Theoretical Mathematics & Applications, vol.4, no.2, 2014, 65-77
ISSN: 1792-9687 (print), 1792-9709 (online)
Scienpress Ltd, 2014

# From Sierpiński's conjecture to Legendre's

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**Abstract**

The third unsolved problem that Landau announced in 1912 at the fifth International Congress of Mathematicians at Cambridge, is Legendre's conjecture. It states that:
There is always at least one prime number between two consecutive squares $N^2$ and $(N+1)^2$ for any integer $N > 0$.
In the present article, an elementary proof of this conjecture is given by creating and solving the D conjecture, a modified version of Sierpiński's conjecture that originally states that:
For any integer $N > 1$, there is always at least one prime number in each line of a $N \times N$ matrix filled up from left to right and from bottom to top with the $N^2$ integers from 1 to $N^2$.
While proving the D conjecture, Sierpiński's S conjecture is also proved as well as Oppermann's conjecture which states that:
For any integer $N > 1$ one has : $\pi(N^2+N) > \pi(N^2) > \pi(N^2-N)$, where $\pi(x)$ is the prime counting function. It's this conjecture that proves Legendre's.
Finally, as applications, $p_{m+1} - p_m = O(\sqrt{p_m})$, Andrica's and Brocard's conjectures are proved.

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Article Info: *Received* : May 10, 2014. *Revised* : June 21, 2014.
*Published online* : June 30, 2014.

**Mathematics Subject Classification:** 11A41
**Keywords:** Landau; Legendre; Opperman; Sierpiński; Andrica; Brocard; conjecture

# 1 Introduction

The unsolved Sierpiński's S conjecture (1958) [1] states that:
For any integer $N > 1$, there is always at least one prime number in each line of a $N \times N$ matrix filled up from bottom to top and from left to right with the $N^2$ integers from 1 to $N^2$.

Let's then write Sierpiński's S(N) matrix:

Table 1. Sierpiński's matrix S(N)

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| N | ... | ... | ... | Low | Opp. | Conj. | ... | ... | $N^2$ |
| N-1 | ... | ... | ... | ... | ... | ... | ... | ... | $N^2 - N$ |
| ... | ... | ... | ... | ... | ... | ... | ... | ... | ... |
| ... | ... | ... | ... | ... | Sierp. | Conj. | ... | ... | ... |
| ... | ... | ... | ... | ... | ... | ... | ... | ... | ... |
| 4 | 3N+1 | ... | ... | ... | ... | ... | ... | ... | 4N |
| 3 | 2N+1 | ... | ... | ... | ... | ... | ... | ... | 3N |
| 2 | N+1 | ... | ... | ... | ... | ... | ... | ... | 2N |
| 1 | 1 | 2 | 3 | ... | ... | ... | ... | ... | N |
| L/C | C1 | C2 | ... | ... | ... | ... | ... | C(N-1) | C(N) |

In this Table 1, with the definitions of the conjectures mentioned in the introduction, one can see that all the lines of the matrix S(N) correspond to Sierpiński's conjecture and that the top line only corresponds to the lower part of Oppermann's conjecture [2].
Considering only the integers $N > 1$, one can see that for $N = 2$ et $N = 3$, Sierpiński's matrices are:

Table 2. Sierpiński's matrix S(2)

| | |
|---|---|
| 3 | 4 |
| 1 | 2 |

Table 3. Sierpiński's matrix S(3)

| | | |
|---|---|---|
| 7 | 8 | 9 |
| 4 | 5 | 6 |
| 1 | 2 | 3 |

and one can check that Sierpiński's conjecture is verified by these two matrices. But they do not show any kind of recursivity.

