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Well-Posedness of N.G-KdV Class (3+1) Equation Under Noncharscteristic Data

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Abstract

In this article, we introduce the well-posedness of N.G-KdV class (3+1) equation which has an important physically phenomena of the propagation of traveling wave with all types solitary waves. We prove, first of all, that the general class of N.G-KdV class (3+1) equation can be reduced for certain data to a semi-linear system of first order partial differential equations. We find the characteristics of this system and show that it is equivalent to a system of ordinary differential equations in which differentiation is along characteristic direction. These equations can be integrated to give the solution of the system provided that the data is not specified on a characteristic. This method of solution is called the generalized characteristics in three dimensions.

Mathematics Subject Classification: Partial differential equation

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1 Introduction

We establish well- posedness for the nonsingular class by applying the well-known theorems on uniqueness, existence and continuous dependence on the initial conditions for semi-linear systems. As regards the singular class, we divide it further according to the multiplicity of the (essential) characteristic roots of N.G-KdV class (3+1) equation which defined as

$$\begin{aligned}&\alpha_1 u_{xxt} + \alpha_2 u_{xxx} + \alpha_3 u_{xtt} + \alpha_4 u u_{xxt} + \alpha_5 u u_{xxx} + \alpha_6 u u_{xtt} + \alpha_7 u_x u_{xt} + \alpha_8 u_x u_{xx} + \alpha_9 u_x u_{tt}\\ &+\alpha_{10} u_t u_{xx} + \alpha_{11} u_t u_{xt} + \alpha_{12} u_{xxxxx} + \alpha_{13} u_{xxxxt} + \alpha_{14} u_{xxxtt} + \alpha_{15} u_{xtttt} + \alpha_{16} u_{xxttt} + \alpha_{17} u_{ttttt}\\ &= \sigma^2 u_{yy} + \sigma^2 u_{zz}\end{aligned} \tag{1.1}$$

We introduce reduction to a semi-linear system of first partial differential equations.

1.1 Reduction to a semi-linear system of partial differential equations

Consider the initial value problem which corresponds to the equation (1.1). Let the initial curve which supports the data be non-characteristic, as shall be defined in the next section, and without loss of generality let this curve be the usual one $t = 0$, i.e.,

$$\begin{aligned}&\alpha_1 u_{xxt} + \alpha_2 u_{xxx} + \alpha_3 u_{xtt} + \alpha_4 u u_{xxt} + \alpha_5 u u_{xxx} + \alpha_6 u u_{xtt} + \alpha_7 u_x u_{xt} + \alpha_8 u_x u_{xx}\\ &+\alpha_9 u_x u_{tt} + \alpha_{10} u_t u_{xx} + \alpha_{11} u_t u_{xt} + \alpha_{12} u_{xxxxx} + \alpha_{13} u_{xxxxt} + \alpha_{14} u_{xxxtt} + \alpha_{15} u_{xtttt}\\ &+\alpha_{16} u_{xxttt} + \alpha_{17} u_{ttttt} = \sigma^2 u_{yy} + \sigma^2 u_{zz}\end{aligned} \tag{1.1.1}$$

with the initial value problem

$$\begin{aligned} &u(x,y,z,0)= f(x,y,z), \quad u_t(x,y,z,0)=g(x,y,z),\\ &u_{tt}(x,y,z,0)=H(x,y,z), \\ &u_{ttt}(x,y,z,0)=\psi(x,y,z), \quad u_{tttt}(x,y,z,0)=\phi(x,y,z) \end{aligned} \tag{1.1.2}$$

We introduce now the following.

Lemma 1.1.1 *The initial value problem for N.G-KdV class* (3+1) *equation* (1.1.1) *with non -characteristic initial data may be reduced to a non-characteristic initial value problem for a first order system of partial differential equations.*

Proof. Re-writing equation (1.1.1) in the form

$$\begin{aligned} &F(u,p,q,r,s,\tau,v,\omega,\mu,\upsilon,\gamma,\eta,M,N,I,h,L,k,E,O,\pi,\xi,\beta,\hbar)=\\ &\quad \alpha_1\beta+\alpha_2\hbar+\alpha_3\xi+\alpha_4 u\beta+\alpha_5 u\hbar+\alpha_6 u\xi+\alpha_7 qs+\alpha_8 q\tau+\alpha_9 qr+\alpha_{10} p\tau\\ &\quad +\alpha_{11} ps+\alpha_{12} v+\alpha_{13}\omega+\alpha_{14}\mu+\alpha_{15}\gamma+\alpha_{16}\upsilon+\alpha_{17}\eta-\sigma^2 h-\sigma^2 k=0 \end{aligned} \tag{1.1.3}$$

Where

$$\begin{aligned} &p=u_t, q=u_x, r=u_{tt}, s=u_{xt}, \tau=u_{xx}, v=u_{xxxxx}, \omega=u_{xxxxt}, \mu=u_{xxxtt},\\ &\upsilon=u_{xxttt}, \gamma=u_{xtttt}, \eta=u_{ttttt},, M=u_{tttt}, N=u_{tttx}, I=u_{ttxx}, h=u_{txxx},\\ &L=u_{ttt}, k=u_{yy}, E=u_{yt}, O=u_{zz}, \pi=u_{zt}, \xi=u_{ttx}, \beta=u_{txx}, \hbar=u_{xxx} \end{aligned} \tag{1.1.4}$$

We use the initial conditions equation (1.1.2) and differentiating (1.1.3) with respect to t yields, we obtain the following equation.

$$\begin{aligned} \frac{\partial F}{\partial t}&=\frac{\partial F}{\partial u}\frac{\partial u}{\partial t}+\frac{\partial F}{\partial p}\frac{\partial p}{\partial t}+\frac{\partial F}{\partial q}\frac{\partial q}{\partial t}+\frac{\partial F}{\partial r}\frac{\partial r}{\partial t}+\frac{\partial F}{\partial s}\frac{\partial s}{\partial t}+\frac{\partial F}{\partial \tau}\frac{\partial \tau}{\partial t}+\frac{\partial F}{\partial v}\frac{\partial v}{\partial t}+\frac{\partial F}{\partial \omega}\frac{\partial \omega}{\partial t}\\ &+\frac{\partial F}{\partial \mu}\frac{\partial \mu}{\partial t}+\frac{\partial F}{\partial \upsilon}\frac{\partial \upsilon}{\partial t}+\frac{\partial F}{\partial \gamma}\frac{\partial \gamma}{\partial t}+\frac{\partial F}{\partial \eta}\frac{\partial \eta}{\partial t}+\frac{\partial F}{\partial M}\frac{\partial M}{\partial t}+\frac{\partial F}{\partial N}\frac{\partial N}{\partial t}+\frac{\partial F}{\partial I}\frac{\partial I}{\partial t}+\frac{\partial F}{\partial h}\frac{\partial h}{\partial t}\\ &+\frac{\partial F}{\partial L}\frac{\partial L}{\partial t}+\frac{\partial F}{\partial k}\frac{\partial k}{\partial t}+\frac{\partial F}{\partial E}\frac{\partial E}{\partial t}+\frac{\partial F}{\partial O}\frac{\partial O}{\partial t}+\frac{\partial F}{\partial \pi}\frac{\partial \pi}{\partial t}+\frac{\partial F}{\partial \xi}\frac{\partial \xi}{\partial t}+\frac{\partial F}{\partial \beta}\frac{\partial \beta}{\partial t}+\frac{\partial F}{\partial \hbar}\frac{\partial \hbar}{\partial t}\\ &=\alpha_1\beta_t+\alpha_2\hbar_t+\alpha_3\xi_t+\alpha_4[u_t\beta+u\beta_t]+\alpha_5[u_t\hbar+u\hbar_t]+\alpha_6[u_t\xi+u\xi_t]\\ &\quad+\alpha_7[q_t s+qs_t]+\alpha_8[q_t\tau+q\tau_t]+\alpha_9[q_t r+qr_t]+\alpha_{10}[p_t\tau+p\tau_t]+\alpha_{11}[p_t s+ps_t]\\ &\quad+\alpha_{12}v_t+\alpha_{13}\omega_t+\alpha_{14}\mu_t+\alpha_{15}\gamma_t+\alpha_{16}\upsilon_t+\alpha_{17}\eta_t-\sigma^2 h_t-\sigma^2 k_t=0, \end{aligned} \tag{1.1.5}$$

but

$$\begin{array}{lll} u_t = p & h = \hbar_t & s_t = \xi \\ \upsilon_t = \gamma_x & \xi_t = N & v_t = \omega_x \\ \tau_t = \beta & q_t = s & \omega_t = \upsilon_x \\ \beta_t = I & r_t = L & \gamma_t = \eta_z \end{array} \qquad (1.1.6)$$

