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Hidden properties of the Navier-Stokes equations. Double solutions. Origination of turbulence.

L.I. Petrova[1]

Abstract

The Navier-Stokes equations, which describe flows of fluids and gases, possess hidden properties that are discovered when studying the consistency of the conservation law equations involved into the set of Navier-Stokes equations. Under such an investigation one obtains a nonidentical evolutionary relation for entropy as a state functional. This relation discloses peculiarities of the solutions to the Navier-Stokes equations due to which the Navier-Stokes equations can describe not only the change of physical quantities (such as energy, pressure, density) but also processes such as origination of waves, turbulent pulsations.

From the evolutionary relation it follows that the Navier-Stokes equations possess solutions of two types, namely, the solution that is not a function and the solution that is a discrete function. The solutions of the first type are defined on nonintegrable manifold (like a tangent one) and describe the non-equilibrium state of a flow. And the solutions of the second type are defined on integrable structure and describe the locally equilibrium state of a flow. The transition from the solutions of the first type to ones of the second type describes the process of origination of turbulence.

[1] Moscow State University, Russia. E-mail: ptr@cs.msu.su.

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These results are obtained due to the skew-symmetric forms the basis of which are nonintegrable manifolds.

1 Introduction

As it is known, the Navier-Stokes equations describe a flow of a fluid or a gas. They are comprised of the conservation law equations for energy, momentum and mass.

The Navier-Stokes equations usually are used for description of physical quantities (such as energy, pressure, density).

But, the Navier-Stokes equations posses the specific properties that enable to describe not only the change of physical quantities but also processes such as a nonequilibrium, transitions to the state of locally equilibrium, origination of various structures and formations such as waves, vortices, turbulent pulsations and so on.

The problem of realization of these possibilities of Navier-Stokes equations consists in the fact that these properties of Navier-Stokes equations are hidden ones, since they do not directly follow from Navier-Stokes equations.

They are discovered when analyzing the consistency of the conservation law equations involved into the set of Navier-Stokes equations.

The hidden properties are related to the peculiarities of the functions that describe physical quantities. Since the functions desired relate to a one material medium (flow), it has to exist a connection between them. This connection is described by state functional that specifies the material medium state.

When analyzing the consistency of the conservation law equations involved into the set of Navier-Stokes equations, one obtains the evolutionary relation for entropy, which is a state functional that specifies a state flow.

This relation for the state functional discloses hidden properties of the Navier-Stokes equations and specific features of their solutions.

The evolutionary relation, which contains the differential of the entropy and the evolutionary skew-symmetric form depended on the characteristics of flows and external actions, turns out to be a nonidentical one.

From the nonidentical relation it follows that *the equations that made up the set of Navier-Stokes equations (also the derivatives obeying the Navier-Stokes equations) appear to be inconsistent.* This points to the fact that the tangent manifold of the Navier-Stokes equations and the accompanying manifold (a manifold made up by the particle trajectories) proofs to be nonintegrable. This implyies that corresponding solutions to the Navier-Stokes equations are not functions (their derivatives do not form a differential). Such solutions, which are defined on nonintegrable manifolds, describe the non-equilibrium state of a flow. This follows from the evolutionary relation. Since the relation is nonidentical, from that one cannot obtain the state functional (entropy), and this points out to the absence of the state function and the non-equilibrium state of a flow.

Further, nonidentical relation describes the process of going to agreement of the equations made up the set of the Navier-Stokes equations and obtaining an internal consistency inherent the material medium under consideration.

From the nonidentical relation it follows that, if flows of fluids and gases (as material medium) possesses any degrees of freedom, it can realize the conditions (the conditions of the degenerate transformation) under which the integrable structure with interior differential is realized. In this case, the so-called generalized solution, which is a discrete function, will be the solution to the Navier-Stokes equations.

Under degenerate transformation the identical relation can be obtained from the nonidentical relation.

From the identical relation one can obtain the state function, and this fact will point out to the transition of a flow into the locally equilibrium state.

The transition from the solution that is not a function to the generalized solution (a transition from nonintegrable manifold to integrable structure) describes the transition of flow gas or fluid from the non-equilibrium state to the locally equilibrium state. Such transition is accompanied by origination of observable formation. The turbulent pulsations is an example of such formations. This discloses a mechanism of the turbulence origination.

