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On Some Results in Fuzzy Metric Spaces

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Abstract

In this paper, we prove common fixed point theorems in fuzzy metric spaces by employing the notion of reciprocal continuity and occasionally weakly compatible mappings. Our result improves recent results of Singh & Jain [13] in the sense that all maps involved in the theorems are discontinuous even at common fixed point.

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Keywords: Common fixed point; compatible maps; continuous t-norm; fuzzy metric space; reciprocal continuity; semi-compatible maps and occasionally weakly compatible maps

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1 Introduction

After Zadeh [16] introduced the concept of fuzzy sets in 1965, many authors have extensively developed the theory of fuzzy sets and its applications. Specially to mention, fuzzy metric spaces were introduced by Deng [3], Erceg [4], Kaleva and Seikkala [8], Kramosil and Michalek [10]. In this paper we use the concept of fuzzy metric space introduced by Kramosil and Michalek [10] and modified by George and Veeramani [5] to obtain Hausdorff topology for this kind of fuzzy metric space.

Recently Singh et. al. [13] introduced the notion of semi-compatible maps in fuzzy metric space and compared this notion with the notion of compatible map, compatible map of type (α), compatible map of type (β) and obtain some fixed point theorems in complete fuzzy metric space in the sense of Grabiec [6].

In the present paper, we prove fixed point theorems in complete fuzzy metric space by replacing continuity condition with a weaker condition called reciprocal continuity.

2 Preliminaries

In this section we recall some definitions and known results in fuzzy metric space.

Definition 2.1. [13] A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is called a *t-norm* if $([0, 1], *)$ is an abelian topological monoid with unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for $a, b, c, d \in [0, 1]$.

Examples of t-norms are $a * b = ab$ and $a * b = \min\{a, b\}$.

Definition 2.2. [13] The 3-tuple (X, M, *) is said to be a *Fuzzy metric space* if X is an arbitrary set, * is a continuous t-norm and M is a Fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions :

for all $x, y, z \in X$ and $s, t > 0$.

(FM-1) $M(x, y, 0) = 0$,

(FM-2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,

(FM-3) $M(x, y, t) = M(y, x, t)$,

(FM-4) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$,

(FM-5) $M(x, y, .) : [0, \infty) \to [0, 1]$ is left continuous,

(FM-6) $\lim_{t\to\infty} M(x, y, t) = 1$.

Note that $M(x, y, t)$ can be considered as the degree of nearness between x and y with respect to t. We identify $x = y$ with $M(x, y, t) = 1$ for all $t > 0$. The following example shows that every metric space induces a Fuzzy metric space.

Example 2.1. [5] Let (X, d) be a metric space. Define $a * b = \min\{a, b\}$ and $M(x, y, t) = \frac{t}{t + d(x, y)}$ for all $x, y \in X$ and all $t > 0$. Then (X, M, *) is a Fuzzy metric space. It is called the Fuzzy metric space induced by d.

Definition 2.3. [6] A sequence $\{x_n\}$ in a Fuzzy metric space (X, M, *) is said to be a *Cauchy sequence* if and only if for each $\varepsilon > 0$, $t > 0$, there exists $n_0 \in N$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \ge n_0$.

The sequence $\{x_n\}$ is said to *converge* to a point x in X if and only if for each $\varepsilon > 0$, $t > 0$ there exists $n_0 \in N$ such that $M(x_n, x, t) > 1 - \varepsilon$ for all $n \ge n_0$.

A Fuzzy metric space (X, M, *) is said to be *complete* if every Cauchy sequence in it converges to a point in it.

Definition 2.4. [14] Self mappings A and S of a Fuzzy metric space (X, M, *) are said to be *compatible* if and only if $M(ASx_n, SAx_n, t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Sx_n, Ax_n \to p$ for some p in X as $n \to \infty$.

Definition 2.5. [11] Two self maps A and B of a fuzzy metric space (X, M, *) are said to be weak compatible if they commute at their coincidence points, i.e.
$Ax = Bx$ implies $ABx = BAx$.

Definition 2.6. Self maps A and S of a Fuzzy metric space (X, M, *) are said to be occasionally weakly compatible (owc) if and only if there is a point x in X which is coincidence point of A and S at which A and S commute.