By inserting equation (1.1.6) into equation (1.1.5), we have

$$\begin{aligned} \frac{\partial F}{\partial t} &= (\alpha_1 + (\alpha_4 + \alpha_5)u)I + \alpha_2 h + (\alpha_3 + \alpha_6 u)N \\ &+ [\alpha_4\beta + \alpha_5\hbar + (\alpha_6 + \alpha_{11})\xi + \alpha_{10}\beta]p + s(\alpha_7 s + \alpha_8\tau + \alpha_9 r) \\ &+ (\alpha_{10}\tau + \alpha_9 s)r + q(\alpha_9 L + \alpha_8\beta) + \alpha_{12}\omega_x + \alpha_{13}\mu_x + \alpha_{14}\upsilon_x \\ &+ \alpha_{15}\eta_x + \alpha_{16}\gamma_x + \alpha_{17}\eta_t - \sigma^2 k_t - \sigma^2 O_t = 0 \end{aligned} \qquad (1.1.7)$$

Thus equations (1.1.6) and (1.1.5) can be combined to form the following system:

$$\begin{aligned} &(\alpha_1 + (\alpha_4 + \alpha_5)u)I + \alpha_2 h + (\alpha_3 + \alpha_6 u)N + [\alpha_4\beta + \alpha_5\hbar + (\alpha_6 + \alpha_{11})\xi + \alpha_{10}\beta]p \\ &+ s(\alpha_7 s + \alpha_8\tau + \alpha_9 r) + (\alpha_{10}\tau + \alpha_9 s)r + q(\alpha_9 L + \alpha_8\beta) + \alpha_{12}\omega_x + \alpha_{13}\mu_x + \alpha_{14}\upsilon_x \\ &+ \alpha_{15}\eta_x + \alpha_{16}\gamma_x + \alpha_{17}\eta_t - \sigma^2 E_y - \sigma^2\pi_z = 0 \end{aligned}$$

$$\begin{array}{lllll} u_t = p & h = \hbar_t & s_t = \xi & & \\ \upsilon_t = \gamma_x & \xi_t = N & v_t = \omega_x & & \\ \tau_t = \beta & q_t = s & \omega_t = \upsilon_x & & \\ \beta_t = I & r_t = L & \gamma_t = \eta_z & k_t = E_y & O_t = \pi_z \end{array} \qquad (1.1.8)$$

This is a system of first order partial differential equation in the dependent variable $u, p, q, r, s, \tau, v, \omega, \mu, \upsilon, \gamma, \eta, M, N, I, h, L, k, E, O, \pi, \xi, \beta$ and $\hbar$. The initial conditions may be obtained from equations (1.1.2) and amount to the specification of $u, p, q, r, s, \tau, v, \omega, \mu, \upsilon, \gamma, \eta, M, N, I, h, L, k, E, O, \pi, \xi, \beta$ and $\hbar$. However, η is not known explicitly, but since the initial conditions are assumed specified on non-characteristic curve, then η may always be determined. Thus initial condition $t = 0$ become

$$u(x,y,z,0)=f(x,y,z) \qquad p(x,y,z,0)=g(x,y,z) \qquad q(x,y,z,0)=f_x(x,y,z)$$
$$\tau(x,y,z,0)=f_{xx}(x,y,z) \qquad s(x,y,z,0)=g_x(x,y,z) \qquad r(x,y,z,0)=H(x,y,z)$$
$$v(x,y,z,0)=f_{xxxxx}(x,y,z) \qquad \omega(x,y,z,0)=g_{xxxx}(x,y,z) \qquad \mu(x,y,z,0)=H_{xxx}(x,y,z)$$
$$L(x,y,z,0)=R(x,y,z) \qquad M(x,y,z,0)=K(x,y,z) \qquad \upsilon(x,y,z,0)=R_{xx}(x,y,z)$$
$$\gamma(x,y,z,0)=K_x(x,y,z) \qquad N(x,y,z,0)=R_x(x,y,z) \qquad I(x,y,z,0)=H_{xx}(x,y,z)$$
$$h(x,y,z,0)=g_{xxx}(x,y,z) \qquad k(x,y,z,0)=C(x,y,z) \qquad E(x,y,z,0)=g_y(x,y,z)$$
$$O(x,y,z,0)=\theta(x,y,z) \qquad \pi(x,y,z,0)=g_z(x,y,z) \qquad \xi(x,y,z,0)=H_x(x,y,z)$$
$$\beta(x,y,z,0)=g_{xx}(x,y,z) \qquad \hbar(x,y,z,0)=f_{xxx}(x,y,z)$$
$$\eta(x,y,z,0)=G(f,g,f_x,f_{xx},g_x,H,f_{xxxxx},g_{xxxx},H_{xxx},R,K,R_{xx},K_x,R_x,H_{xx},g_{xxx},C,g_y,$$
$$\theta,g_z,H_x,g_{xx},f_{xxx}) \tag{1.1.9}$$

The system (1.1.8) can be expressed in the matrix form:

$$U_t + A_{24\times 24}U_x + B_{24\times 24}U_y + D_{24\times 24}U_z + C_{24\times 24} = 0 \tag{1.1.10}$$
$$U(x,y,z,0) = w(x,y,z).$$

where $U, A_{24\times 24}, B_{24\times 24}, D_{24\times 24}$ and $C_{24\times 24}$ are

$$U^T = [u,p,q,r,s,\tau,v,\omega,\mu,\upsilon,\gamma,\eta,M,N,I,h,L,k,E,O,\pi,\xi,\beta,\hbar] \tag{1.1.11}$$
$$C^T_{25\times 25} = [-p,-r,-s,-L,-\xi,-\beta,0,0,0,0,0,J,-\eta,-\gamma,-\upsilon,-\mu,-M,0,0,0,0,-N,-I,-h]$$

With

$$J=\frac{1}{\alpha_{17}}[(\alpha_1+(\alpha_4+\alpha_5)u)I+\alpha_2 h+(\alpha_3+\alpha_6 u)N+[\alpha_4\beta+\alpha_5\hbar+(\alpha_6+\alpha_{11})\xi$$
$$+\alpha_{10}\beta]p+s(\alpha_7 s+\alpha_8\tau+\alpha_9 r)+(\alpha_{10}\tau+\alpha_9 s)r+q(\alpha_9 L+\alpha_8\beta)], \qquad \alpha_{17}\neq 0,$$

$$A_{24\times 24}=\begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&\frac{\alpha_{12}}{\alpha_{17}}&\frac{\alpha_{13}}{\alpha_{17}}&\frac{\alpha_{14}}{\alpha_{17}}&\frac{\alpha_{16}}{\alpha_{17}}&\frac{\alpha_{15}}{\alpha_{17}}&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{bmatrix},$$

$$B_{24\times 24}=\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-\sigma^2}{\alpha_{17}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$$D_{24\times 24}=\begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\dfrac{-\sigma^2}{\alpha_{17}}&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0\\
0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{bmatrix}$$