Such peculiarities of the Navier-Stokes equations, as it follows from the evolutionary relation, are connected with the properties of conservation laws that appear to be noncommutative.

It should be emphasized that the specific properties of the Navier-Stokes equations are those that are inherent the mathematical physics equations which describe material media (systems) like the thermodynamical, gas-dynamic, cosmological systems, the systems of charged particles, and so on. The evolutionary relation obtained from the equations for such material systems is a relation for such functionals as the action functional, entropy, the Pointing vector, the Einstein tensor, and so on.

The application of the mathematical apparatus of skew-symmetric differential forms enables to disclose peculiarities of the solutions to the Navier-Stokes equations and physical meaning of these solutions. In doing so, the skew-symmetric differential forms, which basis is *nonintegrabe* manifolds, were used in addition to exterior skew-symmetric forms. Such skew-symmetric forms, which are obtained (as it was established by the author) from differential equations, possess a nontraditional mathematical apparatus, which includes nonidentical relations and degenerate transformations, and this fact enables to describe discrete transitions [1,2].

Such a mathematical formalism, which non of mathematical formalisms possess, enables to describe processes like the origination of physical structures and observable formations and the turbulence origination.

2 Studying an integrability of the Navier-Stokes equations. Functional and physical properties of the solutions

The investigation of the integrability of Navier-Stokes equations, which depends on the conjugacy of the derivatives obeying the Navier-Stokes equations and on the consistency of the conservation law equations involved into the set of Navier-Stokes equations, enables to understand peculiarities of the solutions to Navier-Stokes equations.

From the Navier-Stokes equations it does not follows (directly) that the derivatives obeying the equation have to be conjugacy. And, since these equations describe actual processes (with nonpotential characteristics), this points out that the tangent manifold of the Navier-Stokes equations is nonintegrable.

Derivatives of such a manifold are not consistent do not made up a differential. This means that the solutions of the Navier-Stokes equations obtained from these derivatives cannot be functions. About the properties of such solutions and generalized solutions (which are the discrete functions) will be said below.

When investigating the integrability of Navier-Stokes equations, the emphasis will be on the analysis of the consistency of the conservation law equations that made up the set of the Navier-Stokes equations. This analysis enables not only to study the integrability of the Navier-Stokes equations, but it also enables to understand the mechanism of evolutionary processes that lead to the turbulence origination.

2.1 Analysis of consistency of the conservation law equations. Evolutionary relation for the state functional

Integrability and the properties of solutions of the Navier-Stokes equations will be investigated for the case of gas-dynamic system, namely, flow of a viscid heat-conducting gas. In addition, the Euler equations, which describe a flow of ideal (inviscid) gas and solutions of which possess the same properties as the solutions of the Navier-Stokes equations will be investigated.

It is known that the Navier-Stokes equations (and Euler equations) are a set of the conservation laws energy, linear momentum and mass [3].

Let us now analyze the consistency of the equations for energy and linear momentum.

We introduce two frames of reference: the first is an inertial one and the second is an accompanying one that is connected with the manifold made up by the trajectories of elements of a gas-dynamic system. (The Euler and Lagrangian coordinate systems cam be regarded as examples of such frames of reference.)

In the inertial frame of reference the energy equation can be reduced to the form:

$$\frac{Dh}{Dt} - \frac{1}{\rho}\frac{Dp}{Dt} = A_1 \tag{1}$$

where D/Dt is the total derivative with respect to time, $\rho = 1/V$ and h are respectively the density and enthalpy of the gas. A_1 is an expression that depends on the flow characteristics and energetic actions.

In the case of viscous heat-conducting gas described the Navier-Stokes equations the expression A_1 can be written as (see [3], Chapter 6, formula (6.2.4))

$$A_1 = \frac{1}{\rho}\frac{\partial}{\partial x_i}\left(-\frac{q_i}{T}\right) - \frac{q_i}{\rho T}\frac{\partial T}{\partial x_i} + \frac{\tau_{ki}}{\rho}\frac{\partial u_i}{\partial x_k} \tag{2}$$

Here q_i is the heat flux and τ_{ki} is the viscous stress tensor.