Definition 2.7. [13] Suppose A and S be two maps from a Fuzzy metric space (X, M, *) into itself. Then they are said to be semi-compatible if $\lim_{n\to\infty} ASx_n = Sx$, whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x \in X$.

Definition 2.8. [12] Suppose A and S be two maps from a Fuzzy metric space (X, M, *) into itself. Then they are said to be reciprocal continuous if $\lim_{n\to\infty} ASx_n = Ax$ and $\lim_{n\to\infty} SAx_n = Sx$ whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x \in X$.

If A and S are both continuous then they are obviously reciprocally continuous but the converse need not be true.

Lemma 2.1. [6] Let (X, M, *) be a fuzzy metric space. Then for all $x, y \in X$, $M(x, y, .)$ is a non-decreasing function.

Lemma 2.2. [11] Let (X, M, *) be a fuzzy metric space. If there exists

$k \in (0, 1)$ such that for all $x, y \in X$, $M(x, y, kt) \geq M(x, y, t)\ \forall\ t > 0$, then $x = y$.

Lemma 2.3. [16] Let $\{x_n\}$ be a sequence in a fuzzy metric space (X, M, *). If there exists a number $k \in (0, 1)$ such that

$$M(x_{n+2}, x_{n+1}, kt) \geq M(x_{n+1}, x_n, t) \quad \forall \quad t > 0 \quad \text{and} \quad n \in N.$$

Then $\{x_n\}$ is a Cauchy sequence in X.

3 Main Results

In the following theorem we replace the continuity condition by weaker notion of reciprocal continuity to get more general form of result 4.1, 4.2 and 4.9 of [13].

Theorem 3.1. Let A, B, S and T be self maps on a complete fuzzy metric space (X,M,*) where * is a continuous t-norm defined by $a * b = \min\{a, b\}$ satisfying :

(3.1) $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$,

(3.2) (B,T) is occasionally weak compatible,

(3.3) for all $x, y \in X$ and $t > 0$, $M(Ax, By, t) \geq \Phi(M(Sx, Ty, t))$, where $\Phi : [0,1] \rightarrow [0,1]$ is a continuous function such that $\Phi(1) = 1$, $\Phi(0) = 0$ and $\Phi(a) > a$ for each $0 < a < 1$.

If (A,S) is semi-compatible pair of reciprocal continuous maps then A, B, S and T have a unique common fixed point.

Proof. Let $x_0 \in X$ be any arbitrary point. Then for which there exists $x_1, x_2 \in X$ such that $Ax_0 = Tx_1$ and $Bx_1 = Sx_2$. Thus we can construct a sequences $\{y_n\}$ and $\{x_n\}$ in X such that $y_{2n+1} = Ax_{2n} = Tx_{2n+1}$, $y_{2n+2} = Bx_{2n+1} = Sx_{2n+2}$ for $n = 0, 1, 2, 3, \ldots\ldots$.

By contractive condition, we get

$$\begin{aligned} M(y_{2n+1}, y_{2n+2}, t) &= M(Ax_{2n}, Bx_{2n+1}, t) \\ &\geq \Phi(M(Sx_{2n}, Tx_{2n+1}, t)) \\ &= \Phi(M(y_{2n}, y_{2n+1}, t)) \\ &> M(y_{2n}, y_{2n+1}, t). \end{aligned}$$

Similarly, we get

$$M(y_{2n+2}, y_{2n+3}, t) > M(y_{2n+1}, y_{2n+2}, t).$$

In general,

$$\begin{aligned} M(y_{n+1}, y_n, t) &\geq \Phi(M(y_n, y_{n-1}, t)) \\ &> M(y_n, y_{n-1}, t). \end{aligned}$$

Therefore $\{M(y_{n+1}, y_n, t)\}$ is an increasing sequence of positive real numbers in $[0,1]$ and tends to limit $l \leq 1$. We claim that $l = 1$.

If $l < 1$ then $M(y_{n+1}, y_n, t) \geq M(y_n, y_{n+1}, t)$.