This completes the proof of lemma. □

Lemma 1.1.2 *The system of equation* (1.1.10) *under the non singular linear transformation*

$$S=\begin{pmatrix}\alpha^* & \beta^* & \gamma^* & 0\\ \delta^* & \mu^* & \omega^* & 0\\ \phi^* & \eta^* & \psi^* & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

This reduces to the system $U_t+A_1U_\zeta+C_{24\times 24}=0.$

Proof. Let

$$x\to\zeta^*=\alpha^*x+\beta^*y+\gamma^*z,$$
$$y\to\varepsilon^*=\delta^*x+\mu^*y+\omega^*z,$$
$$z\to\xi^*=\phi^*x+\eta^*y+\psi^*z,$$

and $t \to t' = t$. Using this transformation, we have

$$U_x = \frac{\partial U}{\partial \zeta^*}\frac{\partial \zeta^*}{\partial x} + \frac{\partial U}{\partial \varepsilon^*}\frac{\partial \varepsilon^*}{\partial x} + \frac{\partial U}{\partial \xi^*}\frac{\partial \xi^*}{\partial x} = U_{\zeta^*}\alpha^* + U_{\varepsilon^*}\delta^* + U_{\xi^*}\phi^*,$$

$$U_y = \frac{\partial U}{\partial \zeta^*}\frac{\partial \zeta^*}{\partial y} + \frac{\partial U}{\partial \varepsilon^*}\frac{\partial \varepsilon^*}{\partial y} + \frac{\partial U}{\partial \xi^*}\frac{\partial \xi^*}{\partial y} = U_{\zeta^*}\beta^* + U_{\varepsilon^*}\mu^* + U_{\xi^*}\eta^*,$$

$$U_z = \frac{\partial U}{\partial \zeta^*}\frac{\partial \zeta^*}{\partial z} + \frac{\partial U}{\partial \varepsilon^*}\frac{\partial \varepsilon^*}{\partial z} + \frac{\partial U}{\partial \xi^*}\frac{\partial \xi^*}{\partial z} = U_{\zeta^*}\gamma^* + U_{\varepsilon^*}\omega^* + U_{\xi^*}\psi^*.$$

Substituting these values in the system of equation (1.1.10) we have

$$U_t + A_1 U_{\zeta^*} + A_2 U_{\varepsilon^*} + A_3 U_{\xi^*} + C_{24\times 24} = 0, \qquad (1.1.12)$$

where $A_1 = \alpha^* A_{24\times 24} + \beta^* B_{24\times 24} + \gamma^* D_{24\times 24}$, $A_2 = \delta^* A_{24\times 24} + \mu^* B_{24\times 24} + \omega^* D_{24\times 24}$ and $A_3 = \phi^* A_{24\times 24} + \eta^* B_{24\times 24} + \psi^* D_{24\times 24}$.

Since any solution of this equation depends on the matrices, then the space of solution is 9-dimensional space span by the matrices $\alpha^*, \beta^*, \gamma^*, \delta^*, \mu^*, \omega^*, \phi^*, \eta^*, \psi^*$. Also the properties of eigenvalues and eigenvectors under linear transformations denote by [1]. So we shall study the solution on a subspace of the space of all solutions. This subspace satisfies the condition, $\alpha^*, \beta^*, \gamma^* = I$ and $A_2, A_3 = 0$. Then the equation is reduced to

$$U_t + A_1 U_{\zeta^*} + C_{24\times 24} = 0, \qquad (1.1.13)$$

where $A_1 = A_{24\times 24} + B_{24\times 24} + D_{24\times 24}$ and defined the form

$$A_{24\times 24}+B_{24\times 24}+D_{24\times 24}=\begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&\frac{\alpha_{12}}{\alpha_{17}}&\frac{\alpha_{13}}{\alpha_{17}}&\frac{\alpha_{14}}{\alpha_{17}}&\frac{\alpha_{16}}{\alpha_{17}}&\frac{\alpha_{15}}{\alpha_{17}}&0&0&0&0&0&0&\frac{-\sigma^2}{\alpha_{17}}&0&\frac{-\sigma^2}{\alpha_{17}}&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0\\
0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0\\
0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{bmatrix} \quad (1.1.14)$$

Definition 1.1.3 *The system of equation* (1.1.13) *is called quasi-linear if* A_1 *depend on* U *. If* A_1 *independent of* U *, and* C *depends on* U *, but not linearly, the system is called Semi-linear. If* C *is also a linear function of* U *, the system is called linear.*

Theorem 1.1.4 *The initial value problem of equation* (1.1.1) *for the general class of N.G-KdV class (3+1) equation with non- characteristic data can be reduced to a non-characteristic initial value problem for a first order semi-linear system of partial differential equations.*

Proof . Making use of the definition (1.1.3) of the semi-linear system and the result of lemma (1.1.2) the theorem is proved. □

2 Characteristics of the system

Definition 2.1 *A characteristic of the system* (1.1.13) *is a curve along which the values of* U *, combined with the equation* (1.1.13) *is insufficient to determine the derivatives of the normal this curve.*

The problem of determining the derivatives of U normal to our data is easily resolved by considering the effect on system (1.1.13) of a change of coordinates

$$t \to t \quad \text{and} \quad \zeta^* \to \Phi(\zeta^*, t) = \text{constant}. \tag{2.1}$$

Then the system (1.1.13) reduces under equation (2.1) to

$$\left[\frac{\partial U}{\partial t} + \frac{\partial U}{\partial \Phi}\frac{\partial \Phi}{\partial t}\right] + A_1\left[\frac{\partial U}{\partial \Phi}\frac{\partial \Phi}{\partial \zeta^*}\right] + C_{24\times 24}(U) = 0$$

i.e., $$\left[\mathrm{I}\frac{\partial \Phi}{\partial t} + A_1\frac{\partial \Phi}{\partial \zeta^*}\right]\left[\frac{\partial U}{\partial \Phi}\right] + \left[\frac{\partial U}{\partial t}\right] + C_{24\times 24}(U) = 0 \tag{2.2}$$

where $\left[\frac{\partial U}{\partial \Phi}\right]$ is the normal derivative of U to, this normal derivative is determined

if $\det\left[\mathrm{I}\frac{\partial \Phi}{\partial t} + A_1\frac{\partial \Phi}{\partial \zeta^*}\right] \neq 0.$

Combining this result with definition (2.1), then characteristic of the system (1.1.13) is given by the equation

$$\det\left[\mathrm{I}\frac{\partial \Phi}{\partial t} + A_1\frac{\partial \Phi}{\partial \zeta^*}\right] = 0. \tag{2.3}$$

Putting $\lambda = \dfrac{-\frac{\partial \Phi}{\partial t}}{\frac{\partial \Phi}{\partial \zeta^*}} = \dfrac{\partial \zeta^*}{\partial t}$, then equation (2.3) can be written as

$$\det(A_1 - \lambda \mathrm{I}) = 0 \tag{2.4}$$

Equation (2.4) is called the characteristic equation of the system (1.1.13), where λ is now an eigenvalue of the matrix A_1.

The above analysis leads to the following.

Theorem 2.2 *The characteristic of the system* (1.1.13) *which corresponds to the N.G-KdV class (3+1) equation (1.1.1) is given by the roots of the equation*

$$\lambda^{19}(\alpha_{12}-\alpha_{13}\lambda+\alpha_{14}\lambda^2-\alpha_{16}\lambda^3+\alpha_{15}\lambda^4-\alpha_{17}\lambda^5)=0 \qquad (2.5)$$

where $\lambda=\left(\frac{\partial\zeta}{\partial t}\right)$.

