(\frac{4t^{9\alpha}\Gamma(4\alpha)}{\Gamma(1+9\alpha)} + \frac{8t^{11\alpha}\Gamma(4\alpha)}{\Gamma(1+11\alpha)} + ... + \frac{23040t^{35\alpha}\Gamma(4\alpha)\Gamma(1+15\alpha)\Gamma(1+31\alpha)}{\Gamma(1+19\alpha)\Gamma(1+35\alpha)\Gamma^9(1+3\alpha)}\Big).$$

The solution $u(t)$ using the standard ADM can be approximated as defined in (27)

$$\begin{aligned}\phi_2(t) &= u_0(t)+u_1(t)+u_2(t)\\ &= \frac{t^\alpha}{\Gamma(1+\alpha)}+\frac{2t^{3\alpha}}{\Gamma(1+3\alpha)}+\frac{8}{\Gamma(4\alpha)}\Big(t^{5\alpha}\Gamma(4\alpha)(\frac{-2}{\Gamma(1+5\alpha)}+...+t^{2\alpha}(\frac{-4-\frac{5\Gamma(1+3\alpha)}{\Gamma^3(1+\alpha)}}{\Gamma(1+7\alpha)}+...)\Big)\\ &+\frac{64}{\Gamma(4\alpha)}\Big(\frac{4t^{9\alpha}\Gamma(4\alpha)}{\Gamma(1+9\alpha)}+\frac{8t^{11\alpha}\Gamma(4\alpha)}{\Gamma(1+11\alpha)}+...+\frac{23040t^{35\alpha}\Gamma(4\alpha)\Gamma(1+15\alpha)\Gamma(1+31\alpha)}{\Gamma(1+19\alpha)\Gamma(1+35\alpha)\Gamma^9(1+3\alpha)}\Big).\end{aligned}$$

The behavior of the approximate solution using our modified method (IADM) compared with ADM and approximate solution using ADM* is given in figures 2.1 and 2.2 at $\alpha=1$ and $\alpha=0.75$, respectively.
The convergence analysis of the approximate solution using our modified method IADM compared with ADM is given in tables 2.1 and 2.2, respectively.

Method	$\frac{\|\|u_1\|\|}{\|\|u_0\|\|}$	$\frac{\|\|u_2\|\|}{\|\|u_1\|\|}$	$\frac{\|\|u_3\|\|}{\|\|u_2\|\|}$
IADM	0.036352	0.059154	0.0042309
ADM	0.051605	0.008104	0.0175610

Table 2.1: The convergence behavior of the truncated solutions at $\alpha = 1.0$ using IADM and ADM.

Method	$\frac{\|\|u_1\|\|}{\|\|u_0\|\|}$	$\frac{\|\|u_2\|\|}{\|\|u_1\|\|}$	$\frac{\|\|u_3\|\|}{\|\|u_2\|\|}$
IADM	0.116501	0.17505	0.059415
ADM	0.218764	0.07414	0.116187

Table 2.2: The convergence behavior of the truncated solutionsn at $\alpha = 0.75$ using IADM and ADM.

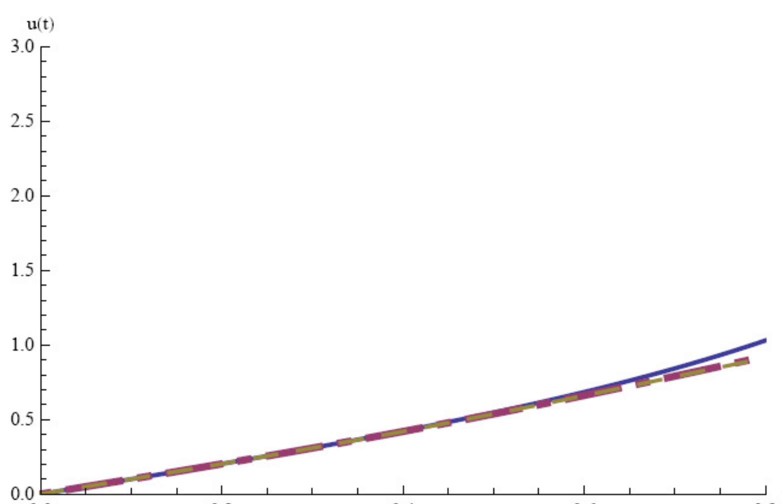

Figure 2.1: The exact solution (solid line), IADM solution $\varphi_2(t)$, (dot-dashed line), ADM* (dashed line) and ADM solution (dot line) at $\alpha = 1$.