## 2 Preliminary Notes

Now, let's define a new matrix that we name D(N) and in which we introduce some recursivity with the help of the recursive relation:

$$(N+1)^2 = N^2 + (2N+1)$$

In order to do that, we simply add to Sierpiński's S(N) matrix, two lines upwards with the 2N numbers immediately greater than $N^2$ and then, one column rightwards filled up with zeros except the number $(N+1)^2$ at its top, as follows:

Table 4. Matrix D(N) (from column 1 to column N+1)

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| N+2 | A | ... | ... | ... | ... | ... | B | $(N+1)^2$ |
| N+1 | C | High | Opp. | Conj. | ... | D | $N^2+N$ | 0 |
| N | E | Low | Opp. | Conj. | F | $N^2-1$ | $N^2$ | 0 |
| N-1 | ... | ... | ... | ... | ... | ... | $N^2-N$ | 0 |
| ... | ... | ... | ... | ... | ... | ... | ... | 0 |
| k | ... | ... | Sierp. | Conj. | ... | ... | kN | 0 |
| ... | ... | ... | ... | ... | ... | ... | ... | 0 |
| 4 | 3N+1 | ... | ... | ... | ... | ... | 4N | 0 |
| 3 | 2N+1 | ... | ... | ... | ... | ... | 3N | 0 |
| 2 | N+1 | ... | ... | ... | ... | ... | 2N | 0 |
| 1 | 1 | 2 | ... | ... | ... | ... | N | 0 |
| L/C | C1 | C2 | ... | ... | ... | ... | C(N) | C(N+1) |

This ties three independent conjectures into one real matrix. With this model, matrix D(N+1) is:

Table 5. Matrix D(N+1)

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| N+3 | ... | Low | Opp. | Conj. | ... | Legend. | Conj. | $(N+2)^2$ |
| N+2 | ... | High | Opp. | Conj. | ... | Legend. | Conj. | 0 |
| N+1 | A | Low | Opp. | Conj. | ... | B | $(N+1)^2$ | 0 |
| N | $N^2$ | C | High | Opp. | Conj. | D | $N^2+N$ | 0 |
| N-1 | ... | ... | E | Low | Opp. | F | $N^2-1$ | 0 |
| ... | ... | ... | ... | ... | ... | ... | ... | 0 |
| k | ... | ... | ... | ... | ... | ... | k(N+1) | 0 |
| ... | ... | ... | ... | ... | ... | ... | ... | 0 |
| 4 | 3N+4 | ... | ... | ... | ... | ... | 4(N+1) | 0 |
| 3 | 2N+3 | ... | ... | ... | ... | ... | 3(N+1) | 0 |
| 2 | N+2 | ... | ... | ... | ... | ... | 2(N+1) | 0 |
| 1 | 1 | 2 | ... | ... | ... | ... | N+1 | 0 |
| L/C | C1 | C2 | ... | ... | ... | ... | C(N+1) | C(N+2) |

Using Table 4 to replace numbers up to $(N+1)^2$ of Table 5, this last one can be written :

Table 6. New matrix D(N+1)

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| N+4 | ... | Low | Opp. | ... | ... | Legend. | Conj. | $(N+2)^2$ |
| N+3 | ... | High | Opp. | ... | ... | Legend. | Conj. | 0 |
| N+2 | A | Low | Opp. | ... | ... | B | $(N+1)^2$ | 0 |
| N+1 | C | High | Opp. | ... | D | $N^2+N$ | 0 | 0 |
| N | E | Low | Opp. | F | $N^2-1$ | $N^2$ | 0 | ↑ |
| N-1 | ... | ... | ... | ... | ... | $N^2-N$ | 0 | 0 |
| ... | ... | ... | ... | ... | ... | ... | 0 | Sierp.'s |
| L k | ... | ... | ... | ... | ... | kN | 0 | matrix |
| ... | ... | ... | ... | ... | ... | ... | 0 | S(N) |
| L4 | 3N+1 | ... | ... | ... | ... | 4N | 0 | 0 |
| L3 | 2N+1 | ... | ... | ... | ... | 3N | 0 | 0 |
| L2 | N+1 | ... | ... | ... | ... | 2N | 0 | 0 |
| L1 | 1 | 2 | ... | ... | ... | N | 0 | ↓ |
| L/C | C1 | C2 | ... | ... | ... | C(N) | C(N+1) | C(N+2) |

Now, let's modify Sierpińsky's conjecture in order to create conjecture D which states that:

For any integer $N > 1$, there is always at least one prime number in each line of a matrix D(N), filled in according to the model in Table 4.