Proof. By using the expression of A_1 as in (1.1.14) and expanding $\det(A_1-\lambda\mathrm{I})=0$, then obviously equation (2.5) follows and the theorem is proved.□

Definition 2.3

(1) *If all the roots of equation* (2.4) *are real and distinct the system of equation* (1.1.13) *is called "totally hyperbolic".*

(2) *If some of the roots of equation* (2.4) *are complex, the system is called "ultra-hyperbolic".*

(3) *If all the roots of equation* (2.4) *are complex, the system* (1.1.13) *is "elliptic".*

(4) *The system* (1.1.13) *is hyperbolic if equation* (2.4) *has at least one real root.*

In the following we shall concentrate only on the case in which the system is hyperbolic.

2.1 Normal form of the first order system

In the previous section we have demonstrated that the characteristic of equation (1.1.13) are given by the eigenvalues of eigenvalue problem

$$A_1X = \lambda X .$$

It is now convenient to transform the system (1.1.13) to a simple form in which the differentiation should be in one direction only, i.e., directed along a characteristic of the system. This new system is called the *normal form* of equation (1.1.13).

For doing this let the eigenvectors corresponding to the eigenvalues λ_i of

A_1 span E^{24} and let T be the matrix in which each column is one of these eigenvectors, then T is nonsingular. Suppose that

$$U = TV \tag{2.1.1}$$

Inserting this transformation into (1.1.13) then

$$(TV)_t + A_1(TV)_\zeta + C_{24\times 24} = 0, \quad TV(\zeta, 0) = H(\zeta). \tag{2.1.2}$$

Hence

$$TV_t + T_tV + A_1TV_\zeta + A_1T_\zeta V + C_{24\times 24} = 0. \tag{2.1.3}$$

Multiplying both sides of equation by T^{-1}

$$V_t + T^{-1}A_1TV_\zeta + \tilde{C} = 0, \tag{2.1.4}$$

such that

$$\tilde{C} = T^{-1}C + T^{-1}A_1T_\zeta V + T^{-1}T_tV. \tag{2.1.5}$$

Since A_1 is a matrix of constant coefficients, equation (1.1.13), and then the eigenvalues of A_1 don't depend on ζ, t and U consequently T doesn't depend on ζ, t and U this implies that $T_t = 0 = T_\zeta$.

Since $T^{-1}A_1T = D$ is diagonal, then equation (2.1.4) can be written as

$$V_t + DV_\zeta + \tilde{C} = 0, \quad D = diag(\lambda_1, \lambda_2, ..., \lambda_{24}), \tag{2.1.6}$$

with the initial condition

$$V(\zeta, 0) = T^{-1}U(\zeta, t) = \psi(\zeta). \tag{2.1.7}$$

Finally, equation (2.1.7) can be written in terms of components and the i-th component, which corresponds to the i-th characteristic, has the form

$$V_t^i + \lambda^i V_\zeta^i + C^i = 0, \qquad V^i(\zeta, 0) = \psi^i(\zeta). \tag{2.1.8}$$

From the theory of a single first order partial differential equation, it follows that on the characteristic traces for the equation, the equation reduces to an ordinary differential equation. Hence, $V^{i}+\lambda^{i}V_{\zeta}^{i}$ is a directional derivative in the direction λ^{i}. Thus, every equation in the form (2.1.6) and (2.1.7) contains a differentiation in one direction only which is the characteristic direction. The form of equation (2.1.6) and (2.1.7) is the normal form of the system (1.1.13).

Example 2.1.1 Consider the initial value problem

$$\begin{aligned}
&u_{tt}=u_{xx}+u_{yy}+u_{zz} \qquad -\infty<x<\infty \quad ,t\geq 0.\\
&\qquad\qquad\qquad\qquad\qquad -\infty<y<\infty\\
&\qquad\qquad\qquad\qquad\qquad -\infty<z<\infty\\
&u(x,y,z,0)=f(x,y,z) \qquad q(x,y,z,0)=g(x,y,z)\\
&p(x,y,z,0)=f_x(x,y,z) \qquad r(x,y,z,0)=f_{xx}(x,y,z)\\
&s(x,y,z,0)=g_x(x,y,z) \qquad k(x,y,z,0)=f_{yy}(x,y,z)\\
&o(x,y,z,0)=g_y(x,y,z) \qquad h(x,y,z,0)=f_{zz}(x,y,z)\\
&A(x,y,z,0)=g_z(x,y,z) \qquad \tau(x,y,z,0)=G(f,g,f_x,f_{xx},g_x,f_{yy},g_y,f_{zz},g_z)
\end{aligned} \tag{2.1.9}$$

To find the solution of this equation by using the method of characteristics, we firstly reduce it to a system of first order partial differential equations. Thus let

$$F(u,p,q,r,s,\tau,k,o,h,A)=\tau-r-k-h=0, \tag{2.1.10}$$

where

$$\begin{aligned}
&q=u_t, \quad p=u_x, \quad r=u_{xx}, \quad s=u_{xt},\\
&\tau=u_{tt}, \quad k=u_{yy}, \quad o=u_{yt}, \quad h=u_{zz}, \quad A=u_{zt}.
\end{aligned} \tag{2.1.11}$$

Differentiating equation (2.1.9) with respect to t

$$\begin{aligned}
\frac{\partial F}{\partial t}=&\frac{\partial F}{\partial u}\frac{\partial u}{\partial t}+\frac{\partial F}{\partial p}\frac{\partial p}{\partial t}+\frac{\partial F}{\partial q}\frac{\partial q}{\partial t}+\frac{\partial F}{\partial r}\frac{\partial r}{\partial t}+\frac{\partial F}{\partial s}\frac{\partial s}{\partial t}+\frac{\partial F}{\partial \tau}\frac{\partial \tau}{\partial t}\\
&+\frac{\partial F}{\partial k}\frac{\partial k}{\partial t}+\frac{\partial F}{\partial o}\frac{\partial o}{\partial t}+\frac{\partial F}{\partial h}\frac{\partial h}{\partial t}+\frac{\partial F}{\partial A}\frac{\partial A}{\partial t}
\end{aligned} \tag{2.1.12}$$

Using equation (2.1.11), then equation (2.1.9) reduces to the form

$$\tau_t-r_t-k_t-h_t=0, \tag{2.1.13}$$

where

$$\begin{aligned} &q = u_t, \quad p_t = s, \quad s_t = \tau_x, \quad r_t = s_x, \\ &s_t = \tau_x, \quad k_t = o_y, \quad o_t = \tau_y, \quad h_t = A_z, \quad A_t = \tau_z. \end{aligned} \tag{2.1.14}$$

Combining equations (2.1.13) and (2.1.14), then the original equation (2.1.9) reduces to the system

$$\begin{aligned} &q_t = \tau, \quad p_t = s, \quad s_t = \tau_x, \quad r_t = s_x, \\ &s_t = \tau_x, \quad k_t = o_y, \quad o_t = \tau_y, \quad h_t = A_z, \quad A_t = \tau_z, \end{aligned} \tag{2.1.15}$$

with the initial data

$$\begin{aligned} &u(x,y,z,0) = f(x,y,z) \quad q(x,y,z,0) = g(x,y,z) \\ &p(x,y,z,0) = f_x(x,y,z) \quad r(x,y,z,0) = f_{xx}(x,y,z) \\ &s(x,y,z,0) = g_x(x,y,z) \quad k(x,y,z,0) = f_{yy}(x,y,z) \\ &o(x,y,z,0) = g_y(x,y,z) \quad h(x,y,z,0) = f_{zz}(x,y,z) \\ &A(x,y,z,0) = g_z(x,y,z) \quad \tau(x,y,z,0) = G(f, g, f_x, f_{xx}, g_x, f_{yy}, g_y, f_{zz}, g_z) \end{aligned} \tag{2.1.16}$$

