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A note on finding geodesic equation of two parameter Weibull distribution

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Abstract

The Weibull distribution has received a great deal of attention since 1970. In Russian statistical literature, this distribution is often referred to as the Weibull-Gnedenko distribution. It has been applied to model a wide range of data secured from problems such as the yield strength of Bofors' steel, the fiber strength of Indian cotton, the fatigue life of ST-37 steel, the statures of adult males born in the British Isles, and breadth of beans of Phaseolus vulgaris. Many authors used this distribution in their reliability and quality control work. Instead of using the classical approach by solving a pair of differential equations, in this paper, we adopt the well-known Darboux Theory by solving a partial differential equation to find the geodesic equation of two parameter Weibull distributions.

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1 Introduction

Swedish physicist, Waloddi Weibull, [2,3] used the Weibull Distribution to describe the breaking strength of material. By 1951[4], there were a variety of other applications. Several basic examples of how to apply the Weibull Distribution were presented in the abstract. In Russian statistical literature this distribution is often referred to as the Weibull-Gnedenko distribution. It is one of the three types of limit distribution for the sample maximum established by Gnedenko [5]. As his special, the Weibull distribution may also include the exponential or the Rayleigh distribution. When the shape parameter is less than 1, the hazard function of the Weibull Distribution is a decreasing function. When the shape parameter equals 1, it is a constant. When the shape parameter is greater than 1, it is an increasing function. Many authors have used this situation in reliability and quality control work such as Weibull[4], Kao[6,7] , and Berretoni [8]. The Weibull Distribution received the most attention in 1970. This is evident from the large number of references that can be found from the book of Johnson N.L. Kotz S, and Balakrishnan N [9]. Since Rao C.R. [10] published his first paper, linking statistics with geometrical properties, numerous authors have expanded upon this area. For example, Lauritzen S.L. [11] derived the Gaussian Curvature, Geodesic Equation of Gamma Manifold and Inverse Gaussian Manifold. Chen W. [12] using the Darboux Theory derived a completed version of the Gamma Geodesic Equation. Chen W. [13] has successfully generalized the formula to compute the Gaussian Curvature and clarify its intricate mathematical concept. Uwe Jensen [14] has reviewed the derivation, calculation and simulation results of Rao Distance, applying it to the portfolio theory. In this paper, we adopted the Darboux Theorem to solve a partial differential equation. We found this approach would be easier than the classical method.

2 Darboux theory and geodesic equation

In general, the distance between two points P and Q on a curve of two-manifold can be expressed as

$$ds^2 = E\ du^2 + 2F du\ dv + G\ dv^2 \tag{2.1}$$

However, if we can transform the distance function (2.1) to the following simplified form

$$ds^2 = dz^2 + \sigma^2 dz_1^2 \tag{2.2}$$

it could help us to find the Geodesic Equation more easily. The task of transforming equation (2.2) is equivalent to asking how we can determine two independent functions, $z = z(u,v)$, and $z_1 = z_1(u,v)$, such that equation (2.1) can be transformed into equation (2.2). Since z(u,v) is a function of (u,v), we know from calculus that

$$\begin{aligned} &dz = z_u du + z_v dv, \\ &ds^2 - dz^2 = (E - z_u^2)du^2 + 2(F - z_u z_v)dudv + (G - z_v^2)dv^2. \end{aligned} \tag{2.3}$$

If we assume that (2.2) is valid, then it would be necessary for either the right hand side of (2.3) to be a perfect square, or for the determinant of (2.3) to be equal to zero. That is,

$$(F - z_u z_v)^2 - (E - z_u^2)(g - z_v^2) = 0. \tag{2.4}$$

Equation (2.4) can be rewritten as

$$\frac{Ez_v^2 - 2Fz_u z_v + Gz_v^2}{EG - F^2} = 1. \tag{2.5}$$

For convenience, we usually write the left hand side of (2.5) as $\nabla Z = 1.$ Now, if we could find an arbitrary solution to (2.5), then we could rewrite (2.3) in the following form:

$$ds^2 - dz^2 = (m(u,v)du + n(u,v)dv)^2, \tag{2.6}$$

where both m, n are some known function of u and v.

If we can further find an integration factor $\frac{1}{\sigma}$, such that

$$m(u,v)du + n(u,v)dv = \sigma\, dz_1,$$

then the distance function ds^2 could be transformed into the form (2.2). Summarizing the above procedures, we conclude that in order to find the Geodesic Equation, two steps must be completed :

Step 1: we must find an arbitrary solution of the partial differential equation (2.5);

Step 2: we must find an integration factor of equation (2.6). Darboux has proposed an improved method to combine the two steps into one step; that method is stated as a theorem.

Theorem 1: Assume the given partial differential equation $\nabla Z = 1$ has found an arbitrary solution Z=Z(u,v,a), where a is an arbitrary constant. Then

$$\frac{\partial Z(u,v;a)}{\partial a} = constant$$

is the required Geodesic Equation.

