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On Submanifolds of an Almost r-Paracontact Riemannian Manifold Endowed with a Quarter Symmetric Metric Connection

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Abstract

We define a quarter symmetric metric connection in an almost $r-$paracontact Riemannian manifold and we consider submanifolds of an almost $r-$paracontact Riemannian manifold endowed with a quarter symmetric metric connection. We also obtain Gauss and Codazzi equations, Weingarten equation and curvature tensor for submanifolds of an almost $r-$paracontact Riemannian manifold endowed with a quarter symmetric metric connection.

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1 Introduction

In [1], R. S. Mishra studied almost complex and almost contact submanifolds. In [2], S. Ali and R. Nivas considered submanifolds of a Riemannian manifold with quarter symmetric connection. Some properties of submanifolds of a Riemannian manifold with quarter symmetric semi-metric connection were studied in [3] by L. S. Das. Moreover, in [4], I. Mihai and K. Matsumoto studied submanifolds of an almost $r-$paracontact Riemannian manifold of $P-$Sasakian type.

Let ∇ be a linear connection in an $n-$dimensional differentiable manifold M. The torsion tensor T and the curvature tensor R of ∇ are given respectively by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The connection ∇ is symmetric if its torsion tensor T vanishes, otherwise it is non-symmetric. The connection ∇ is metric connection if there is a Riemannian metric g in M such that $\nabla g = 0,$ otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if it is the Levi-Civita connection.

In [5], S. Golab introduced the idea of a quarter-symmetric linear connection if its torsion tensor T is of the form

$$T(X,Y) = u(Y)\phi X - u(X)\phi Y,$$

where u is a 1-form and ϕ is a tensor field of the type (1,1). In [6], R. S. Mishra and S. N. Pandey considered a quarter symmetric metic connection and studied some of its properties. In [7], [8], [9], [10] and [11], some kinds of quarter symmetric metric connections were studied.

Let M be an $n-$dimensional Riemannian manifold with a positive definite metric g. If there exist a tensor field ϕ of type (1, 1), r vector fields $\xi_1, \xi_2, \xi_3, ..., \xi_r (n > r)$, r 1-forms $\eta^1, \eta^2, \eta^3, ..., \eta^r$ such that

(1.1) $$\eta^\alpha(\xi_\beta) = \delta^\alpha_\beta, \qquad \alpha, \beta \in (r) = \{1, 2, 3, ..., r\}$$

(1.2) $$\phi^2(X) = X - \eta^\alpha(X)\xi_\alpha,$$

(1.3) $$\eta^\alpha(X) = g(X, \xi_\alpha), \quad \alpha \in (r).$$

(1.4) $$g(\phi X, \phi Y) = g(X, Y) - \sum_\alpha \eta^\alpha(X)\eta^\alpha(Y),$$

where X and Y are vector fields on M, then the structure $\sum = (\phi, \xi_\alpha, \eta^\alpha, g)_{\alpha \in (r)}$ is said to be an almost $r-$paracontact Riemannian structure and M is an almost $r-$paracontact Riemannian manifold [7].

From (1.1) through (1.4), we have

(1.5) $$\phi(\xi_\alpha) = 0, \qquad \alpha \in (r)$$

(1.6) $$\eta^\alpha o \phi = 0, \qquad \alpha \in (r)$$

(1.7) $$\phi(X, Y) \overset{def}{=} g(\phi X, Y) = g(X, \phi Y).$$

An almost $r-$ paracontact Riemannian manifold M with structure $\sum = (\phi, \xi_\alpha, \eta^\alpha, g)_{\alpha \in (r)}$ is said to be $s-$paracontact type if

$$\psi(X, Y) = (\overset{*}{\nabla}_Y \eta^\alpha)(X), \qquad \alpha \in (r)$$

On almost $r-$ paracontact Riemannian manifold M with a structure $\sum = (\phi, \xi_\alpha, \eta^\alpha, g)_{\alpha \in (r)}$ is said to be $P-$Sasakian if it also satisfies

$$(\overset{*}{\nabla}_Z \psi)(X, Y) = -\sum_\alpha \eta^\alpha(X)[g(Y, Z) - \sum_\beta \eta^\beta(Y)\eta^\beta(Z)]$$
$$- \sum_\alpha \eta^\alpha(Y)[g(X, Z) - \sum_\beta \eta^\beta(X)\eta^\beta(Z)]$$

for all vector fields X, Y and Z on M [12].