Clearly, the system of equations (2.1.15) and (2.1.16) can be written in the matrix form

$$U_t = AU_x + BU_y + DU_z + CU, \qquad U(x,y,z,0) = H(x,y,z), \tag{2.1.17}$$

where

$$A = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \end{bmatrix}, \quad B = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \end{bmatrix}$$

$$C=\begin{bmatrix}0&0&1&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0&0\\0&0&0&0&1&0&0&0&0&0\\0&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\end{bmatrix},\quad D=\begin{bmatrix}0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&1\\0&0&0&0&0&1&0&0&0&0\end{bmatrix},$$

and $U^T=[u,p,q,r,s,\tau,k,o,h,A]$. Thus the characteristic roots are $(0,0,0,0,0,0,0,0,1,-1)$. Then, the eigenvectors can be written in the form of the matrix T

$$T=\begin{pmatrix}1&0&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&-1&1\\0&0&0&0&1&0&0&0&1&1\\0&0&0&0&0&1&0&0&-1&1\\0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&0&1\end{pmatrix}$$

Clearly, the inverse of this matrix exists and has the form

$$T^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

then we have

$$T^{-1}AT = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$T^{-1}BT = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$T^{-1}CT=\begin{bmatrix}0&0&1&0&0&0&0&0&0&0\\0&0&0&0&1&0&0&0&1&1\\0&0&0&0&0&1&0&0&-1&1\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\end{bmatrix},$$

$$T^{-1}DT=\begin{bmatrix}0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&-1&0&0&1&0\\0&0&0&0&0&-1&0&0&1&-2\\0&0&0&0&0&-1&0&0&1&1\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&1\\0&0&0&0&0&1&0&0&-1&1\end{bmatrix},$$

then using the lemma (1.1.2). The equation (2.1.17) reduce to the form

$$U_t = A_1 U_{\xi^*} + CU,$$

Now, let $U = TV$, then the system of equation (2.1.17) reduces to

$$(TV)_t = A_1 (TV)_{\xi^*} + C(TV). \tag{2.1.18}$$

Since T doesn't depend on x, y, z and t , then equation (2.1.18) implies that

$$V_t = \left(T^{-1}A_1T\right)V_{\xi^*} + \left(T^{-1}CT\right)V, \tag{2.1.19}$$

then using the Lemma 1.1.2. The equation (2.1.19) reduce to the form

$$V_t = \left(T^{-1}A_1T\right)V_{\zeta^*} + \left(T^{-1}CT\right)V, \qquad V(\zeta^*,0) = H(\zeta^*). \tag{2.1.20}$$

Such that $A_1 = A + B + D$, i.e.,

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \\ V_8 \\ V_9 \\ V_{10} \end{bmatrix}_t = \begin{bmatrix} 0 \\ 0 \\ 0 \\ V_5 - V_6 + 2V_9 + V_{10} \\ -V_{10} \\ V_5 - V_6 + V_8 + 2V_9 + 2V_{10} \\ V_8 \\ V_6 - V_9 + V_{10} \\ V_{10} \\ V_6 - V_9 + V_{10} \end{bmatrix} + \begin{bmatrix} V_3 \\ V_5 + V_9 + V_{10} \\ V_6 - V_9 + V_{10} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

with the initial data $V(\zeta^*, 0) = T^{-1}U(\zeta^*, 0)$, i.e.,

$$V = [f, f_x, g, -g_z + f_{zz} + f_{xx}, -g_z - f_{zz} + g_x, -g_z + f_{zz}, f_{yy}, g_y]^T. \tag{2.1.21}$$

Next, the first component yields, we find that

$$\frac{\partial V_1}{\partial t} = V_3 = g(\zeta^*). \tag{2.1.22}$$

So, by integrating equation (2.1.22), we get that

$$V_1 = \int_0^t g(\zeta^*) d\zeta^* = \frac{1}{2} \int_{x+y+z-t}^{x+y+z+t} g(\zeta^*) d\zeta^*. \tag{2.1.23}$$

Since $V_1(x, y, z, 0) = u(x, y, z, 0) = f(x, y, z)$, equation (2.1.23) implies that

$$V_1\big|_P = u(x, y, z, t) = \int_0^t g(\zeta^*) d\zeta^* = \frac{1}{2} \int_{x+y+z-t}^{x+y+z+t} g(\zeta^*) d\zeta^*. \tag{2.1.24}$$

3 Well-posedness of N.G-KdV class (3+1) equation

This section is devoted to the proof of the well-posedness of N.G-KdV class (3+1) equation (1.1.1), under characteristic data, by using the method of characteristics. For this purpose, we first establish an integral formula for the solution of this equation.

3.1 Integral formula of the solution

It has been proved in subsection (1.1), that the equation (1.1.1) of N.G-KdV class equation can be reduced to the semi-linear system of first order partial differential equations.

$$U_t + A_1 U_{\zeta^*} + C_{24\times 24} = 0, \qquad U(\zeta^*, 0) = H(\zeta^*),$$

and it has been shown that the latter reduces to the normal form

$$V_t + MV_{\zeta^*} + \tilde{C} = 0, \qquad V(\zeta^*, 0) = \psi(\zeta^*),$$

where $M = diag(\lambda_1, \lambda_2, ..., \lambda_{24})$ and $\tilde{C}$ is defined in (2.1.5).

Thus the ith component is

$$V_t^i + \lambda^i V_{\zeta^*} = \tilde{\tilde{C}}^i, \qquad V^i(\zeta^*, 0) = \psi^i(\zeta^*) \qquad (i = 1, 2, ...), \qquad (3.1.1)$$

Such that $\tilde{\tilde{C}} = -\tilde{C}$. Along the characteristics, equations (3.1.1) are ordinary differential equations, since the differentiation is now in one direction only. This is the root of establishing the integral formula.

Definition 3.1.1 (Domain of determinacy) *Consider the linear or semi-linear system* $U_t + A_1 U_{\xi^*} + C = 0$. *The domain of determinacy for this system is defined as the set of all points* $p(\zeta^*, t)$ *which can be connected to the initial interval by characteristic trajectories.*

Now, if $p(\zeta^*, t)$ is any point in the domain of determinacy of the system (3.1.1), then integrating along the characteristic pq_i , we have

$$V^i(p) = V^i(q_i) + \int_{q_i}^{p} \tilde{\tilde{C}}^i(V^i)\, d\eta, \qquad (3.1.2)$$

where q are those points the initial interval, connected to p by the i-th characteristic for all $i = 1, 2, ...$, then equation (3.1.2) converted to

$$V^i(p) = \psi^i(q) + \int_{q_i}^{p} \tilde{\tilde{C}}^i (V^i) d\eta, \tag{3.1.3}$$

This is the integral formula of the underlying system.