We will now prove this conjecture by induction.

# 3 Main Result : Proof by induction

## 3.1 For N=2 and N=3

One can easily check that conjecture D is verified when N=2 and 3:

Table 7. Matrix D(2)

| | | |
|---|---|---|
| 7 | 8 | 9 |
| 5 | 6 | 0 |
| 3 | 4 | 0 |
| 1 | 2 | 0 |

Table 8. Matrix D(3)

| | | | |
|---|---|---|---|
| 13 | 14 | 15 | 16 |
| 10 | 11 | 12 | 0 |
| 7 | 8 | 9 | 0 |
| 4 | 5 | 6 | 0 |
| 1 | 2 | 3 | 0 |

and one can check that conjecture D is verified by these two matrices which show a beginning of recursivity that we will use in the next step.

## 3.2 From N to N+1

Now, we suppose that conjecture D is verified for a value $N > 3$ and we will prove that it is still true for N+1.

## 3.3 Extension of matrix D(N+1) of Table 6

By using recursively the principle used to transform matrix D(N+1) of Table 5 into the new matrix D(N+1) of Table 6, we get at the end of the process, from any $N > 2$ down to $N = 2$ :

Table 9. Matrix D(N+1) of Table 6 recursively transformed

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| ... | Low | Opp. | for | N+2 | ... | Legend. | Conj. | $(N+2)^2$ |
| ... | High | Opp. | for | N+1 | ... | Legend. | Conj. | 0 |
| A | Low | Opp. | for | N+1 | ... | B | $(N+1)^2$ | 0 |
| C | High | Opp. | for | N | D | $N^2 + N$ | 0 | 0 |
| E | Low | Opp. | for N | F | ... | $N^2$ | 0 | 0 |
| ... | High | Opp. | for | N-1 | ... | 0 | 0 | 0 |
| ... | Low | Opp. | for | N-1 | $(N-1)^2$ | 0 | 0 | 0 |
| ... | High | Opp. | for | N-2 | 0 | 0 | 0 | 0 |
| ... | Low | Opp. | for | N-2 | ... | ... | ... | ... |
| ... | High | Opp. | for | ... | ... | ... | ... | ... |
| 21 | 22 | 23 | 24 | 25 | Low | Opp. | for 5 | 0 |
| 17 | 18 | 19 | 20 | 0 | High | Opp. | for 4 | 0 |
| 13 | 14 | 15 | 16 | 0 | Low | Opp. | for 4 | 0 |
| 10 | 11 | 12 | 0 | 0 | High | Opp. | for 3 | 0 |
| 7 | 8 | 9 | 0 | 0 | Low | Opp. | for 3 | 0 |
| 5 | 6 | 0 | 0 | 0 | High | Opp. | for 2 | 0 |
| 3 | 4 | 0 | 0 | 0 | Low | Opp. | for 2 | 0 |
| 1 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| C1 | C2 | C3 | C4 | C5 | ... | C(N) | C(N+1) | C(N+2) |

## 3.4 Conditional proof of Oppermann's conjecture

If we suppose that conjecture D is true for matrix D(N), it means that Sierpiński's, Oppermann's and Legendre's conjectures are true for N, and particularly, it means that both lines N and N+1 of matrix D(N) of Table 4 contain at least one prime number. As line parts CD and EF of these two lines are parts of Oppermann's conjecture for N and become respectively parts of lines N and N+1 of matrix D(N+1) of Table 6, these lines also contain at least one prime number. Oppermann's conjecture, which is already verified for N=2 and N=3 in matrices D(2) and D(3) of Tables 7 and 8, is therefore proved for D(N+1), conditionally to the validity of conjecture D for matrix D(N).