The conditions are equivalent respectively to $\phi X = \overset{*}{\nabla}_X \xi_\alpha$, for all $\alpha \in (r)$ and

$$(\overset{*}{\nabla}_Y \phi)(X) = -\sum_\alpha \eta^\alpha(X)[Y - \eta^\alpha(Y)\xi_\alpha] - [g(X,Y) - \sum_\alpha \eta^\alpha(X)\eta^\alpha(Y)]\sum_\beta \xi_\beta.$$

In this paper, we study quarter symmetric metric connection in an almost $r-$ paracontact Riemannian manifold. We consider hypesurfaces and submanifolds of almost $r-$paracontact Riemannian manifold endowed with a quarter symmetric metric connection. We also obtain Gauss and Codazzi equations for hypersurfaces and curvature tensor and Wiengarten equation for submanifolds of almost $r-$paracontact Riemannian manifold with respect to quarter symmetric metric connection.

2 Preliminaries

Let M^{n+1} be an $(n+1)-$dimensional differentiable manifold of class C^∞ and M^n be the hypersurface in M^{n+1} by the immersion $\tau : M^n \to M^{n+1}$. The differential $d\tau$ of the immersion τ is denoted by B. The vector field X in the tangent space of M^n corresponds to a vector field BX in that of M^{n+1}. Suppose that the enveloping manifold M^{n+1} is an almost $r-$ paracontact Riemannian manifold with metric $\tilde{g}$. Then the hypersurface M^n is also an almost $r-$paracontact Riemannian manifold with induced metric g defined by

$$g(\phi X, Y) = \tilde{g}(B\phi X, BY),$$

where X and Y are the arbitrary vector fields and ϕ is a tensor of type (1,1). If the Riemannian manifolds M^{n+1} and M^n are both orientable, we can choose a unique vector field N defined along M^n such that

$$\tilde{g}(BX, N) = 0$$

and

$$\tilde{g}(N,N)=1$$

for arbitrary vector field N in M^n. We call this vector field the normal vector field to the hypersurface M^n.

We now define a quarter symmetric metric connection $\tilde{\nabla}$ by ([7],[8])

(2.1) $$\tilde{\nabla}_{\tilde{X}}\tilde{Y}=\dot{\tilde{\nabla}}_{\tilde{X}}\tilde{Y}+\tilde{\eta}^{\alpha}(\tilde{Y})\tilde{\phi}\tilde{X}-\tilde{g}(\tilde{\phi}\tilde{X},\tilde{Y})\tilde{\xi}_{\alpha}$$

for arbitrary vector fields $\tilde{X}$ and $\tilde{Y}$ tangents to M^{n+1}, where $\tilde{\nabla}$ denotes the Levi-Civita connection with respect to Riemannian metric $\tilde{g}$, $\tilde{\eta}^{\alpha}$ is a 1-form, $\tilde{\phi}$ is a tensor of type (1,1) and $\tilde{\xi}_{\alpha}$ is the vector field defined by

$$\tilde{g}(\tilde{\xi}_{\alpha},\tilde{X})=\tilde{\eta}^{\alpha}(\tilde{X})$$

for an arbitrary vector field $\tilde{X}$ of M^{n+1}. Also

$$\tilde{g}(\tilde{\phi}\tilde{X},\tilde{Y})=\tilde{g}(\tilde{X},\tilde{\phi}\tilde{Y}).$$

Now, suppose that $\sum=(\tilde{\phi},\tilde{\xi}_{\alpha},\tilde{\eta}^{\alpha},\tilde{g})_{\alpha\in(r)}$ is an almost $r-$ paracontact Riemannian structure on M^{n+1}, then every vector field X on M^{n+1} is decomposed as

$$\tilde{X}=BX+l(X)N,$$

where l is a 1-form on M^{n+1}. For any vector field X on M^n and normal N, we have $b(BX)=b(X)$, $\phi(BX)=B\phi(X)$ and $\eta^{\alpha}(BX)=\eta^{\alpha}(X)$, where b is a 1-form on M^n.