In the case of ideal gas described by the Euler equations we have $A_1 = 0$.

Expressing enthalpy in terms of internal energy e with the help of formula $h = e + p/\rho$ and using the thermodynamic relation $Tds = de + pdV$, equation (1) of the conservation law for energy can be reduced to the form

$$\frac{Ds}{Dt} = A_1 \tag{3}$$

Here s is the entropy.

Since the total derivative with respect to time is that along the trajectory, in the accompanying frame of reference *the equation of the conservation law for energy* takes the form:

$$\frac{\partial s}{\partial \xi^1} = A_1 \tag{4}$$

where ξ^1 is the coordinate along the trajectory.

In the accompanying frame of reference *the equation of conservation law for linear momentum* can be presented as

$$\frac{\partial s}{\partial \xi^\nu} = A_\nu \tag{5}$$

where ξ^ν is the coordinate in the direction normal to the trajectory. In the case of two-dimensional flow of ideal gas one can obtain the following expression for the coefficient A_ν (see [3], Chapter 6, formula (6.7.12)):

$$A_\nu = \frac{\partial h_0}{\partial \nu} + ({u_1}^2 + {u_2}^2)^{1/2}\zeta - F_\nu + \frac{\partial U_\nu}{\partial t} \tag{6}$$

where $\zeta = \partial u_2/\partial x - \partial u_1/\partial y$.

In the case of viscous gas the expression A_ν includes additional terms related to viscosity and heat-conductivity.

One can see that in the accompanying frame of reference the equations for energy and linear momentum are reduced to the equations for derivatives of entropy s. In this case equation (4) obtained from the energy equation defines the derivative of entropy along the trajectory, and equation (5), assigned to the equation for linear momentum, defines the derivatives of entropy in the direction normal to trajectory.

Equations (4) and (5) can be convoluted into the relation

$$ds = \omega \tag{7}$$

where $\omega = A_\mu d\xi^\mu$ is the first degree skew-symmetric differential form and $\mu = 1, \nu$. (A summing over repeated indices is carried out.) Since the conservation law equations are evolutionary ones, the relation obtained is also an evolutionary relation. In this case the skew-symmetric form ω is evolutionary one as well.

Relation (7) has been obtained from the conservation law equation for energy and linear momentum. In this relation the form ω is that of the first degree. Taking into account the conservation law equations for angular momentum and mass, the evolutionary relation may be written as

$$d\psi = \omega^p \tag{8}$$

where the form degree p takes the values $p = 1, 2, 3$.

[Skew-symmetric forms (such as ω, for example), which are obtained from differential equations, are defined on nonintegrable (accompanying) manifolds as opposed to exterior forms, which are defined on integrable manifolds or structures. Such skew-symmetric forms which are evolutionary ones, possess the properties that enable one to investigate differential equations [1]. From those one can obtain closed inexact exterior forms, which are invariants and describe physical structures. This gives a possibility to understand the mechanism of origination of various physical structures [2]].

The evolutionary relation (7) possesses the properties that enable one to investigate the integrability of the Navier-Stokes equations and the properties of their solutions. (Relation (8) possesses the same properties.)

2.2 Nonidentity of the evolutionary relation. Inconsistency of the conservation law equations made up the set of Navier-Stokes equations

Evolutionary relation (7) has a certain peculiarity. This relation appears to be nonidentical. This relates to the fact that this relation involves the skew-symmetric differential form ω, which is unclosed and cannot be a differential like the left-hand side of this relation. The evolutionary form ω is not closed since the differential of evolutionary form ω and its commutator are nonzero.

The differential of evolutionary form ω is expressed as $d\omega = \sum K_{1\nu} d\xi^1 d\xi^\nu$, where $K_{1\nu}$ are components of the form commutator. Without accounting for terms that are connected with the deformation of the manifold made up by the trajectories, the commutator can be written as

$$K_{1\nu} = \frac{\partial A_\nu}{\partial \xi^1} - \frac{\partial A_1}{\partial \xi^\nu} \tag{9}$$

The coefficients A_μ of the form ω have been obtained either from the equation of the conservation law for energy or from that for linear momentum. This means that in the first case the coefficients depend on the energetic action and in the second case they depend on the force action. In actual processes energetic and force actions have different nature and appear to be inconsistent. The commutator of the form ω constructed of the derivatives of such coefficients is nonzero. Since the commutator of the form ω is nonzero, this means that the differential of the form ω is nonzero as well. Thus, the form ω proves to be unclosed and is not a differential. In the left-hand side of relation (7) it stands a differential, whereas in the right-hand side it stands an unclosed form that is not a differential. Such a relation cannot be an identical one.