On letting $n \to \infty$, we get

$$\lim_{n\to\infty} M(y_{n+1}, y_n, t) \geq \Phi(\lim_{n\to\infty} M(y_n, y_{n-1}, t))$$

i.e. $l \geq \Phi(l) = l$, a contradiction. Now for any positive integer p,

$$M(y_n, y_{n+p}, t) \geq M(y_n, y_{n+1}, t/p) * M(y_{n+1}, y_{n+2}, t/p) * \ldots * M(y_{n+p-1}, y_{n+p}, t/p).$$

Letting $n \to \infty$, we get

$$\lim_{n\to\infty} M(y_n, y_{n+p}, t) \geq 1 * 1 * 1 * \ldots * 1 = 1.$$

Thus,

$$\lim_{n\to\infty} M(y_n, y_{n+p}, t) = 1.$$

Thus $\{y_n\}$ is a Cauchy sequence in X. Since X is complete, $\{y_n\}$ converges to a point z in X. Hence the subsequences $\{Ax_{2n}\}$, $\{Sx_{2n}\}$, $\{Tx_{2n+1}\}$ and $\{Bx_{2n+1}\}$ also converge to z.

Now since A and S are reciprocal continuous and semi-compatible then we have

$$\lim_{n\to\infty} ASx_{2n} = Az, \quad \lim_{n\to\infty} SAx_{2n} = Sz \text{ and } \lim_{n\to\infty} M(ASx_{2n}, Sz, t) = 1.$$

Therefore we get $Az = Sz$.

Now we will show $Az = z$. For this suppose $Az \neq z$. Then by contractive condition, we get

$$M(Az, Bx_{2n+1}, t) \geq \Phi(M(Sz, Tx_{2n+1}, t)).$$

Letting $n \to \infty$, we get

$$M(Az, z, t) \geq \Phi(M(Az, z, t)) > M(Az, z, t),$$

a contradiction, thus $z = Az = Sz$.

Since $A(X) \subseteq T(X)$, there exists $u \in X$ such that $z = Az = Tu$.

Putting $x = x_{2n}$ and $y = u$ in (3.3) we get,

$$M(Ax_{2n}, Bu, t) \geq \Phi(M(Sx_{2n}, Tu, t)).$$

Letting $n \to \infty$, we get

$$M(z, Bu, t) \geq \Phi(M(z, z, t)) = \Phi(1) = 1,$$

i.e. $z = Bu = Tu$ and the occasionally weak-compatibility of (B, T) gives $TBu = BTu$, i.e. $Tz = Bz$.

Again by contractive condition and assuming $Az \neq Bz$, we get $Az = Bz = z$.

Hence finally, we get

$z = Az = Bz = Sz = Tz$, i.e. z is a common fixed point of A, B, S and T.

The uniqueness follows from contractive condition. This completes the proof. ◊

Now we prove an another common fixed point theorem with different contractive condition:

Theorem 3.2. Let A, B, S and T be self maps on a complete fuzzy metric space (X, M, *) satisfying:

(3.4) $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$,

(3.5) (B, T) is occasionally weak compatible,

(3.6) for all $x, y \in X$ and $t > 0$,

$$M(Ax, By, t) \geq \Phi\{\min(M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), M(Ax, Ty, t))\},$$

where $\Phi : [0,1] \to [0,1]$ is a continuous function such that $\Phi(1) = 1$, $\Phi(0) = 0$ and $\Phi(a) > a$ for each $0 < a < 1$. If (A, S) is semi-compatible pair of reciprocal continuous maps then A, B, S and T have a unique common fixed point.

Proof. Let $x_0 \in X$ be any arbitrary point. Then for which there exists $x_1, x_2 \in X$ such that $Ax_0 = Tx_1$ and $Bx_1 = Sx_2$. Thus we can construct sequences $\{y_n\}$ and $\{x_n\}$ in X such that $y_{2n} = Ax_{2n} = Tx_{2n+1}$, $y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$ for n = 0, 1, 2, 3, … .

By contractive condition, we get

$$\begin{aligned}
M(y_{2n+1}, y_{2n+2}, t) &= M(Ax_{2n}, Bx_{2n+1}, t)\\
&\geq \Phi\{\min(M(Sx_{2n}, Tx_{2n+1}, t), M(Ax_{2n}, Sx_{2n}, t),\\
&\qquad M(Bx_{2n+1}, Tx_{2n+1}, t), M(Ax_{2n}, Tx_{2n+1}, t))\}\\
&= \Phi\{\min(M(y_{2n-1}, y_{2n}, t), M(y_{2n}, y_{2n-1}, t),\\
&\qquad M(y_{2n+1}, y_{2n}, t), M(y_{2n}, y_{2n}, t))\}\\
&= \Phi\{\min(M(y_{2n-1}, y_{2n}, t), M(y_{2n+1}, y_{2n}, t))\}\\
&= \Phi\{M(y_{2n+1}, y_{2n}, t)\}\\
&\geq M(y_{2n-1}, y_{2n}, t).
\end{aligned}$$