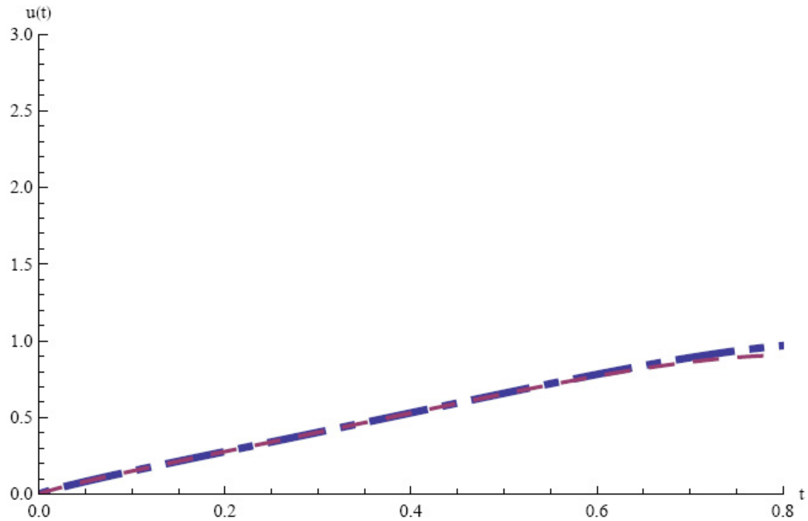

Figure 2.2: The IADM solution $\varphi_2(t)$, (dot-dashed line), ADM* solution (dashed line) and ADM solution (dot line) at $\alpha = 0.75$.

Example 3:

Fisher equation appears in many scientific fields such as financial mathematics and economics in which it estimates the relationship between nominal and real interest rates under inflation. Here in this example we deal with its fractional version.

Consider the initial value problem for non-linear fractional Fisher differential equation [18]

$$D_t^{\alpha} u(x,t) = u_{xx}(x,t) + 6u(x,t)[1-u(x,t)], \quad t>0, \quad 0<\alpha\leq 1, \tag{34}$$

subject to the initial condition $u(x,0)=\frac{1}{(1+e^x)^2}$.
The exact solution of this problem at $\alpha=1$ is $u(x,t)=\frac{1}{(1+e^{x+5t})^2}$.
In order to obtain the numerical solutions by using our treatment (IADM), we follow the following steps:
1: Rewrite Eq.(34) in the following operator form

$$L^{\alpha} u(x,t) = Ru(x,t) + Nu(x,t), \tag{35}$$

where $L^{\alpha}=\frac{\partial^{\alpha}}{\partial t^{\alpha}}$, $Ru=u_{xx}+6u$ and $N(u)=-6u^2$.
2: Apply the inverse operator which is defined by

$$L^{-\alpha}(.) = \frac{1}{\Gamma(1+\alpha)} \int_0^t (.)(d\tau)^{\alpha}, \tag{36}$$

to both sides of (35) gives

$$u(x,t) = u(x,0) + L^{-\alpha}\Big[Ru(x,t) + Nu(x,t)\Big]. \tag{37}$$

Substituting by Eqs.(14) and (15) in Eq.(37) gives

$$\sum_{n=0}^{\infty} u_n(x,t) = u(x,0) + L^{-\alpha}\Big[R\sum_{n=0}^{\infty} u_n(x,t) + \sum_{n=0}^{\infty} A_n\Big], \tag{38}$$

substituting by initial condition, then the components $u_n(x,t)$ of the solution $u(x,t)$ can be written as

$$u_0(x,t) = \frac{1}{(1+e^x)^2}, \qquad u_{n+1}(x,t) = L^{-\alpha}\Big[Ru_n + A_n\Big], \qquad n\geq 0, \tag{39}$$

where A_n can be obtained by the formula (16) at $s=1$.
3: To obtain the components $u_n(x,t)$ of the solution using the iteration formula (39), follow the following

$$u_0(x,t) = \frac{1}{(1+e^x)^2} = h_0(x) \quad \text{and} \quad A_0 = -6(h_0(x))^2,$$