Noticing that Oppermann's conjecture (just conditionally proved), applies to lines N to N+4 of matrix D(N+1) of Table 6, these five lines therefore contain, still conditionally, at least one prime number.

## 3.5 Conditional proof of Sierpińński's conjecture

Still because Oppermann's conjecture is conditionally proved, it also applies to all the lines of matrix D(N+1) of Table 9 except line 1, but as this line 1 always contains the prime number 2 when $N > 1$, one can therefore conclude that in matrix D(N+1) of Table 9, and still conditionally, all lines contain at least one prime number.

Now, we will do the reverse operation that we did to get Table 9 from Table 6, operation that was exactly to expand the N lines of the N by N matrix of Sierpińński into 2N lines for which we have just shown that each of them contains at least one prime number. We can therefore say that this reverse operation consists, ignoring zeros, to force the *first N* lines of Table 9, each of them containing at least one prime number, into the N lines *N+1 to 2N* of Table 9, each of them also containing at least one prime number. At the end of this process, the N prime numbers of the *first N* lines of Table 9 have been forced into the N lines *N+1 to 2N* of Table 9 which, at the start of the process, contained already at least one prime number. Therefore, in the new matrix D(N+1) of Table 6, the number of prime numbers by line varies from at least 1 in line N (which remains unchanged in the process) to exactly $\pi(N)$ in ligne 1. This proves Sierpińński's conjecture for matrix D(N+1), conditionally to the validity of conjecture D for matrix D(N).

## 3.6 Proof of conjecture D

As we have seen that, conditionally to the validity of conjecture D for matrix D(N), lines *N to N+4* of the new matrix D(N+1) of Table 6 contain at least one prime number according to the conditionally proved Oppermann's conjecture, and that lines *1 to N* of this new matrix D(N+1) contain at least one prime number according to the conditionally proved Sierpiński's conjecture, one can therefore conclude that *all* lines of the new matrix D(N+1) of Table 6 contain at least one prime number. Conjecture D is therefore unconditionally proved.

### 3.7 Unconditional proofs of Oppermann's and Sierpiński's conjectures

Oppermann's and Sierpiński's conjectures which were only proved conditionally to the validity of conjecture D for matrix D(N) are now unconditionally proved, as conjecture D has been unconditionally proved in step 3.6.

### 3.8 Proof of Legendre's conjecture

Finally, as Oppermann's conjecture has been unconditionally proved in step 3.7, Legendre's conjecture is also unconditionally proved.

## 4 Consequences

The above four proved conjectures allow other proofs of conjectures. Three of these other proofs are given hereafter.

### 4.1 Conjecture $d_m = p_{m+1} - p_m = O(\sqrt{p_m})$

Proof - As Oppermann's and Sierpiński's conjectures have been proved, we can therefore say that in lines 1 to N+1 of the following extended matrix S(N), there is always at least one prime number:

Table 10. Extended matrix of Sierpiński S(N)

| N+1 | $N^2+1$ | ... | ... | High | Opp. | Conj. | ... | $p_{m+1}$ | $N^2+N$ |
|---|---|---|---|---|---|---|---|---|---|
| N | $p_m$ | ... | ... | Low | Opp. | Conj. | ... | ... | $N^2$ |
| N-1 | ... | ... | ... | ... | ... | ... | ... | ... | $N^2-N$ |
| N-2 | ... | ... | ... | ... | ... | ... | ... | ... | $N^2-2N$ |
| ... | ... | ... | ... | ... | ... | ... | ... | ... | ... |
| Line k | ... | ... | ... | ... | Sierp. | Matrix | ... | ... | ... |
| ... | ... | ... | ... | ... | ... | ... | ... | ... | ... |
| L4 | 3N+1 | ... | ... | ... | ... | ... | ... | 4N-1 | 4N |
| L3 | 2N+1 | ... | ... | ... | ... | ... | ... | ... | 3N |
| L2 | N+1 | ... | ... | ... | ... | ... | ... | ... | 2N |
| L1 | 1 | ... | ... | ... | ... | ... | ... | ... | N |