The nonidentity of the evolutionary relation points to the fact that the conservation law equations for energy and linear momentum (entered into the set of Navier-Stokes equations) appear to be inconsistent. (It should be underlined that one has to take into account the consistency of the conservation law equations for energy and linear momentum with the equation of conservation law for mass.)

The integrability of the Navier-Stokes equations and the properties of their solutions depend on the inconsistency of the equations made up the set of Navier-Stokes equations.

Here the attention should be called to the following. Since the conservation law equations are inconsistent, this means that the conservation laws are noncommutative. Below it will be shown that the peculiarities of the Navier-Stokes equations reflect an influence of the noncommutativity of conservation laws conservation laws on the processes proceeding in a gas-dynamic system.

2.3 Inexact solutions to the Navier-Stokes equations (the solutions that are not functions)

Since the equations made up the set of Navier-Stokes equations are inconsistent, they cannot be contracted into an identical relation (which is built by differentials) and integrated directly. This means that that the solutions to equations are not functions, which depend only on variables.

They will depend on a commutator of the form ω which enters into the evolutionary relation. (If the commutator be equal to zero, the evolutionary relation would be identical and the equations would be integrated directly).

(Hereafter these solutions will be referred to as the solutions of the first type or "inexact solutions". However, one has to keep in mind that these solutions are not approximate ones. Inexactness is related to the fact that they cannot be represented analytically because they are not functions. Inexact solutions describe the quantities that are not the inherent quantities of a gas-dynamic system. The inconsistency of these quantities, as will be said below, brings a system into a non-equilibrium state).)

2.4 Physical meaning of inexact solutions. Non-equilibrium state of a gas-dynamic system

Inexact solutions have a physical meaning. They describe a nonequilibrium state of gas-dynamic system. This follows from the evolutionary relation.

Evolutionary relation (7) has an unique physical meaning because this relation includes a differential of entropy s, which is a state functional. The entropy entered into the evolutionary relation is the functional, which characterizes the state of gas-dynamic system. (Here, it should be called attention to the fact that the entropy, which enters into the evolutionary relation for a gas-dynamic system, depends on space-time coordinates rather then on thermodynamical variables like the entropy entered into the thermodynamical relation. The state of gas-dynamic system is characterized by the entropy, which depends on space-time variables. And the entropy that depends on thermodynamical variables characterizes a state of thermodynamic system. In the gas-dynamic system the entropy depended on thermodynamical variables characterizes only the state of a gas rather then the state of gas-dynamic system itself.)

If from relation (7) the differential of entropy could be obtained, this would point to the fact that entropy is a state function. And this would mean that the state of a gas-dynamic system is a equilibrium one.

But, since relation (7) is a nonidentical relation, from that one cannot obtain the differential of entropy and find the state function. This means that the gas-dynamic system is in a non-equilibrium state.

One can see that *the solutions of the Navier-Stokes equations, which are not functions, describe a nonequilibrium state of gas-dynamic system.*

The nonequilibrium means that in a gas-dynamic system an internal force acts. It is evident that the internal force is described by the commutator of skew-symmetric form ω, on which the inexact solutions of the Euler and Navier-Stokes depend. (If the evolutionary form commutator be zero, the evolutionary relation would be identical, and this would point out to the equilibrium state, i.e. the absence of internal forces.) Everything that gives a contribution into the commutator of the evolutionary form ω leads to emergence of internal force that causes the non-equilibrium state of a gas-dynamic system.

From the analysis of the expression A_μ in formulas (2) and (6) one can see that the terms, which are related to the multiple connectedness of the flow domain, the nonpotentiality of the external forces and the nonstationarity of the flow contribute into the commutator (see, formula (6)). In the case of a viscous non-heat-conducting gas, the terms related to the transport processes will contribute to the commutator (see, formula (2)). (In a general case the term related to physical-chemical processes will make a contribution into the commutator.)