3.1.2 Uniqueness

To prove the uniqueness of the solution of the system

$$U_t + A_1 U_{\zeta^*} + C_{24\times 24} = 0, \qquad U(\zeta^*, 0) = H(\zeta^*),$$

where A_1 and $C_{25\times 25}$ are given by equation (1.1.13), it is important to note that it can be rewritten in the form

$$U_t + A_1 U_{\zeta^*} + \mathrm{E}_{24\times 24} U = 0, \qquad U(\zeta^*, 0) = H(\zeta^*), \tag{3.1.4}$$

since by the expression of $C_{24\times 24}$, it can be shown that $C_{24\times 24} = \mathrm{E}_{24\times 24} U$, where U is given by equation (1.1.11) and

$$\mathrm{E}_{24\times24} = \begin{bmatrix}
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{(\alpha_4\beta+\alpha_5\hbar+(\alpha_6+\alpha_{11})\xi+\alpha_{10}\beta)}{\alpha_{17}} & \frac{(\alpha_9 L+\alpha_8\beta)}{\alpha_{17}} & \frac{(\alpha_{10}\tau+\alpha_9 s)}{\alpha_{17}} & \frac{(\alpha_7 s+\alpha_8\tau+\alpha_9 r)}{\alpha_{17}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{(\alpha_3+\alpha_6 u)}{\alpha_{17}} & \frac{(\alpha_1+(\alpha_4+\alpha_5)u)}{\alpha_{17}} & \frac{\alpha_2}{\alpha_{17}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

Lemma 3.1.3 [2] *If* $W(\xi^*,t)$ *is a solution of the linear system*

$$W_t + A_1 W_{\xi^*} + \mathrm{E}_{24\times 24} W = 0, \qquad W(\xi^*,0) = 0, \qquad (3.1.5)$$

where A_1 *is a symmetric matrix, then* $W = 0$.

Using the result of Lemma 3.1.3, then the uniqueness of the solution of the original system 3.1.4 can be proved.

Theorem 3.1.4 *If* U *is a solution of the semi-linear system* (3.1.4), *then* U *is unique*.

Proof. The semi-linear system (3.1.4) can be reduced, by nonsingular linear transformation to the normal form:

$$V_t + MV_{\xi^*} + \tilde{C}(V) = 0, \qquad V(\zeta^*,0) = \psi(\zeta^*), \qquad (3.1.6)$$

where $U = \mathrm{K}V$, $M = diag(\lambda_1,\lambda_2,...,\lambda_{24})$, $\tilde{C} = \Upsilon^{-1}E_{24\times 24}\Upsilon$ and λ_i for every $i = 1,2,...,24$ are the eigenvalues of the matrix A_1. Hence, to prove the uniqueness of the system (3.1.4) it suffices, without loss of generality, to prove that the solution of equation (3.1.6) is unique.

Let V_1 and V_2 be two solutions of equation (3.1.6) and $W = V_1 - V_2$, then W satisfies

$$W_t + MW_{\xi^*} + \tilde{C}(V_1) - \tilde{C}(V_2) = 0, \quad W(\xi^*,0) = 0$$

Using the mean-value theorem, we have

$$\tilde{C}(V_1) - \tilde{C}(V_2) = k(V_1,V_2)(V_1 - V_2) = k(V_1,V_2)W.$$

Then equation (3.1.7) reduce to

$$W_t + MW_{\xi^*} + k(V_1,V_2)W = 0, \qquad W(\xi^*,0) = 0, \qquad (3.1.7)$$

then the previous equation (3.1.7) with A_1 diagonal and k doesn't depend on W . Now since $W(\xi^*,0) = 0$ then by using lemma (3.1.3) $W(\xi^*,t) = 0$, i.e., $V_1 = V_2$. Consequently the solution of the equation (3.1.4) is unique. □

Example 3.1.5 Consider the initial value problem

$$u_{xxt}+u_{xtt}+uu_{xxt}+u_t u_{xt}+u_{xtttt}+u_{ttttt}-u_{yy}-u_{zz}=0, \tag{3.1.8}$$

$$\begin{aligned} &u(x,y,z,0)=f(x,y,z), \qquad u_t(x,y,z,0)=g(x,y,z)\\ &u_{tt}(x,y,z,0)=H(x,y,z), \qquad u_{ttt}(x,y,z,0)=\varphi(x,y,z)\\ &u_{tttt}(x,y,z,0)=\Omega(x,y,z), \end{aligned} \tag{3.1.9}$$

It is clear that the equation (3.1.8) belongs to the N.G-KdV class (3+1) equation (1.1). To prove that the solution of this problem is unique, we reduce the problem into a system of first order partial differential equations. Thus let

$$F(u,p,q,r,s,\tau,\omega,\mu,\upsilon,\gamma,\eta,M,N,I,h,L,k,E,O,\pi,\xi,\beta)=0, \tag{3.1.10}$$

$$\begin{aligned} &p=u_t, \quad q=u_x, \quad r=u_{tt}, \quad s=u_{xt}, \quad \tau=u_{xx}, \quad \omega=u_{xxxxt}, \quad \mu=u_{xxxtt},\\ &\upsilon=u_{xxttt}, \quad \gamma=u_{xtttt}, \quad \eta=u_{ttttt}, \quad M=u_{tttt}, \quad N=u_{xttt}, \quad I=u_{ttxx}, \quad h=u_{txxx},\\ &L=u_{ttt}, \quad k=u_{yy}, \quad E=u_{yt}, \quad O=u_{zz}, \quad \pi=u_{zt}, \quad \xi=u_{ttx}, \quad \beta=u_{txx}, \end{aligned} \tag{3.1.11}$$

then by using equation (3.1.11) into equation (3.1.8)

$$\xi+\beta(1+u)+ps+\gamma+\eta-k-O=0. \tag{3.1.12}$$

Differentiating equation (3.1.10) with respect to t and using equation (3.1.11), we get

$$\begin{aligned} \frac{\partial F}{\partial t}=&\frac{\partial F}{\partial u}u_t+\frac{\partial F}{\partial p}p_t+\frac{\partial F}{\partial q}q_t+\frac{\partial F}{\partial r}r_t+\frac{\partial F}{\partial s}s_t+\frac{\partial F}{\partial \tau}\tau_t+\frac{\partial F}{\partial \omega}\omega_t+\frac{\partial F}{\partial \mu}\mu_t+\frac{\partial F}{\partial \upsilon}\upsilon_t\\ &+\frac{\partial F}{\partial \gamma}\gamma_t+\frac{\partial F}{\partial \eta}\eta_t+\frac{\partial F}{\partial M}M_t+\frac{\partial F}{\partial N}N_t+\frac{\partial F}{\partial I}I_t+\frac{\partial F}{\partial L}L_t+\frac{\partial F}{\partial k}k_t+\frac{\partial F}{\partial E}E_t\\ &+\frac{\partial F}{\partial O}O_t+\frac{\partial F}{\partial \pi}\pi_t+\frac{\partial F}{\partial \xi}\xi_t+\frac{\partial F}{\partial \beta}\beta_t, \end{aligned}$$

then equation (3.1.10) reduce to the form

$$\xi_t+\beta u_t+\beta_t(1+u)+ps_t+p_t s+\gamma_t+\eta_t-k_t-O_t=0, \tag{3.1.13}$$

But

$$\begin{aligned} &p=u_t, \quad s=q_t, \quad \beta=\tau_t, \quad \beta_t=I, \quad \xi_t=N, \quad p_t=r, \quad s_t=\xi,\\ &\gamma_t=\eta_x, \quad k_t=E_y, \quad O_t=\pi_z, \quad \omega_t=\mu_x, \quad \mu_t=\upsilon_x, \quad \pi_t=r_z, \quad E_t=r_y. \end{aligned} \tag{3.1.14}$$