Proof: Assume the distance between two points P and Q on a curve of two-manifold has the simplified form (2.2). The total differential at a point (u,v) can be written as:

$$\begin{aligned} dz &= z_u du + z_v dv, \\ dz_1 &= z_{1u} du + z_{1v} dv \end{aligned} \tag{2.7}$$

Furthermore, if we take the partial derivative with respect to the constant a for the above two equations and we get

$$d\frac{\partial z}{\partial a} = \frac{\partial^2 z}{\partial u \partial a} du + \frac{\partial^2 z}{\partial v \partial a} dv,$$
$$d\frac{\partial z_1}{\partial a} = \frac{\partial^2 z_1}{\partial u \partial a} du + \frac{\partial^2 z_1}{\partial v \partial a} dv,$$

and (2.8)

$$dz\ d\frac{\partial z}{\partial a} = -\sigma\ dz_1\left(\frac{\partial \sigma}{\partial a}dz_1 + \sigma\ d\frac{\partial z_1}{\partial a}\right)$$

If (2.8) were true, then from the third equation of (2.8) we can conclude that dz_1 must divide evenly on $dz\,\mathrm{d}\frac{\partial z}{\partial \mathrm{a}}$. This means that dz_1 can divide either dz or $d\frac{\partial z}{\partial a}$ evenly. Concerning the first situation that dz_1 can divide dz evenly, then from (2.7) we have

$$\begin{vmatrix} z_u & z_v \\ z_{1u} & z_{1v} \end{vmatrix} = 0 \tag{2.9}$$

But this means that z and z_1 are functionally dependent. This contradicts equation (2.2), which assumes that z and z_1 are independent . Hence, the only case that can possibly be valid is that dz_1 can divide $d\frac{\partial z}{\partial a}$ evenly. This means that $z_1 = \text{constant}$ $and\ \frac{\partial z}{\partial a} = cons\tan t$ are curves in the same families. This proves that the equation $\frac{\partial z}{\partial a} = cons\tan t$ is the required geodesic equation.

3 Finding the Geodesic Equation of the Weibull bution

From section 4, we have calculated

$$E = \frac{C_1}{u^2}$$

we sometimes also write

$$C_1 = a^2 + b^2,$$

where

$$a = 1-\gamma, \text{ and } \quad b = \frac{\pi}{\sqrt{6}}.$$

$$\gamma = 0.5772157$$

is known as Euler's constant.

$$F = \frac{C_2}{v}, \quad \text{in this case } C_2 = -a, \tag{3.1}$$

$$G = \frac{u^2}{v^2}.$$

Then $\nabla Z = 1$ becomes

$$(a^2+b^2)\frac{v^2}{u^2}z_v^2 + 2avz_u z_v + u^2 z_u^2 = \frac{\pi^2}{6}.$$

In order to find one of the solutions of equation (3.1), we make a transformation from (u,v) to (u_1,v_1) as follow:

$$u = u_1^{-1}, \quad v = e^{-v_1}.$$

Then, through the chain rule, we get

$$\frac{\partial z}{\partial u_1} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial u_1} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial u_1} = -u_1^{-2}z_u, \text{ or } z_u = -u_1^2 z_{u1},$$

$$\frac{\partial z}{\partial v_1} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial v_1} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial v_1} = -e^{-v_1}z_v, \text{ or } z_v = -e^{v_1}z_{v1}$$

The given partial differential equation (3.1) turns out to be

$$(a^2+b^2)z_{v1}^2 + 2az_{u1}z_{v1} + z_{u1}^2 = \frac{\pi^2}{6}u^{-2}_{1}, \tag{3.2}$$

Next, consider making another transformation to polar coordinates:

$$u_1 = r\cos\theta, \quad v_1 = r\sin\theta;$$

then, through the chain rule, we can find the following relation:

$$z_{v1} = \sin\theta\, z_r + \frac{1}{r}\cos\theta\, z_\theta, \qquad z_{u1} = \cos\theta z_r - \frac{1}{r}\sin\theta\, z_\theta$$

After substituting z_{v1}, z_{u1} into equation (3.2), recalling the term and simplifying, we summarize our results as follows:

The coefficients of $z_r z_\theta$:

$$(a^2+b^2)\ \tan 2\theta\ +\ 2a-\tan 2\theta\ =\ 0$$

$$\tan 2\theta=\frac{-2a}{a^2+b^2-1}\ =-1.026573416$$

this means

$$2\theta=-45.75124^{\circ},\ \ or\ \ \theta=-22.8756$$

The coefficient of z_r^2 :

$$(a^2+b^2)\sin^2\theta+2a\cos\theta\sin\theta+\cos^2\theta=0.821620836$$

The coefficient of z_θ^2 :

$$\frac{a^2+b^2}{r^2}\cos^2\theta+2a(\frac{-1}{r^2}\cos\theta\sin\theta)+\frac{1}{r^2}\sin^2\theta=\frac{2.002059825}{r^2}$$