$$u_1(x,t) = L^{-\alpha}\left[6u_0(x,t) + \frac{\partial^2 u_0(x,t)}{\partial x^2} + A_0\right] = \frac{10e^x}{(1+e^x)^3\Gamma(1+\alpha)} t^{\alpha} = h_1(x) t^{\alpha},$$

i.e., $h_1(x) = \dfrac{10e^x}{(1+e^x)^3\Gamma(1+\alpha)}$ and $A_1 = -12h_0(x)h_1(x)t^\alpha$,

$$u_2(x,t) = L^{-\alpha}\left[6u_1(x,t) + \frac{\partial^2 u_1(x,t)}{\partial x^2} + A_1\right] = \frac{50e^x(2e^x-1)}{(1+e^x)^4\Gamma(1+2\alpha)}t^{2\alpha} = h_2(x)t^{2\alpha},$$

following the same procedure we obtain

$$u_3(x,t) = \frac{50\alpha e^x((5-6e^x-15e^{2x}+20e^{3x})\Gamma^2(1+\alpha)-12e^x\Gamma(1+2\alpha))}{(1+e^x)^6\Gamma^3(1+\alpha)\Gamma(1+2\alpha)}\times$$
$$\Gamma(1+\alpha)\Gamma(1+2\alpha)\,{}_2F_1[1,1+2\alpha,2+3\alpha,1]t^{3\alpha}.$$

So, the solution can be approximated by $\phi_n(x,t)$ as defined in (27)

$$\begin{aligned}\phi_2(x,t) &= u_0(x,t)+u_1(x,t)+u_2(x,t)\\ &= \frac{1}{(1+e^x)^2} + \frac{10e^x}{(1+e^x)^3\Gamma(1+\alpha)}t^\alpha + \frac{50e^x(2e^x-1)}{(1+e^x)^4\Gamma(1+2\alpha)}t^{2\alpha}.\end{aligned}$$

In order to obtain the numerical solution of the fractional Fisher equation (34) using ADM in which we use the Riemann-Liouville fractional derivatives and the Adomian polynomials defined in (17), we compute the first components of the solution as follows

$$u_0(x,t) = \frac{1}{(1+e^x)^2},$$

$$u_1(x,t) = \frac{10e^x}{(1+e^x)^3\Gamma(1+\alpha)}t^\alpha,$$

$$u_2(x,t) = \frac{50e^x(2e^x-1)}{(1+e^x)^4\Gamma(1+2\alpha)}t^{2\alpha},$$

$$u_3(x,t) = \frac{50e^x((5-6e^x-15e^{2x}+20e^{3x})\Gamma^2(1+\alpha))-12e^x\Gamma(1+2\alpha))}{(1+e^x)^6\Gamma^3(1+\alpha)\Gamma(1+3\alpha)}t^{3\alpha}.$$

So, the solution can be approximated by $\phi_n(x,t)$ which is defined by (27)

$$\begin{aligned}\phi_2(x,t) &= u_0(x,t)+u_1(x,t)+u_2(x,t)\\ &= \frac{1}{(1+e^x)^2} + \frac{10e^x}{(1+e^x)^3\Gamma(1+\alpha)}t^\alpha + \frac{50e^x(2e^x-1)}{(1+e^x)^4\Gamma(1+2\alpha)}t^{2\alpha}.\end{aligned}$$

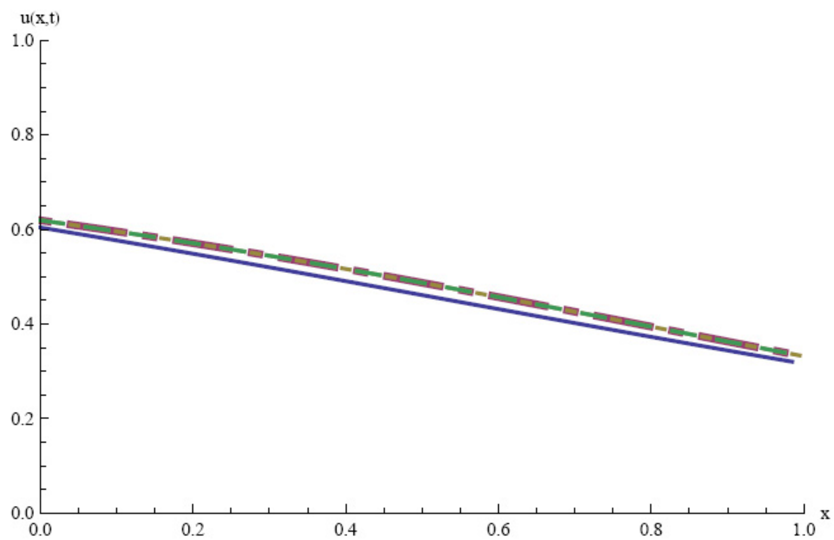

Figure 3.1: The exact solution (solid line), IADM solution, $\phi_2(x,t)$ (dot-dashed line), ADM* solution (dashed line) and ADM solution (dot line) at $\alpha = 1$.

