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Hölder Inequality with Variable Index

Richeng Liu[1]

Abstract

By using the Young inequality, we considered the classic Hölder inequality and extended it to the case that the conjugate indices p, q were essentially bounded Lebesgue measurable functions.

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1 Introduction

We know, Hölder inequality [1] is very important in mathematical analysis and real analysis, and has a very wide range of applications in differential equations and many other branches of mathematics, in particular in estimates of solutions for partial differential equations. It is no exaggeration to say that if there

[1] Northeast Petroleum University; Mathematics College; Department of Information and Computing Sciences; No.149 Fazhan Road; High-tech Development Zone; Daqing 163318, China. E-mail: richengliu@sina.com

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is no Hölder inequality, then it would be impossible for us to solve many partial differential equations.

The classic Hölder inequality holds for the conjugate indices p , q being constants. While in application, we will also meet the case that the indices are the variables, for example, when we make estimate for solutions of partial differential equations with $p(x)-$Laplace operator (extended as $p-$Laplace operator), the classic Hölder inequality it will become helpless. So we need to extend the classic Hölder inequality to the case for the conjugate indices p , q being variable.

In this paper, we shall extend the classic Hölder inequality to the case that the conjugate indices p , q were essentially bounded Lebesgue measurable functions.

2 Preliminary Notes

Let $\Omega \subseteq R^N$ be a Lebesgue measurable set and $L(\Omega)$ be the set of all Lebesgue measurable functions on Ω . Let $p(x)(>1)$, $q(x)(>1)$ denote the Lebesgue measurable functions which are almost everywhere bounded on Ω . Let $p_1 = \operatorname{essinf}\ p(x)$, $p_2 = \operatorname{esssup} p(x)$, $q_1 = \operatorname{essinf} q(x)$, $q_2 = \operatorname{esssup} q(x)$, where essinf and esssup denote the essential infimum and the essential supremum of $f(x) \in L(\Omega)$ on Ω, defined as

$$\operatorname{essinf}\ f(x) := \sup_{mE=0} \inf_{x\in\Omega\backslash E} f(x), \operatorname{esssup} f(x) := \inf_{mE=0} \sup_{x\in\Omega\backslash E} f(x), \quad f(x) \in L(\Omega).$$

Lemma 2.1. For essinf and esssup , there exist null subsets E and F $(i.e. mE = 0$ and $mF = 0)$ such that

$$\inf_{x\in\Omega\backslash E} f(x) = \operatorname{essinf}\ f(x),$$

and

$$\sup_{x\in\Omega\backslash F} f(x) = \operatorname{esssup} f(x).$$

This can be seen in [2].

3 Main Results

Theorem 2.1. Suppose $p(x)$ and $q(x)$ are almost everywhere conjugate on Ω, i.e. $\frac{1}{p(x)}+\frac{1}{q(x)}=1$ *a.e.* on Ω. If the two constants α and β satisfying the following conditions

$$[\int_\Omega |f(x)|^{p(x)}\,\mathrm{d}x]^{1/\alpha}=\max\{[\int_\Omega |f(x)|^{p(x)}\,\mathrm{d}x]^{1/p_1},[\int_\Omega |f(x)|^{p(x)}\,\mathrm{d}x]^{1/p_2}\}$$

and

$$[\int_\Omega |g(x)|^{q(x)}\,\mathrm{d}x]^{1/\beta}=\max\{[\int_\Omega |g(x)|^{q(x)}\,\mathrm{d}x]^{1/q_1},[\int_\Omega |g(x)|^{q(x)}\,\mathrm{d}x]^{1/q_2}\},$$

then $f(x)g(x)\in L^1(\Omega)$, and

$$\int_\Omega |f(x)g(x)|\,\mathrm{d}x\le 2\int_\Omega |f(x)|^{p(x)}\,\mathrm{d}x]^{1/\alpha}[\int_\Omega |g(x)|^{q(x)}\,\mathrm{d}x]^{1/\beta}. \tag{1}$$

Proof. Owing to Lemma 2.1, there exist null subsets E_p, K_p, E_q, K_q such that

$$\inf_{x\in\Omega\backslash E_p} p(x)=\operatorname{essinf} p(x),\quad \sup_{x\in\Omega\backslash K_p} p(x)=\operatorname{esssup} p(x);$$

$$\inf_{x\in\Omega\backslash E_q} q(x)=\operatorname{essinf} q(x),\quad \sup_{x\in\Omega\backslash K_q} q(x)=\operatorname{esssup} q(x).$$

Let $S=E_p\cup K_p\cup E_q\cup K_q$, then S is a null set. By the definitions of essinf and esssup, we have

$$\inf_{x\in\Omega\backslash S} p(x)=\operatorname{essinf} p(x),\quad \sup_{x\in\Omega\backslash S} p(x)=\operatorname{esssup} p(x);$$

$$\inf_{x\in\Omega\backslash S} q(x)=\operatorname{essinf} q(x),\quad \sup_{x\in\Omega\backslash S} q(x)=\operatorname{esssup} q(x).$$

Owing to the act that integral value is identically equal to 0, we assume that the essential infimum and the essential supremum of $f(x)\in L(\Omega)$ on Ω are the supremum and infimum of $f(x)\in L(\Omega)$ on Ω respectively, for simplicity. Write

$$F=\int_\Omega |f(x)|^{p(x)}\,\mathrm{d}x,\quad G=\int_\Omega |g(x)|^{q(x)}\,\mathrm{d}x.$$

By Young inequality, we have

$$\frac{|f(x)g(x)|}{F^{1/p(x)}G^{1/q(x)}} \le \frac{1}{p(x)}\frac{|f(x)|^{p(x)}}{F} + \frac{1}{q(x)}\frac{|g(x)|^{q(x)}}{G},$$

and further

$$|f(x)g(x)| \le \frac{1}{p(x)}\left(\frac{G}{F}\right)^{1/q(x)} |f(x)|^{p(x)} + \frac{1}{q(x)}\left(\frac{F}{G}\right)^{1/p(x)} |g(x)|^{q(x)}. \tag{2}$$

Let

$$\left(\frac{G}{F}\right)^{1/t} = \max_{x\in\Omega}\left(\frac{G}{F}\right)^{1/q(x)} , \tag{3}$$

then $\frac{1}{t} = \frac{1}{q_1}$ or $\frac{1}{q_2}$. By noticing that

$$\left(\frac{F}{G}\right)^{1/p(x)} = \left(\frac{G}{F}\right)^{1/q(x)}\left(\frac{G}{F}\right)^{-1},$$

we get

$$\left(\frac{F}{G}\right)^{1-(1/t)} = \max_{x\in\Omega}\left(\frac{F}{G}\right)^{1/p(x)}, \tag{4}$$

and $1-\frac{1}{t} = \frac{1}{p_1}$ or $\frac{1}{p_2}$. By virtues of (2)—(4), we obtain

$$|f(x)g(x)| \le \left(\frac{G}{F}\right)^{1/t} |f(x)|^{p(x)} + \left(\frac{F}{G}\right)^{1-(1/t)} |g(x)|^{q(x)}. \tag{5}$$

Upon integrating (5) about x on Ω we get

$$\int_\Omega |f(x)g(x)|\,\mathrm{d}x \le 2F^{1-(1/t)}G^{1/t}. \tag{6}$$

Keep in mind the assignments for $\frac{1}{t}$ and $1-\frac{1}{t}$ in (6), we obtain the desired result.

This completed the proof of the theorem.

References

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