For each $\alpha\in(r)$, we have [8]

(2.2) $$\tilde{\phi}BX=B_{\phi}X+b(X)N,$$

(2.3) $$\tilde{\xi}_{\alpha}=B\xi_{\alpha}+a_{\alpha}N,$$

where ξ_{α} is a vector field and a_{α} is defined as

(2.4) $$a_{\alpha}=m(\xi_{\alpha})=\eta^{\alpha}(N)$$

for each $\alpha\in(r)$ on M^n.

Now, we define $\tilde{\eta}^\alpha$ as

(2.5) $$\tilde{\eta}^\alpha(BX) = \eta^\alpha(X), \qquad \alpha \in (r).$$

Theorem 2.1 *The connection induced on the hypersurface of a Riemannian manifold with a quarter symmetric metric connection with respect to the unit normal is also quarter symmetric metric connection.*

Proof: Let $\dot{\nabla}$ be the induced connection from $\dot{\tilde{\nabla}}$ on the hypersurface with respect to the unit normal N, then we have

(2.6) $$\dot{\tilde{\nabla}}_{BX} BY = B(\dot{\nabla}_X Y) + h(X,Y)N$$

for arbitrary vector fields X and Y on M^n, where h is a second fundamental tensor of the hypersurface M^n. Let ∇ be connection induced on the hypersurface from $\tilde{\nabla}$ with respect to the unit normal N, then we have

(2.7) $$\tilde{\nabla}_{BX} BY = B(\nabla_X Y) + m(X,Y)N$$

for arbitrary vector fields X and Y of M^n, m being a tensor field of type (0, 2) on the hypersurface M^n.

From equation (2.1), we have

$$\tilde{\nabla}_{BX} BY = \dot{\tilde{\nabla}}_{BX} BY + \tilde{\eta}^\alpha(BY)\tilde{\phi}BX - g(\tilde{\phi}BX, BY)(B\xi_\alpha + a_\alpha N).$$

Using (2.5), (2.6) and (2.7) in above equation, we get

(2.8) $$\begin{aligned} B(\nabla_X Y) + m(X,Y)N = {} & B(\dot{\nabla}_X Y) + h(X,Y)N + \eta^\alpha(Y)B\phi Y \\ & + \eta^\alpha(Y)b(X)N - g(\phi X, Y)(B\xi_\alpha + a_\alpha N). \end{aligned}$$

Comparing the tangential and normal vector fields, we get

(2.9) $$\nabla_X Y = \dot{\nabla}_X Y + \eta^\alpha(Y)\phi Y - g(\phi X, Y)\xi_\alpha$$

and

(2.10) $$m(X,Y) = h(X,Y) + \eta^\alpha(Y)b(X) - a_\alpha g(\phi X, Y).$$

Thus,

(2.11) $$\nabla_X Y - \nabla_Y X - [X,Y] = \eta^\alpha(Y)\phi X - \eta^\alpha(X)\phi Y.$$

Hence the connection ∇ induced on M^n is a quarter symmetric metric connection [5]. □

3 Totally umbilical and totally geodesic hypersurfaces

We define $\dot{\nabla}B$ and ∇B respectively by

$$(\dot{\nabla}B)(X,Y)=(\dot{\nabla}_X B)(Y)=(\tilde{\dot{\nabla}}_{BX}BY)-B(\dot{\nabla}_X Y)$$

and

$$(\nabla B)(X,Y)=(\nabla_X B)(Y)=(\tilde{\nabla}_{BX}BY)-B(\nabla_X Y),$$

where X and Y being arbitrary vector fields on M^n.
Then (2.6) and (2.7) take the form

$$(\dot{\nabla}_X B)(Y)=h(X,Y)N$$

and

$$(\nabla_X B)(Y)=m(X,Y)N.$$

These are Gauss equations with respect to induced connection $\dot{\nabla}$ and ∇ respectively.

Let $X_1, X_2, X_3, X_4, \ldots\ldots\ldots, X_n$ be n-orthonormal vector fields, then the function

$$\frac{1}{n}\sum_{i=1}^{n}h(X_i,X_i)$$

is called the mean curvature of M^n with respect to Riemannian connection $\dot{\nabla}$ and

$$\frac{1}{n}\sum_{i=1}^{n}m(X_i,X_i)$$

is called the mean curvature of M^n with respect to the quarter symmetric semi-metric connection ∇.