Combining equation (3.1.13) and (3.1.14) then, we get

$$N+\beta p+I(1+u)+p\xi+rs+\eta_x+\eta_t-E_y-\pi_z=0, \qquad (3.1.15)$$

with the initial data

$$\begin{array}{ll}
u(x,y,z,0)=f(x,y,z) & p(x,y,z,0)=g(x,y,z)\\
q(x,y,z,0)=f_x(x,y,z) & r(x,y,z,0)=H(x,y,z)\\
s(x,y,z,0)=g_x(x,y,z) & \tau(x,y,z,0)=f_{xx}(x,y,z)\\
L(x,y,z,0)=\varphi(x,y,z) & \xi(x,y,z,0)=H_x(x,y,z)\\
\beta(x,y,z,0)=g_{xx}(x,y,z) & M(x,y,z,0)=\Omega(x,y,z)\\
N(x,y,z,0)=\varphi_x(x,y,z) & I(x,y,z,0)=H_{xx}(x,y,z)\\
\gamma(x,y,z,0)=\Omega_x(x,y,z) & k(x,y,z,0)=f_{yy}(x,y,z)\\
E(x,y,z,0)=g_y(x,y,z) & O(x,y,z,0)=f_{zz}(x,y,z)\\
\pi(x,y,z,0)=g_z(x,y,z) & h(x,y,z,0)=g_{xxx}(x,y,z)
\end{array}$$

$$\eta(x,y,z,0)=G=[f,g,f_x,h,g_x,f_{xx},\varphi,h_x,g_{xx},g_{xxx},\Omega,\varphi_x,h_{xx},\Omega_x,f_{yy},g_y,f_{zz},g_z]$$

The system (3.1.14) and (3.1.15) can be written in the matrix form

$$U_t+AU_x+BU_y+DU_z+CU=0, \qquad (3.1.16)$$

then by using the Lemma 1.1.2. The equation (3.1.16) reduce to the form

$$U_t+A_1U_{\xi^*}+CU=0, \qquad U(\xi^*,0)=H(\xi^*), \qquad (3.1.17)$$

where

$$U^T=[u,p,q,r,s,\tau,\omega,\mu,\upsilon,\gamma,\eta,M,N,I,h,L,k,E,O,\pi,\xi,\beta],$$

then

$$\begin{aligned}(CU)^T=[&-p,-r,-s,-L,-\xi,-\beta,0,0,0,0,N+\beta p+I(1+u)+p\xi+rs,-\eta,\\ &-\gamma,-\upsilon,0,-M,0,0,0,0,-N,-I]\end{aligned} \qquad (3.1.18)$$

where $A_1=A+B+D$,

$$A_1 = \begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&-1&0&-1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0\\
0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0\\
0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{bmatrix}$$

The eigenvalues of A_1 given by $\det(A_1 - \lambda I) = 0$, thus

$$\lambda_i = [0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1], \qquad (3.1.19)$$

the eigenvectors correspond these eigenvalues (3.1.19) are the solution of $A_1X = \lambda X$, then T and T^{-1} can be easily obtained and yields

$$T^{-1}A_1T = diag\,(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1)\,.$$

Now, applying the nonsingular transformation $U = TV$, to the system of equation (3.1.10) this leads to the characteristic form, i.e.,

$$V_t + GV_{\xi^*} + \tilde{C} = 0, \qquad V(\xi^*,0) = \phi(\xi^*), \qquad (3.1.20)$$

where $G = diag\,(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1)$ and

$$\tilde{C} = [-p,-r,-s,-L,-\xi,-\beta,I,-I,I,-I,J+I,-\eta,-\gamma,-\upsilon,0,-M,0,0,0,0,-N,-I]^T.$$

Hence to prove the uniqueness of the system (3.1.14) and (3.1.15) it is sufficient to prove the uniqueness of (3.1.20). Thus, let V_1 and V_2 be two solutions of equation (3.1.20).

Let $W = V_1 - V_2$. Then W satisfies the initial value problem

$$W_t + GW_{\xi^*} + \tilde{C}(V_1) - \tilde{C}(V_2) = 0, \qquad W(\xi^*, 0) = 0. \qquad (3.1.21)$$

By using the differentiation of $\tilde{C}$ from equation (3.1.18) with the relations

$$J + I = N + \beta p + I(1+u) + p\xi + rs + I$$
$$\beta_1 p_1 - \beta_2 p_2 = (\beta_1 - \beta_2) p_1 + (p_1 - p_2)\beta_2 = \beta p_1 + p\beta_2,$$

and

$$p_1\xi_1 - p_2\xi_2 = (p_1 - p_2)\xi_1 + (\xi_1 - \xi_2) p_2 = p\xi_1 + \xi p_2,$$
$$r_1 s_1 - r_2 s_2 = (r_1 - r_2)s_1 + (s_1 - s_2)r_2 = rs_1 + sr_2,$$
$$I_1 u_1 - I_2 u_2 = (I_1 - I_2)u_1 + (u_1 - u_2)I_2 = Iu_1 + uI_2,$$

where $p_1 = u_{1t}, p_1 = u_{2t}, p = W_t$ …etc, . Then

$$C(V_1) - C(V_2) =$$
$$[-p, -r, -s, -L, -\xi, -\beta, I, -I, I, -I, N + \beta p_1 + p\beta_2 + Iu_1 + uI_2 + p\xi + rs_1 + sr_2 + 2I,$$
$$-\eta, -\gamma, -\upsilon, 0, -M, 0, 0, 0, 0, -N, -I].$$

Hence $\tilde{C}(V_1) - \tilde{C}(V_2) = \tilde{M}W$, where

$$\tilde{M}=\begin{bmatrix}
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
u_2 & \beta_2+\xi_1 & 0 & s_1 & r_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2+u_1 & 0 & 0 & 0 & 0 & 0 & 0 & p_2 & p_1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$$W_t+GW_{\xi^*}+K(V_1,V_2)W=0, \quad W(\xi^*,0)=0.$$

In the case of this system is linear in W and the matrix $\tilde{M}$ is symmetric and $W(\xi^*,0)=0$, then the hypotheses of lemma (3.1.3) has at most one solution. Consequently, the equation (3.1.8) has at most one solution also.

4 Existence

The existence theory for hyperbolic system of quasi-linear partial differential equations

$$U_t+AU_x+BU_y+DU_z+C=0, \qquad U(x,y,z,0)=H(x,y,z)$$

has been studied in one dimension by many [3], [4], [5], for the analytic problem, now we introduce this initial value problem in (3+1) dimensions, i.e., when A, B, D and C are analytic in x, y, z, t and H is analytic in x, y and z, then the solution exists and depends continuously on data in the small (i.e., for suitably narrow neighborhood of $x, y, z = 0$ and $t = 0$) by the Caushy-Kowalewsky theorem [2]. This result was extended by Lax [6] who was able to show firstly that for analytic data the solution exists not only in the small but it can be continued analytically until it reaches the boundary of the domain of analyticity. Secondly, by approximating a non- analytic problem by a sequence of analytic problems and using the above results, the solution of a non-analytic initial value problem which is now a generalized solution is shown to get Lax proved that if all the matrices A, B, D, C and T (where T is the matrix of eigenvectors of A) have continuous first derivatives and the first derivative of $H(x, y, z)$ is almost everywhere continuous at all regular points of the system, i.e., points that don't lie on characteristics through points of discontinuity of the initial data.

Now we will study the existence theory for hyperbolic system of semi-linear of equation (3.1.4) which has the normal form

$$V_t + MV_{\xi^*} + \tilde{C} = 0 \qquad V(\xi^*, 0) = \psi(\xi^*), \tag{4.1}$$

where $M = diag(\lambda_1, \lambda_2, ..., \lambda_{24})$, and $\xi^* = \xi^*(x, y, z)$.

Now, we introduce the next lemma and then prove the existence for equation (4.1).