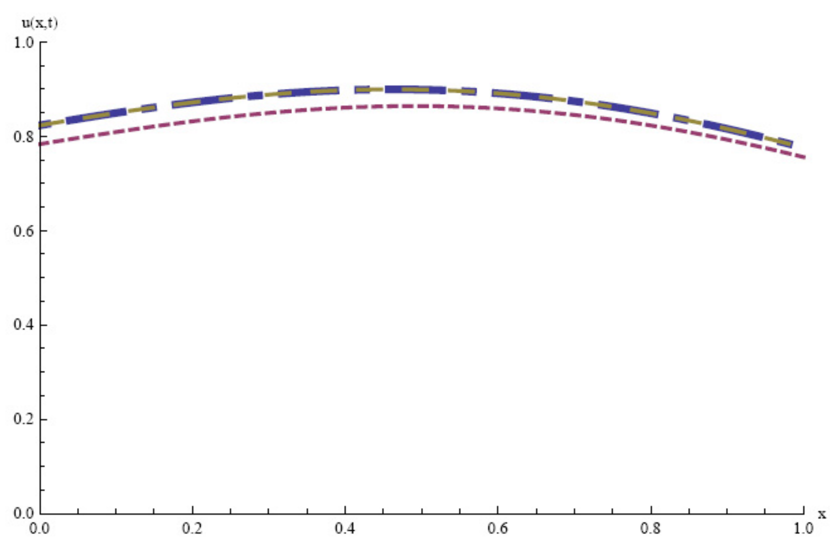

Figure 3.2: The IADM solution, $\phi_2(x,t)$ (dot-dashed line), ADM* solution (dashed line) and ADM solution (dot line) at $\alpha = 0.75$.

The behavior of the approximate solutions using our modified method IADM compared with ADM and approximate solution using ADM* (ADM with Jumarie derivative) are given in figures 3.1 and 3.2 at $\alpha = 1$ and $\alpha = 0.75$, respectively, with $t = 0.25$.

$\alpha = 1.0$				$\alpha = 0.75$			
x	$\frac{\|\|u_1\|\|}{\|\|u_0\|\|}$	$\frac{\|\|u_2\|\|}{\|\|u_1\|\|}$	$\frac{\|\|u_3\|\|}{\|\|u_2\|\|}$	x	$\frac{\|\|u_1\|\|}{\|\|u_0\|\|}$	$\frac{\|\|u_2\|\|}{\|\|u_1\|\|}$	$\frac{\|\|u_3\|\|}{\|\|u_2\|\|}$
0.00	2.88675	0.72169	0.96225	0.00	3.44076	1.09315	1.22731
0.25	3.24573	0.99092	0.37434	0.25	3.86863	1.50096	0.37674
0.50	3.59377	1.25195	0.05251	0.50	4.28347	1.89634	0.26252
0.75	3.92124	1.49755	0.39212	0.75	4.67379	2.26836	0.78562
1.00	4.22077	1.72220	0.67245	1.00	5.03080	2.60863	1.22714

Table 3.1: The convergence analysis of the truncated solution using IADM of fractional Fisher equation at different values of x at $t = 0.5$.

$\alpha = 1.0$				$\alpha = 0.75$			
x	$\frac{\|\|u_1\|\|}{\|\|u_0\|\|}$	$\frac{\|\|u_2\|\|}{\|\|u_1\|\|}$	$\frac{\|\|u_3\|\|}{\|\|u_2\|\|}$	x	$\frac{\|\|u_1\|\|}{\|\|u_0\|\|}$	$\frac{\|\|u_2\|\|}{\|\|u_1\|\|}$	$\frac{\|\|u_3\|\|}{\|\|u_2\|\|}$
0.00	2.88675	0.72169	0.96225	0.00	3.44076	1.09315	1.46989
0.25	3.24573	0.99092	0.37434	0.25	3.86863	1.50096	0.58014
0.50	3.59377	1.25195	0.05251	0.50	4.28347	1.89634	0.09071
0.75	3.92124	1.49755	0.39213	0.75	4.67379	2.26836	0.64100
1.00	4.22077	1.72220	0.67245	1.00	5.03080	2.60863	1.10637

Table 3.2: The convergence analysis of the truncated solution using ADM of fractional Fisher equation at different values of x at $t = 0.5$.