As all numbers of the column on the right are composite, for any couple of lines N and N+1, the maximum possible distance $d_m$ between two consecutive prime numbers $p_m$ and $p_{m+1}$ is:

$$d_m = p_{m+1} - p_m \leq (N-1) + (N-1) = 2N - 2 \tag{1}$$

Verifying this on lines (3,4) and (N,N+1), we get respectively:

$$d_m = p_{m+1} - p_m \leq (4N-1) - (2N+1) = 2N - 2$$
$$d_m = p_{m+1} - p_m \leq (N^2+N-1) - (N^2-N+1) = 2N - 2$$

As for lines N and N+1, Oppermann's proved conjecture implies that:

$$(N-1)^2 < N^2 - N < p_m < N^2 < p_{m+1} < N^2 + N < (N+1)^2$$

one also has, taking only the positive square roots:

$$(N-1) < \sqrt{p_m} < N < \sqrt{p_{m+1}} < (N+1) \tag{2}$$

which shows that:

$$\sqrt{p_{m+1}} - \sqrt{p_m} < (N+1) - (N-1) = 2 \tag{3}$$

But as (2) contains:

$$N < \sqrt{p_{m+1}} \tag{4}$$

and as (3) can also be written:

$$\sqrt{p_{m+1}} < \sqrt{p_m} + 2 \tag{5}$$

from relations (1), (4) and (5) applied in that order, we get:

$$d_m = p_{m+1} - p_m \leq 2N - 2 < 2\sqrt{p_{m+1}} - 2 < 2(\sqrt{p_m} + 2) - 2$$

or :

$$d_m = p_{m+1} - p_m < 2\sqrt{p_m} + 2$$

which proves the limit searched for by Hoheisel [3] and others since 1930:

$$d_m = p_{m+1} - p_m = O(\sqrt{p_m})$$

where O( ) is the big O of Landau's notation.

## 4.2 Andrica's conjecture

This conjecture [4] states that for any $m > 0$:

$$\sqrt{p_{m+1}} - \sqrt{p_m} < 1$$

Proof - As $p_{m+1} - p_m > 0$, with relation (3) we have:

$$0 < \sqrt{p_{m+1}} - \sqrt{p_m} < 2$$

which gives, by division by $\sqrt{p_m}$:

$$0 < \frac{\sqrt{p_{m+1}} - \sqrt{p_m}}{\sqrt{p_m}} < \frac{2}{\sqrt{p_m}} \tag{6}$$

¿From Euclid we know that there are infinitely many primes. This implies that when m tends to infinity, we have:

$$\tfrac{2}{\sqrt{p_m}} \rightarrow 0$$

and from (6) :

$$0 < \tfrac{\sqrt{p_{m+1}} - \sqrt{p_m}}{\sqrt{p_m}} < \tfrac{2}{\sqrt{p_m}} \rightarrow 0$$

which implies that when m tends to infinity:

$$lim_{m \rightarrow \infty}(\sqrt{p_{m+1}} - \sqrt{p_m}) = 0 \tag{7}$$

Finally, as the quantity $\sqrt{p_{m+1}} - \sqrt{p_m}$ reaches a maximum of $\sqrt{11} - \sqrt{7} = 0,67... < 1$ for $m = 4$ before tending to zero as proved above in (7), this proves Andrica's conjecture.