From this we have following definitions:

Definition 3.1 *The hypersurface* M^n *is called totally geodesic hypersurface of* M^{n+1} *with respect to the Riemannian connection* $\dot{\nabla}$ *if* h *vanishes.*

Definition 3.2 *The hypersurface* M^n *is called totally umbilical with respect to connection* $\dot{\nabla}$ *if* h *is proportional to the metric tensor* g.

We call M^n totally geodesic and totally umbilical with respect to quarter symmetric metric connection ∇ according as the function m vanishes and proportional to the metric g respectively.

Now we have following theorems:

Theorem 3.1 *In order that the mean curvature of the hypersurface* M^n *with respect to* $\dot{\nabla}$ *coincides with that of* M^n *with respect to* ∇, *if and only if the vector field* $\tilde{\xi}_\alpha$ *is tangent to* M^{n+1} *and* M^n *is invariant.*

Proof: In view of (2.10), we have

$$m(X_i, X_i) = h(X_i, X_i) + \eta^\alpha(Y_i) b(X_i) - a_\alpha g(\phi X_i, Y_i)$$

Summing up for $i = 1, 2, 3, ..., n$ and dividing by n, we obtain

$$\frac{1}{n}\sum_{i=1}^{n} m(X_i, X_i) = \frac{1}{n}\sum_{i=1}^{n} h(X_i, X_i),$$

if and only if $\alpha = 0$ and $b = 0$. Hence from (2.3), we have

$$\tilde{\xi}_\alpha = B\xi_\alpha .$$

Thus the vector field $\tilde{\xi}_\alpha$ is tangent to M^{n+1} and M^n is invariant. □

Theorem 3.2 *The hypersurface* M^n *will be totally geodesic with respect to Riemannian connection* $\dot{\nabla}$, *if and only if it is totally geodesic with respect to the quarter symmetric metric connection* ∇ *and* $\eta^\alpha(Y) b(X) - a_\alpha g(\phi X, Y) = 0.$

Proof: The proof follows from (2.10) easily. □

4 Gauss, Weingarten and Codazzi equations

In this section we shall obtain Weingarten equation with respect to the quarter symmetric metric connection $\tilde{\tilde{\nabla}}$. For the Riemannian connection $\tilde{\dot{\nabla}}$, these equations are given by [12]

(4.1) $$\tilde{\dot{\nabla}}_{BX} N = -BHX$$

for any vector field X in M^n, where h is a tensor field of type (1,1) of M^n defined by

(4.2) $$g(HX, Y) = h(X, Y).$$

From equation (2.1) (2.2) and (2.4) we have

(4.3) $$\tilde{\tilde{\nabla}}_{B\tilde{X}} N = \tilde{\dot{\nabla}}_{B\tilde{X}} N + a_\alpha B\phi X - Bb(X)\xi_\alpha .$$

Using (4.1) we have

(4.4) $$\tilde{\tilde{\nabla}}_{BX} N = -BMX,$$

where $$MX = HX - a_\alpha \phi X + b(X)\xi_\alpha$$

for any vector field X in M^n. Equation (4.4) is Weingarten equation.

We shall find equation of Gauss and Codazzi with respect to the quarter symmetric metric connection. The curvature tensor with respect to quarter symmetric metric connection $\tilde{\tilde{\nabla}}$ of M^{n+1} is

(4.5) $$\tilde{\tilde{R}}(\tilde{\tilde{X}}, \tilde{\tilde{Y}}), \tilde{\tilde{Z}} = \tilde{\tilde{\nabla}}_{\tilde{\tilde{X}}} \tilde{\tilde{\nabla}}_{\tilde{\tilde{Y}}} \tilde{\tilde{Z}} - \tilde{\tilde{\nabla}}_{\tilde{\tilde{Y}}} \tilde{\tilde{\nabla}}_{\tilde{\tilde{X}}} \tilde{\tilde{Z}} - \tilde{\tilde{\nabla}}_{[\tilde{\tilde{X}}, \tilde{\tilde{Y}}]} \tilde{\tilde{Z}}.$$

Putting $\tilde{\tilde{X}} = BX$, $\tilde{\tilde{Y}} = BY$, and $\tilde{\tilde{Z}} = BZ$, we have

$$\tilde{\tilde{R}}(BX, BY), BZ = \tilde{\tilde{\nabla}}_{BX} \tilde{\tilde{\nabla}}_{BY} BZ - \tilde{\tilde{\nabla}}_{BY} \tilde{\tilde{\nabla}}_{BX} BZ - \tilde{\tilde{\nabla}}_{[BX, BY]} BZ.$$