After rotating -22.8756°, equation (3.2) becomes

$$r^2z_r^2=2.358453-2.436722\ z_\theta^2=A^2$$

We can separate the variables of r and θ, then solve this partial differential equation as follows:

$$z_r^2=\frac{A^2}{r^2},\quad \text{so that}\quad z=\pm A\ \ln\ r.$$

On the other hand, we can derive from

$$2.358453-2.436722\ z_\theta^2=A^2$$

$$z=\pm\frac{\sqrt{2.358453\text{-}A^2}}{1.561}\theta,$$

we can now summarize the above two results and write one of the general solutions that we found

$$Z=\pm A\ \ln r\ \pm\frac{\sqrt{2.358453\text{-}A^2}}{1.561}\theta \tag{3.3}$$

From previous relations, we know that (r,θ) and (u_1,v_1) are related to $u_1^2+v_1^2=r^2$, and $\tan\ \theta\ =\frac{v_1}{u_1}$, hence, after substituting into equation (3.3) we get

$$Z = \pm A \ \ln \sqrt{u_1^2 + v_1^2} \pm \frac{\sqrt{2.358453 - A^2}}{1.561} \tan^{-1} \frac{v_1}{u_1}.$$

Making a further substitution:

$$u_1 = u^{-1}, \quad \text{and} \quad v_1 = -\log v$$

we can then easily find that

$$u_1^2 + v_1^2 = u^{-2} + (\log v)^2, \quad \text{and} \quad \frac{v_1}{u_1} = \log v^{-u}.$$

Finally, we find one of the general solution of equation (3.2)

$$Z = \pm A \ \ln \ (u^{-2} + (\log v)^2)^{\frac{1}{2}} \pm \frac{\sqrt{2.358453 - A^2}}{1.561} \tan^{-1}(\log v^{-u}).$$

The Geodesic Equation of the Weibull Distribution can then be written as

$$\frac{\partial Z}{\partial A} = B,$$

or

$$\pm \ \frac{1}{2} \ln \ (u^{-2} + (\log v)^2) \pm \frac{A \tan^{-1}(\log v^{-u})}{1.561\sqrt{2.358453 - A^2}} = B.$$

where A, B are arbitrary constants.

4 List the fundamental tensor

The probability density function for the Weibull Distribution is given by

$$f(x :, u, v) = \frac{u x^{u-1}}{v^u} \exp(-(\frac{x}{v})^u) I_{(0,\infty)}(x);$$

where v is the scale parameter and u is the shape parameter. (4.1)

$$\ln f = \ln u + (u-1) \ln x - u \ln v - (\frac{x}{v})^u.$$

From equation (4.1), we derive the metric tensor components for the Weibull case

$$E = -E(\frac{\partial^2 \ln f(x)}{\partial u^2}) = \frac{\Gamma^{(2)}(2)+1}{u^2},$$

$$F = -E(\frac{\partial^2 \ln f(x)}{\partial v \partial u}) = \frac{-\Gamma^{'}(2)}{v},$$

$$G = -E(\frac{\partial^2 \ln f}{\partial v^2}) = \frac{u^2}{v^2}$$

$$\psi(2) = \psi(1) + 1 = -0.5772156649 + 1 = 0.422784335 = \Gamma^{'}(2)$$

$$\Gamma^{(2)}(2) = \psi^{'}(2) + (\Gamma^{'}(2))^2 = 0.644934067 + 0.422784335^2 = 0.823680661$$

$$C_1 = \Gamma^{(2)}(2)+1\,; \qquad C_2 = -\Gamma^{'}(2) \qquad C_3 = C_1 - C_2^2 = 1.644934067$$

In the above derivation, we applied the following integral results

$$E((\ln(\frac{x}{v}))^2(\frac{x}{v})^u) = \int_0^\infty (\ln\frac{x}{v})^2(\frac{x}{v})^u \frac{ux^{u-1}}{v^u} e^{-(\frac{x}{v})^u} dx \quad = \frac{\Gamma^{(2)}(2)}{u^2};$$

$$E((\ln(\frac{x}{v}))(\frac{x}{v})^u) = \int_0^\infty (\ln\frac{x}{v})(\frac{x}{v})^u \frac{ux^{u-1}}{v^u} e^{-(\frac{x}{v})^u} dx \quad = \frac{\Gamma^{'}(2)}{u}$$

we define the nth derivative of the gamma function :

$$\Gamma^{(n)}(x) = \int_0^\infty e^{-t} t^{x-1} (\ln t)^n \, dt, \qquad x > 0.$$

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