## 4.3 Brocard's conjecture (1904)

This conjecture states that for $m \geq 2$ :

$$\pi(p_{m+1}^2) - \pi(p_m^2) \geq 4$$

Proof - As the minimum distance between two primes is $d_{min} = p_{m+1} - p_m = 2$ for the case of twin primes and that, for any N, we have:

$$(N+1) - (N-1) = 2$$

$$(N+1)^2 - (N-1)^2 = 4N \tag{8}$$

it is thus possible to consider the minimum case of twin primes where for N even:

$$p_{m+1} = (N+1)$$

$$p_m = (N-1)$$

so that from (8):

$$p_{m+1}^2 - p_m^2 = 4N$$

Sierpiński's matrix S(N) extended up to line N+7 for an even N between twin primes, can then be written:

Table 11. SierpińsTable 11. Sierpińsski's matrix S(N) extended up to line N+7 for twin primes

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| ... | ... | ... | ... | ... | ... | ... | $(N+3)^2$ | $N^2+7N$ |
| ... | ... | ... | ... | ... | ... | ... | ... | $N^2+6N$ |
| ... | ... | ... | $(N+2)^2$ | ... | ... | ... | ... | $N^2+5N$ |
| ... | ... | ... | ... | ... | ... | ... | ... | $N^2+4N$ |
| $p_{m+1}^2$ | ... | ... | ... | ... | ... | ... | ... | $N^2+3N$ |
| ... | ... | ... | Leg.'s | proved | Conj. | ... | ... | $N^2+2N$ |
| ... | ... | ... | Leg.'s | proved | Conj. | ... | ... | $N^2+N$ |
| ... | ... | ... | Sierp.'s | proved | Conj. | ... | ... | $N^2$ |
| $p_m^2$ | ... | ... | Sierp.'s | proved | Conj. | ... | ... | $N^2-N$ |
| ... | ... | ... | ... | ... | ... | ... | ... | $N^2-2N$ |
| ... | ... | ... | $(N-2)^2$ | ... | ... | ... | ... | $N^2-3N$ |
| ... | ... | ... | ... | ... | ... | ... | ... | $N^2-4N$ |
| ... | ... | ... | ... | ... | ... | ... | $(N-3)^2$ | $N^2-5N$ |
| ... | ... | ... | ... | ... | ... | ... | ... | ... |
| ... | ... | ... | Sierp.'s | proved | Conj. | ... | ... | ... |
| ... | ... | ... | ... | ... | ... | ... | ... | ... |
| 3N+1 | ... | ... | ... | ... | ... | ... | ... | 4N |
| 2N+1 | ... | ... | ... | ... | ... | ... | ... | 3N |
| $p_{m+1}$ | ... | ... | ... | ... | ... | ... | ... | 2N |
| 1 | 2 | 3 | ... | ... | ... | ... | $p_m$ | N(even) |

Here, we have to consider four points. First point, Table 11 is obtained for the case of twin primes. Second point, in Table 11 the square numbers $(N-3)^2$, $(N-2)^2$, $(N-1)^2 = p_m^2$, $(N+1)^2 = p_{m+1}^2$, $(N+2)^2$ and $(N+3)^2$ of general equation $z = (N + or - \sqrt{C})^2$ where z is integer only when the column number C is a square number, are located on a parabola of horizontal axis that is described by the equation $y = L = (N+1) + or - 2\sqrt{C}$ where C is the column number and y=L is the line number. Third point, the already proved Sierpińsski's and Legendre's conjectures provide at least one prime number per line in Table 11. Fourth point, no couple of two consecutive primes can be nearer than twin primes. With these four points and as $p_{m+1}^2$ and $p_m^2$ are located four lines away in Table 11, we can conclude that there is always at least 4 prime numbers between $p_m^2$ and $p_{m+1}^2$. This proves Brocard's conjecture.

# 5 Conclusion

This article provides the proofs of six conjectures: Oppermann's, Sierpiński's, Legendre's, Andrica's, Brocard's and that on $d_m = p_{m+1} - p_m$. It shows a method to solve the overlapping three first conjectures by linking them in a larger one, the D conjecture, that has the property of recursivity.

**Acknowledgements.** Many thanks to the Editors of this journal to have welcomed my article as well as to the team of reviewers.

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