By virtue of (2.7), (4.4), and (2.11), we get

(4.6) $$\begin{aligned} \tilde{\tilde{R}}(BX, BY)BZ = B\{&R(X,Y)Z + m(X,Z)MY - m(Y,Z)MX\} \\ &+\{(\nabla_X m)(Y,Z) - (\nabla_Y m)(X,Z) \\ &+ m(\eta^\alpha(Y)\phi X - \eta^\alpha(X)\phi Y)\}N, \end{aligned}$$

where

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

is the curvature tensor of the quarter symmetric metric connection ∇.

Substituting

$$\tilde{R}(\tilde{X},\tilde{Y},\tilde{Z},\tilde{U}) = g(\tilde{R}(\tilde{X},\tilde{Y})\tilde{Z},\tilde{U})$$

and

$$R(X,Y,Z,U) = g(R(X,Y)Z,U)$$

in (4.6), we can easily obtain

$$\begin{aligned} \tilde{R}(BX,BY,BZ,BU) = R(X,Y,Z,U) + m(X,Z)m(Y,U) \\ - m(Y,Z)m(X,U) \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} \tilde{R}(BX,BY,BZ,N) = (\nabla_X m)(Y,Z) - (\nabla_Y m)(X,Z) \\ + m(\eta^\alpha(Y)\phi X - \eta^\alpha(X)\phi Y, Z). \end{aligned} \tag{4.8}$$

Equations (4.7) and (4.8) are the equation of Gauss and Codazzi with respect to the quarter symmetric metric connection.

5 Submanifolds of codimension 2

Let M^{n+1} be an $(n+1)-$ dimensional differentiable manifold of differentiability class C^∞ and M^{n-1} be $(n-1)-$ dimensional manifold immersed in M^{n+1} by immersion $\tau : M^{n-1} \to M^{n+1}$. We denote the differentiability $d\tau$ of the immersion τ by B, so that the vector field X in the tangent space of M^{n-1} corresponds to a vector field BX in that of M^{n+1}. Suppose that M^{n+1} is an almost paracontact Riemannian manifold with metric tensor $\tilde{g}$. Then the submanifold M^{n-1} is also an almost paracontact Riemannian manifold with metric tensor g such that

$$\tilde{g}(B\phi X, BY) = g(\phi X, Y)$$

for any arbitrary vector fields X, Y in M^{n-1} [2].

If the manifolds M^{n+1} and M^{n-1} are both orientable such that

$$\tilde{g}(B\phi X, N_1) = \tilde{g}(B\phi X, N_2) = \tilde{g}(N_1, N_2) = 0$$

and

$$\tilde{g}(N_1, N_1) = \tilde{g}(N_2, N_2) = 1$$

for arbitrary vector field X in M^{n-1} [3].

We suppose that the enveloping manifold M^{n+1} admits a quarter symmetric metric connection ∇ given by [7].

$$\tilde{\nabla}_{\tilde{X}}\tilde{Y} = \tilde{\dot{\nabla}}_{\tilde{X}}\tilde{Y} + \tilde{\eta}^{\alpha}(\tilde{Y})\tilde{\phi}\tilde{X} - \tilde{g}(\tilde{\phi}\tilde{X}, \tilde{Y})\tilde{\xi}_{\alpha}$$

for arbitrary vector fields $\tilde{X}, \tilde{Y}$ in M^{n-1} denotes the Levi–Civita connection with respect to the Riemannian metric $\tilde{g}$, $\tilde{\eta}^{\alpha}$ is a 1-form.