Example 4: It is well known that the non-linear Klein-Gordon equation has many applications in physics. It is equation of motion of a quantum scalar or pseudoscalar field, a field whose quanta are spin less particles. It is a relativistic version of the Schrödinger equation. In this example we solve the non-linear fractional Klein-Gordon equation using IADM and ADM.

Consider the non-linear fractional Klein-Gordan differential equation [5]

$$D_t^{\alpha} u(x,t) = u_{xx} + au + bu^2 + cu^3, \qquad 0 < \alpha \leq 1, \tag{40}$$

for some constants, $a = -1$, $b = 0$ and $c = 1$, subject to the initial condition

$$u(x,0) = -\text{sech}(x).$$

In this example, we present the series solution of Eq.(40) using IADM as follows:

1: Rewrite Eq.(40) in the operator form

$$L^{\alpha} u(x,t) = R(u) + N(u), \tag{41}$$

where $L^{\alpha} = \frac{\partial^{\alpha}}{\partial t^{\alpha}}$, $R(u) = u_{xx} - u$ and $N(u) = u^3$.

2: Apply the inverse operator which is defined by (36) to both sides of Eq.(41) gives

$$u(x,t) = u(x,0) + L^{-\alpha}\Big[Ru(x,t) + N(u(x,t))\Big]. \tag{42}$$

Substituting by Eqs.(14) and (15) gives

$$\sum_{n=0}^{\infty} u_n(x,t) = u(x,0) + L^{-\alpha}\Big[R(\sum_{n=0}^{\infty} u_n) + \sum_{n=0}^{\infty} A_n\Big]. \tag{43}$$

Substituting by initial condition, therefore, the components $u_n(x,t)$ of the solution $u(x,t)$ can be written as

$$\begin{aligned} u_0(x,t) &= -\operatorname{sech}(x), \\ u_{n+1}(x,t) &= L^{-\alpha}\Big[\frac{\partial^2 u_n(x,t)}{\partial x^2} - u_n(x,t) + A_n\Big], \qquad n \geq 0, \end{aligned} \tag{44}$$

where A_n can be obtained by the formula (16) at $(s = 1)$.

3: To obtain the components $u_n(x,t)$ using the iteration formula (44), follow the following

$$u_0(x,t) = -\operatorname{sech}(x) = h_0(x), \quad A_0 = (h_0(x))^3,$$

therefore, the first component $u_1(x,t)$ is given from Eq.(44) as follows

$$u_1(x,t) = L^{-\alpha}\left[\frac{\partial^2 u_0(x,t)}{\partial x^2} - u_0(x,t) + A_0\right] = \frac{\operatorname{sech}^3(x)}{\Gamma(1+\alpha)} t^{\alpha} = h_1(x) t^{\alpha},$$

from that, $h_1(x) = \frac{\operatorname{sech}^3(x)}{\Gamma(1+\alpha)}$, $\quad A_1 = 3(h_0(x))^2 h_1(x) t^{\alpha}$,
therefore, the second component $u_2(x,t)$ is given from Eq.(44) as follows

$$\begin{aligned} u_2(x,t) &= L^{-\alpha}\Big[\frac{\partial^2 u_1(x,t)}{\partial x^2} - u_1(x,t) + A_1\Big] \\ &= \frac{\alpha \operatorname{sech}^5(x)}{\Gamma(1+\alpha)}(-5 + 4\cosh(2x))\Gamma(1+\alpha)\, {}_2F_1[1, 1+\alpha, 2(1+\alpha), 1] t^{2\alpha}. \end{aligned} \tag{45}$$