One can see that nonequilibrium is caused by not a simple connectedness of the flow domain, nonpotential external (for each local domain of a gas-dynamic system) forces, a nonstationarity of the flow, and transport phenomena. (In common case of the gas-dynamic instability, the thermodynamic, chemical, oscillatory, rotational and translational nonequilibrium will effect).

All these factors lead to emergence of internal forces, that is, to nonequilibrium, and to development of various types of instability. And yet for every type of instability one can find an appropriate term giving contribution into the evolutionary form commutator, which is responsible for this type of instability. Thus, there is an unambiguous connection between the type of instability and the terms that contribute into the evolutionary form commutator in the evolutionary relation. (It can be noted that, for the case of ideal gas, Lagrange derived a condition of the eddy-free stable flow. This condition is as follows: the domain must be simple connected one, forces must be potential and the flow must be stationary. One can see, that, under fulfillment of these conditions, there are no terms that contribute into the commutator).

Here it can be noted that the nonidentity of the relation is connected with a noncommutativity of conservation laws. And this points out to the fact that the noncommutativity of conservation laws is a cause of nonequilibrium state of a gas-dynamic system [4].

Evolutionary relation also describes a variation of non-equilibrium state. This is due to another peculiarity of nonidentical evolutionary relation, namely, this relation is a selfvarying relation.

The manifold, on which the evolutionary form is defined, made up by trajectories of the material medium elements (particles) and, in addition, in the case of non-equilibrium state of a material medium such manifold appears to be a deforming (under the action of internal forces) one. This means that the evolutionary form basis varies. In turn, this leads to variation of the evolutionary form, and the process of intervariation of the evolutionary form and the basis is repeated. Selfvariation of the evolutionary relation goes on by exchange between the evolutionary form coefficients and the manifold characteristics. (This is an exchange between physical quantities and space-time characteristics.) Since one of the objects of evolutionary relation is an unmeasurable quantity, the other cannot be compared with the first one, and hence, the process of mutual variation cannot be terminated. The processes of selfvariation of the evolutionary relation are governed by the evolutionary form commutator. It should be noted that, in addition to the term made up by the skew-symmetric form coefficients, the commutator of metric form of nonintegrable manifold, which is nonzero, enters into the commutator of evolutionary form defined on nonintegrable manifold [1]. (In formula (9), where the expression for the commutator of the evolutionary form ω is presented, this term was not accounted for). The interaction of two terms of the evolutionary form commutator just exerts the process of the evolutionary relation selfvariation.

The process of the evolutionary relation selfvariation describes the process of selfvariation of the gas-dynamic system state. This process proceeds under the internal force action and is described by inexact solutions. As it was already noted, in this case the evolutionary form commutator, on which an inexact solution depends, describes an internal force.

2.5 Realization of exact solutions of the Navier-Stokes equations

The exact solutions of differential equations, which are functions, are allowed only on integrable manifold or on integrable structures.

Since the tangent and accompanying manifolds of the Navier-Stokes equations are nonintegrable ones, the exact solutions of the Navier-Stokes equations are possible only on integrable structures.

Here it may be called attention to the following. Exact solutions describe quantities that are inherent ones of material media (gas-dynamic system), whereas the solutions, that are not functions, are not quantities that are inherent ones of material media, because they depend on the commutator being connected with nonpotential quantities (any external forces).

It turns out that, under additional conditions, from the Navier-Stokes equations integrable structures with exact solutions can be realized. These conclusions follow from the analysis of the nonidentical evolutionary relation.

The Navier-Stokes equations can have exact solutions only in the case if from the evolutionary skew-symmetric form ω in the right-hand side of nonidentical evolutionary relation it is realized a closed skew-symmetric form, which is a differential. (In this case the identical relation is obtained from the nonidentical relation, and this will point out to a consistency of the conservation law equations and an integrability of the Navier-Stokes equations.)

But here there is some delicate point.

From the evolutionary unclosed skew-symmetric form, which differential is nonzero, one can obtain a closed exterior form with a differential being equal to zero only under degenerate transformation, namely, under a transformation that does not conserve differential. (The Legendre transformation is an example of such a transformation.)