Let us now put

(5.1) $$\tilde{\phi}BX = B\phi X + a(X)N_1 + b(X)N_2,$$

(5.2) $$\tilde{\xi}_{\alpha} = B\xi_{\alpha} + a_{\alpha}N_1 + b_{\alpha}N_2,$$

where $a(X)$ and $b(X)$ are 1-forms on M^{n-1}, ξ_{α} is a vector field in the tangent space on M^{n-1}, and a_{α}, b_{α} are functions on M^{n-1} defined by

(5.3) $$\tilde{\eta}^{\alpha}(N_1) = a_{\alpha} \quad , \qquad \eta^{\alpha}(N_2) = b_{\alpha}.$$

Theorem 5.1 *The connection induced on the submanifold M^{n-1} of co-dimension two of the Riemannian manifold M^{n+1} with quarter symmetric metric connection ∇ is also quarter symmetric metric connection.*

Proof: Let $\dot{\nabla}$ be the connection induced on the submanifolds M^{n-1} from the connection $\tilde{\dot{\nabla}}$ on the enveloping manifold with respect to unit normals N_1 and N_2, then we have [1]

(5.4) $$\tilde{\dot{\nabla}}_{BX} BY = B(\dot{\nabla}_X Y) + h(X,Y)N_1 + k(X,Y)N_2$$

for arbitrary vector fields X, Y of M^{n-1} where h and k are second fundamental tensors of M^{n-1}. Similarly, if ∇ be connection induced on M^{n-1} from the quarter symmetric metric connection $\tilde{\nabla}$ on M^{n-1}, we have

(5.5) $$\tilde{\nabla}_{BX} BY = B(\nabla_X Y) + m(X,Y)N_1 + n(X,Y)N_2$$

m and n being tensor fields of type (0, 2) of the submanifold M^{n-1}.

In view of equation (2.1), we have

$$\tilde{\nabla}_{BX} BY = \tilde{\dot{\nabla}}_{BX} BY + \tilde{\eta}^{\alpha}(BY)\tilde{\phi}BX - \tilde{g}(\tilde{\phi}BX, BY)\tilde{\xi}_{\alpha}.$$

In view of (5.1), (5.2) and (5.5), we find

(5.6) $$\begin{aligned} B(\nabla_X Y) + m(X,Y)N_1 + n(X,Y)N_2 &= B(\dot{\nabla}_X Y) + h(X,Y)N_1 \\ &+ k(X,Y)N_2 + \eta^{\alpha}(Y)(B_{\phi}X + a(X)N_1 + b(X)N_2) \\ &- g(\phi X,Y)(B\xi_{\alpha} + a_{\alpha}N_1 + b_{\alpha}N_2), \end{aligned}$$

where $\tilde{\eta}^{\alpha}(BY) = \tilde{\eta}^{\alpha}(Y)$ and $g(B\phi X, BY) = g(\phi X, Y)$.

Comparing tangential and normal vector fields to M^{n-1}, we get

(5.7) $$\nabla_X Y = \dot{\nabla}_X Y + \eta^{\alpha}(Y)\phi X - g(\phi X, Y)\xi_{\alpha},$$

(5.8) (a) $$m(X,Y) = h(X,Y) + a(X)\eta^{\alpha}(Y) - a_{\alpha}g(\phi X, Y),$$

(b) $$n(X,Y) = k(X,Y) + b(X)\eta^{\alpha}(Y) - b_{\alpha}g(\phi X, Y).$$

Thus,

(5.9) $$\nabla_X Y - \nabla_Y X - [X,Y] = \eta^{\alpha}(Y)\phi X - \eta^{\alpha}(X)\phi Y.$$

Hence the connection ∇ induced on M^{n-1} is quarter symmetric metric connection. □

6 Totally umbilical and totally geodesic submanifolds

Let $X_1, X_2, X_3, \ldots\ldots\ldots\ldots X_{n-1}$ be (n-1)-orthonormal vector fields on the submanifold M^{n-1}. Then the function

$$\frac{1}{2(n-1)}\sum_{i=1}^{n-1}\{h(X_i, X_i)+k(X_i, X_i)\}$$

is mean curvature of M^{n-1} with respect to the Riemannian connection $\dot{\nabla}$ and

$$\frac{1}{2(n-1)}\sum_{i=1}^{n-1}\{m(X_i, X_i)+n(X_i, X_i)\}$$

is the mean curvature of M^{n-1} with respect to ∇ [3].

Now we have the following definitions:

Definition 6.1 *If h and k vanish separately, the submanifold M^{n-1} is called totally geodesic with respect to the Riemannian connection $\dot{\nabla}$.*

Definition 6.2 *The submanifold M^{n-1} is called totally umbilical with respect to the $\dot{\nabla}$ if h and k are proportional to the metric g.*

We call M^{n-1} totally geodesic and totally umbilical with respect to the quarter symmetric metric connection ∇ according as the functions m and n vanish separately and are proportional to metric tensor g respectively.