Similarly, we get

$$M(y_{2n+2}, y_{2n+3}, t) \geq M(y_{2n+1}, y_{2n+2}, t).$$

In general,

$$M(y_{n+1}, y_n, t) \geq \Phi(M(y_n, y_{n-1}, t)) \geq M(y_n, y_{n-1}, t).$$

Therefore $\{M(y_{n+1}, y_n, t)\}$ is an increasing sequence of positive real numbers in [0,1] and tends to limit $l \leq 1$ then by the same technique of above theorem we can easily show that $\{y_n\}$ is a Cauchy sequence in X. Since X is complete $\{y_n\}$ converges to a point z in X. Hence the subsequences $\{Ax_{2n}\}$, $\{Sx_{2n}\}$, $\{Tx_{2n+1}\}$ and $\{Bx_{2n+1}\}$ also converge to z.

Now since A and S are reciprocal continuous and semi-compatible then we have

$$\lim_{n\to\infty} ASx_{2n} = Az, \quad \lim_{n\to\infty} SAx_{2n} = Sz \text{ and } \lim_{n\to\infty} M(ASx_{2n}, Sz, t) = 1.$$

Therefore, we get Az = Sz.

Now we will show Az = z. For this suppose $Az \neq z$.

Then by (3.6), we get a contradiction, thus Az = z.

Hence by similar techniques of above theorem we can easily show that z is a common fixed point of A, B, S and T i.e. z = Az = Bz = Sz = Tz. Uniqueness of fixed point can be easily verify by contractive condition. This completes the proof. ◊

We now give an example which not only illustrate our Theorem 3.1 but also shows that the notion of reciprocal continuity of maps is weaker than the continuity of maps.

Example 3.1. Let (X, d) be usual metric space where X = [2, 20] and M be the usual fuzzy metric on (X, M, *) where * = min be the induced fuzzy metric space with

$M(x, y, t) = \frac{t}{t + d(x, y)}$ for $x, y \in X$, $t > 0$.

We define mappings A, B, S and T by

A2 = 2, Ax = 3 if x > 2, S2 = 2, Sx = 6 if x > 2,

Bx = 2 if x = 2 or > 5, Bx = 6 if 2 < x ≤ 5,

Tx = 2, Tx = 12 if 2 < x ≤ 5, $T(x) = \frac{(x+1)}{3}$ if x > 5.

Then A, B, S and T satisfy all the conditions of the above theorem with $\Phi(a) = \frac{7a}{3a+4} > a$, where a = 1/{1+d(Sx, Ty)} and have a unique common fixed point x = 2.

It may be noted that in this example A(X) = {2,3} $\subseteq$ T(X) = [2,7] $\cup$ {12} and B(X) = {2,6} $\subseteq$ S(X) = {2,6}.

Also A and S are reciprocally continuous compatible mappings. But neither A nor S is continuous not even at fixed point x = 2. The mapping B and T are non-compatible but occasionally weak-compatible since they commute at their co-incidence points. To see B and T are non-compatible, let us consider the sequence $\{x_n\}$ in X defined by $\{x_n\} = \left\{5+\frac{1}{n}\right\}$; $n \geq 1$. Then, $\lim_{n\to\infty} Tx_n = 2$, $\lim_{n\to\infty} Bx_n = 2$, $\lim_{n\to\infty} TBx_n = 2$ and $\lim_{n\to\infty} BTx_n = 6$. Hence B and T are non-compatible.

Remark 3.1. The maps A, B, S and T are discontinuous even at the common fixed point x = 2.

4 Conclusion

The known common fixed point theorems involving a collection of maps in fuzzy metric spaces require one of the mapping in compatible pair to be continuous. For example in [2], Chug assume one of the mapping A, B, S or T to be continuous. Similarly Singh et. al. [13, 14] and Khan et. al. [9] assume one of the mappings in compatible pairs of maps is continuous. The present theorem however does not require any of the mappings to be continuous and hence all the results mentioned above can be further improved in the spirit of our Theorem 3.1.

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