Lemma 4.2 *The system of differential equations* (4.1) *can be replaced equivalently by a system of nonlinear integral equations.*

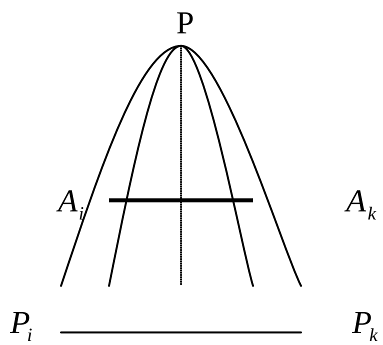
Proof. Let $M_k = \frac{\partial}{\partial t} + \lambda_k \frac{\partial}{\partial \xi^*}$ in the *k*-th component of equation (4.1) then M_k can be regarded as differentiation along the characteristic C_k. Thus, by similar arguments as were used to derive the integral formula (4.1), the system of equation (3.1.3) corresponds to the nonlinear integral equations.

$$V = LV \tag{4.2}$$

$$LV^k(\rho,\tau) = \psi^i(\xi_k^*) + \int_0^p \tilde{\tilde{C}}^k(\xi^*,\eta,\hbar)d\eta, \tag{4.3}$$

This proves the lemma. □

Now, we define the region in which the existence proof is valid. Let H be a closed domain in x, y, z and t space in which all the characteristics C_i followed from a point p in H backwards in t meet a given section J of the initial data line $t = 0$ in the points p_i , as in the following figure



Let S be the set of all functions V with domain H having continuous derivatives and equal to $\psi(x, y, z)$ on $t = 0$. Finally, we define the norm of elements of S to be the largest value of the functions attained in the closed domain H. However, if we choose $\|\psi(\xi^*)\| = N^*$ and restrict admissible functions by choosing $|V| < 2N^*$ then there exists a common upper bound $\ell > 0$ such that [7]:

$$\|\tilde{C}_V^k\| < \ell, \quad \|\tilde{C}_{\xi^*}^k\| < \ell, \quad \text{and} \quad \|\tilde{C}_t^k\| < \ell, \tag{4.4}$$

where $\tilde{C}_V^k$ is a functional gradient of $\tilde{C}^k$ with respect to V . Note that $\tilde{C}_{\xi^*}^k = \tilde{C}_t^k = 0$ for the equation (3.1.4).

Now, we introduce the following theorem:

Theorem 4.3 *Let* $\psi(\xi^*)$, $\tilde{C}$ *have continuous first derivatives, then the system*

$$V_t + MV_{\xi^*} + \tilde{C} = 0 \qquad V(\xi^*,0) = \psi(\xi^*), \qquad (4.5)$$

possesses a solution which has the same differentiability as $\psi(\xi^*)$.

Proof. If we choose α sufficiently small, then equation (4.3) implies that

$$\left\|V^k\right\| \le \left\|\psi(\xi^*)\right\| + \ell\alpha = N^* + \alpha\ell \le 2N^*.$$

The system of equation (4.3) lends itself immediately to a process of solution by iteration and for a suitably narrow strip H_α the desired fixed element will be constructed as the uniform limit, as derivatives with respect to ξ^*, since the $t-$derivatives follows from the known directional derivatives.

Now, the existence and continuity of V in the characteristic direction follows directly system of equation (4.3) and from the continuity of the solution obtained. To prove the existence and continuity of derivatives $\frac{\partial V}{\partial \xi^*}$ we observe, first of all, that the assumed continuous differentiability of $\psi(\xi^*)$ and $\tilde{C}$ implies that all the approximations constructed in proving the existence of a solution, have continuous derivative with respect to ξ^*. Differentiating the $(n+1)$ th approximation,

$$V_{n+1}(\xi^*,\tau) = \psi(\xi^*(0,\tau,\varepsilon)) + \int_0^\tau \tilde{C}(\xi^*,t,V_n)d\eta$$

By differentiate with respect to ε. Thus

$$\frac{\partial V_{n+1}}{\partial \varepsilon} = \psi'(\xi^*(0,\tau,\varepsilon))\frac{\partial \varepsilon}{\partial \xi^*} + \int_0^\tau \left(\frac{\partial \tilde{C}}{\partial V_n}\frac{\partial V_n}{\partial \xi^*}\frac{\partial \xi^*}{\partial \varepsilon} + \frac{\partial \tilde{C}}{\partial \xi^*}\frac{\partial \xi^*}{\partial \varepsilon} \right) d\eta$$

$$= \psi'\frac{\partial \varepsilon}{\partial \xi^*} + \int_0^\tau \left(\tilde{C}_V V_{\xi^*} + \tilde{C}_{\xi^*} \right)\xi^*_\varepsilon d\eta. \qquad (4.6)$$

Similar to the assumption for the system of equation (4.3) we can prove the uniform convergence of the sequence $\left(\frac{\partial V_n}{\partial \xi^*}\right)$ { ξ^* instead of ε }, $n = 1,2,...$, by using the same method which we used to prove the convergence of (V_n). This gives us $\lim_{n\to\infty} \frac{\partial V_n}{\partial \xi^*} = \frac{\partial V}{\partial \xi^*}$, which suffices to prove the existence of the solution of the characteristic system (4.4) locally. We must be sure that the solution exists globally._i.e., in a larger region, we use the line $t = \alpha$ as new initial line and solve the problem by the same procedures, as above, in the strip $\alpha < t < 2\alpha$. We continue stepwise in this way which implies the existence of the solution in an arbitrary large t so long as the assumption of the continuity and bounded remains satisfied the existence of the original system

$$U_t + A_1 U_{\xi^*} + C = 0 \qquad U(\xi^*, 0) = G(\xi^*). \qquad \square$$

Theorem 4.4 *Let* $U(x, y, z, t)$ *and* $W(x, y, z, t)$ *be two solutions of equation (4.1), such that* $U(x, y, z, 0) = \psi(x, y, z)$ *and* $\|\varphi - \psi\| < \delta$. *Then* $\|W - U\| < \varepsilon$ *and* $\varepsilon \to 0$ *as* $\delta \to 0$ (*continuous dependence of the solution on the initial data*).

Proof. Let

$$\varphi(x, y, z) - \psi(x, y, z) = \alpha(x, y, z),$$

where $\|\alpha(x, y, z)\| < \delta$, and

$$U(x, y, z, t) - W(x, y, z, t) = \Omega(x, y, z, t).$$

Then, as in Theorem 4.3, and Ω satisfies the integral equation

$$\Omega(x, y, z, t) = \delta(x, y, z) + \int_0^\tau \tilde{C}_V(x, y, z, \eta, V)(U - W)\, d\eta$$
$$= \delta(x, y, z) + \int_0^\tau \tilde{C}_V(x, y, z, \eta, V)\Omega(x, y, z, \eta)\, d\eta,$$

where V is intermediate value.

Let $\max_{x, y, z, t \in S} |\Omega(x, y, z, t)| = \varepsilon$, then by estimates analogous to that used in the existence proof

$$\varepsilon < \delta + \varepsilon \tau \ell, \quad (\|C\| < \ell), \tag{4.7}$$

replacing Ω in integral equation (4.6) by the right hand side of equation (4.7) and repeating, we obtain

$$\varepsilon < \delta(1+\ell\tau) + \varepsilon\frac{\ell^2\tau^2}{2}.$$

Repeating this operation n times we have

$$\varepsilon < \delta\left[1+\ell\tau+\frac{\ell^2\tau^2}{2}+...+\frac{\ell^{n-1}\tau^{n-1}}{(n-1)}+\varepsilon\frac{\ell^n\tau^n}{n}\right]$$

Now, as $n \to \infty$, we get $\varepsilon < \delta e^{\ell t}$. Thus if t is bounded, then $\delta \to 0$ implies $\varepsilon \to 0$, which proves the theorem. □

5 Conclusion

In this article, the well-posedness of N.G KdV class (3+1) equation was investigated. For this investigation it was convenient to reduce N.G KdV class (3+1) equation to a system of first order partial differential equations. It is found that if $\alpha_{17} \neq 0$ and the data are non characteristic, then N.G KdV class (3+1) equation reduced to a semi-linear system of first order partial differential equations on its characteristics. The proof of this fact was carried out for the case where all the characteristics are real and this proof can be done if some of these characteristics are complex by reducing the system to two systems of real characteristics and the reduction to systems of ordinary differential equations is clearly obtained again.

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