Theorem 6.1 *The mean curvature of M^{n-1} with respect to the Riemannian connection $\dot{\nabla}$ coincides with that of M^{n-1} with respect to the quarter symmetric metric connection ∇ if and only if*

$$\sum_{i=1}^{n}\{\eta^{\alpha}(X_i)(a(X_i)+b(X_i))-(a_{\alpha}+b_{\alpha})g(\phi X_i, X_i)\}=0.$$

Proof: In view of (5.8), we have

$$m(X_i, X_i)+n(X_i, X_i)=h(X_i, X_i)+k(X_i, X_i)+k(X_i, X_i)$$
$$-(a_{\alpha}+b_{\alpha})g(\phi X_i, X_i)+\eta^{\alpha}(X_i)(a(X_i)+b(X_i)).$$

Summing up for $i=1,2,........,(n-1)$ and dividing by $2(n-1)$, we get

$$\frac{1}{2(n-1)}\sum_{i=1}^{n-1}\{m(X_i, X_i)+n(X_i, X_i)\}=\frac{1}{2(n-1)}\sum_{i=1}^{n-1}\{h(X_i, X_i)+k(X_i, X_i)\}$$

if and only if $$\sum_{i=1}^{n}\{\eta^{\alpha}(X_i)(a(X_i)+b(X_i))-(a_\alpha+b_\alpha)g(\phi X_i,X_i)\}=0,$$

which proves our assertion.

Theorem 6.2 *The submanifold* M^{n-1} *is totally geodesic with respect to the Riemannian connection* $\dot{\nabla}$ *if and only if it is totally geodesic with respect to the quarter symmetric metric connection* ∇ *provided that*

$$a(X)\eta^{\alpha}(Y)-a_\alpha g(\phi X,Y)=0$$

and

$$b(X)\eta^{\alpha}(Y)-b_\alpha g(\phi X,Y)=0.$$

Proof: The proof follows easily from equations (5.8) (a) and (b). □

7 Curvature tensor and Weingarten equation

For the Riemannian connection $\dot{\nabla}$, the Weingarten equations are given by [1]

(7.1) (a) $$\dot{\nabla}_{BX}N_1=-BHX+1(X)N_2$$

and

(b) $$\dot{\nabla}_{BX}N_2=-BKX+1(X)N_1$$

where H and K are tensor fields of type (1,1) such that

$$g(HX,Y)=h(X,Y)$$

and

$$g(KX,Y)=k(X,Y).$$

Also, making use of (2.1) and (7.1) (a), we get

$$\tilde{\nabla}_{BX}N_1=-B(HX-a_\alpha\phi X+a(X)\xi_\alpha)+(a_\alpha b(X)-b_\alpha a(X)+1(X))N_2.$$

(7.3) $$\tilde{\nabla}_{BX}N_1=-BM_1X+L(X)N_2,$$

where

$$M_1 X = HX - a_\alpha \phi X + a(X)\xi_\alpha$$

and

$$L(X) = a_\alpha b(X) - b_\alpha a(X) + 1(X).$$

Similarly, from (2.1) and (7.1) (b), we obtain

(7.4) $$\tilde{\nabla}_{BX} N_2 = -BM_2 X - L(X) N_1,$$

where

$$M_2 X = KX - b_\alpha \phi X + b(X)\xi_\alpha.$$

Equations (7.3) and (7.4) are Weingarten equations with respect to the quarter symmetric metric connection.

8 Riemannian curvature tensor for quarter symmetric metric connection

Let $\tilde{R}(\tilde{X},\tilde{Y})\tilde{Z}$ be the Riemannian curvature tensor of the enveloping manifold M^{n+1} with respect to the quarter symmetric metric connection $\tilde{\nabla}$, then

$$\tilde{R}(\tilde{X},\tilde{Y})\tilde{Z} = \tilde{\nabla}_{\tilde{X}}\tilde{\nabla}_{\tilde{Y}}\tilde{Z} - \tilde{\nabla}_{\tilde{Y}}\tilde{\nabla}_{\tilde{X}}\tilde{Z} - \tilde{\nabla}_{[\tilde{X},\tilde{Y}]}\tilde{Z}.$$