So, the solution can be approximated by $\phi_n(x,t)$ as defined in (27)

$$\begin{aligned} \phi_2(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) \\ &= -\operatorname{sech}(x) + \frac{\operatorname{sech}^3(x) t^{\alpha}}{\Gamma(1+\alpha)} \\ &+ \frac{\alpha \operatorname{sech}^5(x) t^{2\alpha}}{\Gamma(1+\alpha)}(-5 + 4\cosh(2x))\Gamma(1+\alpha)\, {}_2F_1[1, 1+\alpha, 2(1+\alpha), 1]. \end{aligned}$$

In order to obtain the numerical solutions for Eq.(40) using ADM, we compute the first components of the solution as follows

$$u_0(x,t) = -\mathrm{sech}(x),$$

$$u_1(x,t) = \frac{\mathrm{sech}^3(x)}{\Gamma(1+\alpha)} t^{\alpha},$$

$$u_2(x,t) = \frac{4^{-\alpha}\sqrt{\pi}(-5+4\cosh(2x))\mathrm{sech}^5(x)}{\alpha\Gamma(\alpha)\Gamma(\frac{1}{2}+\alpha)} t^{2\alpha}.$$

So, the solution can be approximated by $\phi_n(x,t)$ as defined in (27)

$$\begin{aligned}\phi_2(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t)\\ &= -\mathrm{sech}(x) + \frac{\mathrm{sech}^3(x)}{\Gamma(1+\alpha)} t^{\alpha} + \frac{4^{-\alpha}\sqrt{\pi}(-5+4\cosh(2x))\mathrm{sech}^5(x)}{\alpha\Gamma(\alpha)\Gamma(\frac{1}{2}+\alpha)} t^{2\alpha}.\end{aligned}$$

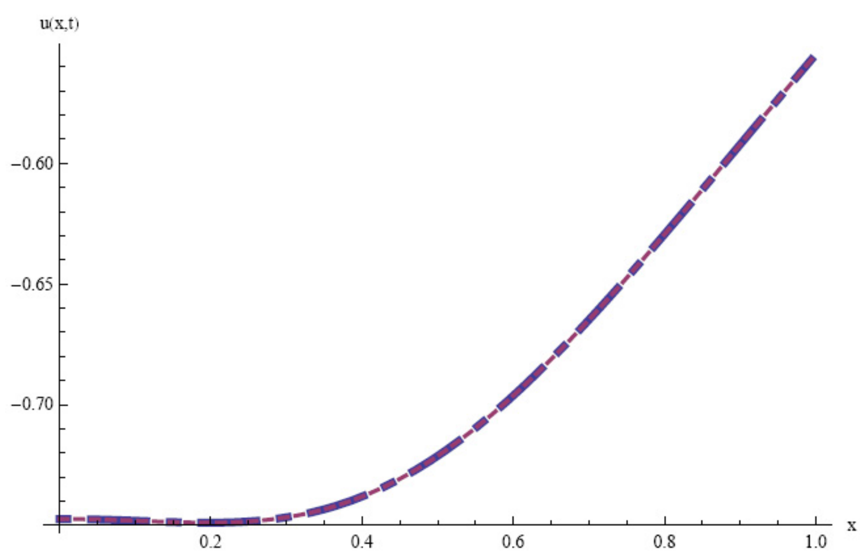

Figure 4.1: The IADM solution, $\phi_2(x,t)$, (dot-dashed line), ADM* solution (dashed line) and ADM solution (dot line) at $\alpha = 1.0$.

The behavior of the approximate solutions using our modified method IADM compared with ADM and approximate solution using ADM* is given in figures 4.1 and 4.2 at $\alpha = 1$ and $\alpha = 0.75$, respectively, with $t = 0.25$.