Degenerate transformations can take place under additional conditions, which are related with degrees of freedom. The vanishing of such functional expressions as determinants, Jacobians, Poisson's brackets, residues and others corresponds to the additional conditions.

The conditions of degenerate transformation specify the integrable structures (pseudostructures) on which the solutions become exact ones. The characteristics (the determinant of coefficients at the normal derivatives vanishes), the singular points (Jacobian is equal to zero), potentials of simple and double layers, and others are such integrable structures.

The conditions of degenerate transformation can be realized under change of nonidentical evolutionary relation, which, as it was noted, appears to be a selfvarying relation.

If the conditions of degenerate transformation are realized, from the unclosed evolutionary form ω (see evolutionary relation (7)) with nonvanishing differential $d\omega \neq 0$, one can obtain the differential form closed on pseudostructure. The differential of this form equals zero. That is, it is realized the transition

$d\omega \neq 0 \to$ (degenerate transformation) $\to d_\pi \omega = 0$, $d_\pi{}^*\omega = 0$

The realization of the conditions of $d_\pi{}^*\omega = 0$ and $d_\pi \omega = 0$ means that it is realized a closed dual form $^*\omega$, which describes some an integrable structure π, and the closed exterior form ω_π which basis is an integrable structure obtained. (It should be noted an integrable structure is a pseudostructure with respect to its metric properties.)

(As it is known, the form dual to a certain exterior skew-symmetric form describes an integrable manifold, which is the basis of exterior form. In the present case, since the dual form is realized only under additional condition, the dual form describes an integrable structure, which is the basis of closed inexact exterior form.)

Thus, it appears that under degenerate transformation the closed inexact (defined only on pseudostructure) exterior form (with the differential being equal to zero) is realized.

Since the form ω_π is a closed on pseudostructure form, this form turns out to be a differential. (It should be emphasized that such a differential is an interior one: it asserts only on pseudostructure, which is defined by the condition of degenerate transformation).

On the pseudostructure π, which is an integrable structure, from evolutionary relation (7) it is obtained the relation

$$ds_\pi = \omega_\pi \tag{10}$$

which occurs to be an identical one, since the form ω_π is a differential.

Thus, on the pseudostructure, which is an integrable structure, from the evolutionary relation $ds = \omega$ it is obtained the identical relation $ds_\pi = \omega_\pi$.

The identity of the relation obtained from the evolutionary relation means that on the integrable structure realized the equations of conservation laws, which made up the Navier-Stokes equations, become consistent. This points out to that the Navier-Stokes equations become locally integrable (only on integrable structure).

On integrable structures the desired quantities of gas-dynamic system (such as the temperature, pressure, density) become functions of only independent variables and do not depend on the commutator (and on the path of integrating). These are generalized solutions, which are the discrete functions, since they are realized only under additional conditions (on the integrable structures). Such solutions may be found by means of integrating (on integrable structures) the the Navier-Stokes equations.

Since generalized solutions are defined only on realized integrable structures, they or their derivatives have discontinuities in the direction normal to integrable structure [5].

Thus, one can see that the Navier-Stokes equations can have the solutions of two types:

1. the inexact solutions that are not functions, i.e., they depend not only on independent variables, and

2. the generalized solutions, which are the discrete functions.

The specific feature is the fact that the solutions to the Navier-Stokes equations are defined on different spatial objects.

The solutions of the Euler equations, which describe the flows of ideal (inviscid) gas, possess the same properties.

Here the following is noteworthy. The degenerate transformation, under which a closed exterior form is obtained from evolutionary form, is realized as a transition from nonintegrable accompanying manifold (on which the evolutionary form is defined) to the integrable structures with a closed form. Mathematically to this it is assigned a transition from one frame of reference to another nonequivalent frame of reference (from accompanying frame of reference to a locally-inertial on obtained integrable structures).

3 Transition of gas-dynamic system from non-equilibrium state to locally-equilibrium state. Origination of vorticity and turbulence

As it has been shown above, under degenerate transformation the identical relation is obtained from nonidentical one.