Replacing $\tilde{X}$ by BX, $\tilde{Y}$ by BY and $\tilde{Z}$ by BZ, we get

$$\tilde{R}(BX,BY)BZ = \tilde{\nabla}_{BX}\tilde{\nabla}_{BY}BZ - \tilde{\nabla}_{BY}\tilde{\nabla}_{BX}BZ - \tilde{\nabla}_{[BX,BY]}BZ.$$

Using (7.3), we obtain

$$\begin{aligned}\tilde{R}(BX,BY)BZ = {} & \tilde{\nabla}_{BX}\{B(\nabla_Y Z) + m(Y,Z)N_1 + n(Y,Z)N_2\} \\ & - \tilde{\nabla}_{BY}\{B(\nabla_X Z) + m(X,Z)N_1 + n(X,Z)N_2\} \\ & - \{B(\nabla_{[X,Y]}Z) + m([X,Y],Z)N_1 + n([X,Y],Z)N_2\}.\end{aligned}$$

Again using (5.5), (7.3), (7.4) and (5.9), we find

$$\begin{aligned}\tilde{R}(BX,BY)BZ &= BR(X,Y,Z)+B\{m(X,Z)M_1Y-m(Y,Z)M_1X\\&+n(X,Z)M_2Y-n(Y,Z)M_2X\}+m\{\eta^\alpha(Y)\phi X\\&-\eta^\alpha(X)\phi Y,Z\}N_1+n\{\eta^\alpha(Y)\phi X-\eta^\alpha(X)\phi Y,Z\}N_2\\&+\{(\nabla_X m)(Y,Z)-(\nabla_Y m)(X,Z)\}N_1+\{(\nabla_X n)(Y,Z)\\&-(\nabla_Y n)(X,Z)\}N_2+L(X)\{m(Y,Z)N_2-n(Y,Z)N_1\}\\&-L(Y)\{m(X,Z)N_2-n(X,Z)N_1\},\end{aligned}$$

where $R(X,Y,Z)$ being the Riemannian curvature tensor of the submanifold with respect to the induced connection ∇ on M^n.

References

[1] Mishra, R. S., Almost complex and almost contact submanifold, *Tensor*, **25**, (1972), 419-433.

[2] Ali, S. and Nivas, R., On submanifolds of a manifold with quarter symmetric connection, *Riv. Mat. Uni. Parma*., **6**(3), (2000), 11-23.

[3] Das, L. S., Nivas, R., Ali, S. and Ahmad, M., Study of submanifolds immersed in a manifold with quarter symmetric semi-metric connection, *Tensor, N. S.*, **65**, (2004), 250-260.

[4] Mihai, I. and Matsumoto, K., Submanifolds of an almost r-paracontact manifold of P– Sasakian type, *Tensor, N. S.*, **48** (2), (1989), 136-142.

[5] Golab, S., On semi-symmetric and quarter-symmetric linear connections, *Tensor*, **29**, (1975), 249-254.

[6] Mishra, R. S and Pandey, S. N., On quarter symmetric metric F-connections, *Tensor, N. S.*, **34**, (1980), 1-7.

[7] Ahmad, M., Jun, J. B. and Haseeb, A., Hypersurfaces of almost r-paracontact Riemannian manifold endowed with quarter symmetric metric connection, *Bull. Korean Math. Soc.*, **46**(3), (2009), 477-487.

[8] Ahmad, M., Ozgur, C. and Haseeb, A., Hypersurfaces of almost r-paracontact Riemannian manifold endowed with a quarter symmetric non-metric connection, *KYUNGPOOK Math, J.*, **49**, (2009), 533-543.

[9] Nivas, R. and Verma, Geeta, On quarter-symmetric non-metric connection in a Riemannian manifold, *J. Rajasthan Acad. Phys. Sci.*, **4**(1), (2005), 57-68.

[10] Pujar, S. S., A note on Lorentzian para Sasakian manifold with an LP-contact non-metric quarter symmetric φ-connection, *Journal of the Indian Math. Soc*., **70**, (2003), 1-4.

[11] Tripathi, M. M., A new connection in a Riemannian manifold, *Int. Elec. J. Geo*. **1**(1), (2008), 15-24.

[12] Chen, B.Y., *Geometry of submanifolds*, Marcel Dekker, *New York*, 1973.