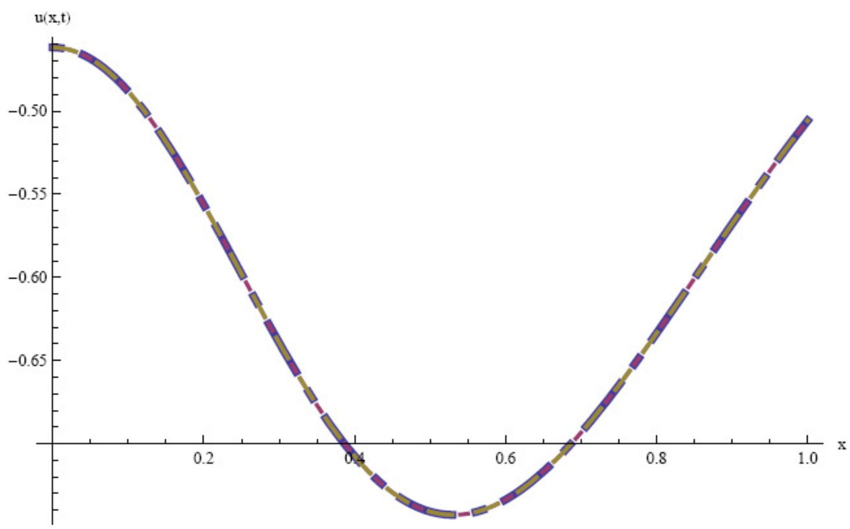

Figure 4.2: The IADM solution $\phi_2(x,t)$ (dot-dashed line), ADM* solution (dashed line) and ADM solution (dot line) at $\alpha = 0.75$.

$\alpha = 1.0$				$\alpha = 0.75$			
x	$\frac{\|\|u_1\|\|}{\|\|u_0\|\|}$	$\frac{\|\|u_2\|\|}{\|\|u_1\|\|}$	$\frac{\|\|u_3\|\|}{\|\|u_2\|\|}$	x	$\frac{\|\|u_1\|\|}{\|\|u_0\|\|}$	$\frac{\|\|u_2\|\|}{\|\|u_1\|\|}$	$\frac{\|\|u_3\|\|}{\|\|u_2\|\|}$
0.00	0.577350	0.288675	2.501850	0.00	0.688153	0.437259	4.70912
0.25	0.542718	0.132829	1.127400	0.25	0.646874	0.201198	2.74185
0.50	0.454056	0.266150	3.321530	0.50	0.541196	0.403140	5.40920
0.75	0.344439	0.759426	1.716570	0.75	0.410542	1.150310	2.88463
1.00	0.242472	1.218280	0.690909	1.00	0.289007	1.845330	1.16639

Table 4.1: The convergence analysis of the truncated solution using IADM of fractional Klein-Gordan equation at different values of x at $t = 0.75$.

$\alpha = 1.0$				$\alpha = 0.75$			
x	$\frac{\|\|u_1\|\|}{\|\|u_0\|\|}$	$\frac{\|\|u_2\|\|}{\|\|u_1\|\|}$	$\frac{\|\|u_3\|\|}{\|\|u_2\|\|}$	x	$\frac{\|\|u_1\|\|}{\|\|u_0\|\|}$	$\frac{\|\|u_2\|\|}{\|\|u_1\|\|}$	$\frac{\|\|u_3\|\|}{\|\|u_2\|\|}$
0.00	0.577350	0.288675	2.50185	0.00	0.688153	0.437259	4.70912
0.25	0.542718	0.132829	1.12740	0.00	0.646874	0.201198	2.74185
0.50	0.454056	0.266150	3.32153	0.50	0.541196	0.403140	5.40920
0.75	0.344439	0.759426	1.71657	0.75	0.410542	1.150310	2.88463
1.00	0.242472	1.218280	0.69091	1.00	0.289007	1.845330	1.16639

Table 4.2: The convergence analysis of the truncated solution using ADM of fractional Klein-Gordan equation at different values of x at $t = 0.75$.

From our obtained results, we can conclude that the solutions using IADM are in excellent agreement with the exact solution, see examples 2 and 3, and satisfy the convergence analysis as described in Theorem 4.1. Also, the proposed method IADM is applicable to solve a wide range of fractional differential equations.

6 Conclusion and remarks

In this article, we constructed the approximate solutions of non-linear differential equations of high fractional order. The main goal of this work has been achieved by introducing an extension of Adomian's polynomials to treatment Adomian decomposition method using the modified Riemann-Liouville fractional derivative. From the introduced comparison we can observe the following notes

1. When (s=1) in Eq.(16), the results using IADM are the same as ADM* results as we saw in the examples of non-linear fractional Fisher and Klein-Gordan equations and more accurate and convergent than ADM results;

2. When (s=2,3,...) in Eq.(16) the results using IADM are more accurate and convergent than ADM* and ADM results as we saw from examples 1 and 2;

3. Our treatment using IADM eliminates the calculations of all inaccurate terms in ADM which deteriorate the convergence rate;

4. The series solution converges slowly as we are far from the initial values but its convergence is acceptable near the initial values.

We support our treatment with some figures to illustrate its accuracy and efficiency and we used Mathematica program to preform the calculations of the included examples.

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