From identical relation one can obtain the differential of entropy ds and find entropy s as a function of space-time coordinates. It is precisely the entropy that will be a gas-dynamic function of state. The availability of gas-dynamic function of state would point out to equilibrium state of a gas-dynamic system. However, since the identical relation is satisfied only under additional conditions, such a state of gas-dynamic system will be a locally-equilibrium one.

One can see that the transition from nonidentical relation to identical one points out to transition of material system from non-equilibrium state into locally-equilibrium state.

As it has been shown above, the transition from nonidentical relation to identical points out to a transition from inexact solutions of the first type to the generalized (exact) solutions.

It turns out that the transition from inexact solutions to exact (generalized) solutions is assigned to the transition of gas-dynamic system from non-equilibrium state to locally-equilibrium state.

Since the non-equilibrium state has been induced by an availability of internal force and in the case of locally-equilibrium state there is no internal force (in local domain of gas-dynamic system), it is evident that under transition of gas-dynamic system from non-equilibrium state into locally-equilibrium state the nonmeasurable quantity, which acts as internal force, changes to a measurable quantity. This manifests itself in the form of arising a certain observable measurable formation. Waves, vortices, turbulent pulsations and so on are examples of such formations.

Exact generalized solutions to the Euler and Navier-Stokes equations describe such observable formations arisen.

[It should be noted that closed dual forms and closed inexact exterior forms, which are realized under degenerate transformations, made up a differential-geometric structure, i.e. a pseudostructure (integrable structure) with conservative quantity (closed exterior form describes a conservative quantity because its differential equals zero). Realization of such differential-geometric structure (under degenerate transformation) points out to emergence of physical structure. The characteristics, the singular points, the envelopes of characteristics, and other structures with conserved quantities are examples of such physical structures. The origination of physical structure reveals as a new measurable and observable formation that spontaneously arises in a gas-dynamic system.]

It is evident that the transition from inexact solutions to exact (generalized) solutions is assigned to the transition of gas-dynamic system from a non-equilibrium state to a locally-equilibrium state, which is accompanied by the emergence of observable formations. Such observable formations are described by generalized solutions of the Euler and Navier-Stoles equations. In this case the discontinuities of a function, which corresponds to generalized solutions, or their derivatives are defined by a quantity that is described by the commutator of unclosed form ω and acts as an internal force. Such a quantity defines the intensity of formations arisen (if the commutator be equal to zero, the intensity of formation would be equal to zero, i.e. the formation could't arise).

The process of arising observable formations discloses a mechanism of such phenomena as an emergence of vorticity and turbulence.

Here it should be emphasized that the conservation laws for energy, linear momentum, and mass, which are noncommutative ones, play a controlling role in these processes [4].

4 Conclusion

The Navier-Stokes equations possess the properties that are inherent in the

the mathematical physics equations which describe material media (systems) like the thermodynamical, gas-dynamic, cosmological systems, the systems of charged particles, and so on.

As opposed to common differential equations, these equations include the conservation law equations for energy, linear momentum, angular momentum, and mass that exert a connection between the change of physical quantities (such as energy, pressure and density) and external actions.

Physical quantities of material media possess specific properties.

Due to external actions physical quantities change and cease to be consistent, namely, inherent quantities of material media, and this leads to emergence of internal forces and a non-equilibrium state of the medium. A role of the conservation laws consists in performing the process that leads changed physical quantities to the consistency, which is internal, inherent to a given material medium (in correspondence to its degrees of freedom).

The peculiarity of the Navier-Stokes equations, as well as the relevant mathematical physics equations, consists in the fact that from the Navier-Stokes equations one obtains the relation for the state functional, which discloses such a role of the conservation laws and describes the mechanism of evolutionary processes that are accompanied by origination of various observable formations (such as waves, vortices, turbulent pulsations).

The difficulties of the Navier-Stokes equations, which did not't allow to disclose such possibilities, were related to the fact that the relevant properties of the Navier-Stokes equations are not reveal explicitly. As it was shown, they are disclosed only under studying the consistency of corresponding conservation law equations. Moreover, these difficulties could be also related to the fact that for studying such a consistency of equations one needs a previously unknown mathematical apparatus of skew-symmetric forms which basis are nonintegrable manifolds.

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