

# Blow up of solutions for the Cauchy problem of the damped sixth-order Boussinesq equation

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## Abstract

This work studies the Cauchy problem of the sixth-order Boussinesq equation with a damping term. By using generalized concavity method, it is proven blow up of the solution.

**Mathematics Subject Classification:** 35A01; 35G20

**Keywords:** Boussinesq equation; blow up; damping term

## 1 Introduction

In this work, we study the Cauchy problem of the sixth-order Boussinesq equation with a damping term

$$u_{tt} - u_{xx} - u_{xxt} - u_{xxxxx} - ku_{xxt} = f(u)_{xx}, \quad x \in \mathbf{R}, \quad t > 0, \quad (1)$$

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$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \mathbb{R}, \quad (2)$$

where  $u(x,t)$  denotes the unknown function,  $f(s)$  is the given nonlinear function,  $u_0(x)$  and  $u_1(x)$  are the given initial value functions,  $k$  is a constant.

Scott Russell's study [1] of solitary water waves motivated the development of nonlinear partial differential equations for the modeling wave phenomena in fluids, plasmas, elastic bodies, etc. It is well known that Boussinesq equation can be written in two basic forms

$$u_{tt} - u_{xx} + \delta u_{xxxx} = (u^2)_{xx}, \quad (3)$$

$$u_{tt} - u_{xx} - u_{xxt} = (u^2)_{xx}, \quad (4)$$

Equation (4) is an important model that approximately describes the propagation of long waves on shallow water like the other Boussinesq equations (with  $u_{xxxx}$ , instead of  $u_{xxt}$ ). In the case  $\delta > 0$  Equation (3) is linearly stable and governs small nonlinear transverse oscillations of an elastic beam (see [2] and references therein). It is called the "good" Boussinesq equation, while this equation with  $\delta < 0$  received the name of the "bad" Boussinesq equation since it possesses the linear instability.

In the Boussinesq equations, the effects of small nonlinearity and dispersion are taken into consideration, but in many real situations, damping effects are compared in strength to the nonlinear and dispersive one. Therefore the damped Boussinesq equations is considered as well

$$u_{tt} - 2bu_{xxt} = -\alpha u_{xxxx} + u_{xx} + \beta(u^2)_{xx}, \quad (5)$$

where  $u_{xxt}$  is the damping term and  $a, b = \text{const} > 0, \beta = \text{const} \in \mathbb{R}$  (see [3] and references therein).

Varlamov [2] investigated the long-time behavior of solutions to initial value, spatially periodic, and initial-boundary value problems for equation (5) in

two space dimensions. Polat et al. [4] established the blow up of the solutions for the initial boundary value problem of the damped Boussinesq equation

$$u_{tt} - bu_{xx} + \delta u_{xxxx} - ru_{xxt} = f(u)_{xx}.$$

The asymptotic behavior and the blow up of solution for a nonlinear evolution equation of four order

$$u_{tt} - a_1 u_{xx} - a_2 u_{xxt} - a_3 u_{xxtt} = \varphi(u_x)_x$$

were established in [5].

Polat and Kaya [6] studied the existence, both locally and globally in time, the asymptotic behavior, and the blow up of solution for the class of nonlinear wave equations with dissipative and dispersive terms

$$u_{tt} - u_{xx} - u_{xxtt} - \lambda u_{xxt} + u = \sigma(u_x)_x.$$

Wang and Chen [7] studied the existence and blow up of the solution for the Cauchy problem of the generalized double dispersion equation

$$u_{tt} - u_{xx} - u_{xxtt} + u_{xxxx} - \alpha u_{xxt} = g(u)_{xx}. \quad (6)$$

Polat and Ertaş [3] extended the result of [7] to the multidimensional version of the problem (6). Also, Polat and Pişkin [8] studied asymptotic behavior of the solution multidimensional version of the problem (6).

Recently, higher order Boussinesq equations have been investigated by authors. Schneider and Eugene [9] considered a class of Boussinesq equation which models the water wave problem with surface tension as follows

$$u_{tt} - u_{xx} - u_{xxtt} - \mu u_{xxxx} + u_{xxxxt} = (u^2)_{xx}, \quad (7)$$

where  $\mu \in R$  and  $u(x, t) \in R$ . Daripa [10] derived the higher-order Boussinesq equation

$$u_{tt} - u_{xx} \pm \alpha u_{xxxx} - \varepsilon^2 \alpha u_{xxxxx} = \alpha (u^2)_{xx}, \quad (8)$$

for two-way propagation of shallow water waves. Various generalizations of Equations (7) and (8) have been proposed and studied from many aspects (see [11-16]).

Throughout this work, we use the notation:  $\|u\|_{L^p} = \|u\|_p$  ( $1 \leq p \leq \infty$ ). In particular  $\|u\|_{L^2} = \|u\|$ .

Now, we state the local existence theorem (see [16] for the proof).

**Theorem 1.** *Suppose that  $s > \frac{1}{2}$ ,  $u_0 \in H^s$ ,  $u_1 \in H^{s-2}$  and  $f \in C^{[s]+1}(R)$ , then the problem (1)-(2) admits a unique local solution  $u(x, t)$  defined on a maximal time interval  $[0, T_0)$  with  $u(x, t) \in C([0, T_0); H^s) \cap C^1([0, T_0); H^{s-2})$ .*

## 2 Blow up of solutions

In this section, we are going to consider the blow up of the solution for problem (1)-(2) by the concavity method. For this purpose, we give the following lemma [17] which is a generalization of Levine's result [18].

**Lemma 2** [17]. *Suppose that a positive, twice differentiable function  $F(t)$  satisfies on  $t \geq 0$  the inequality*

$$F(t)F''(t) - (1 + \nu)(F'(t))^2 \geq -2M_1F(t)F'(t) - M_2(F(t))^2,$$

where  $\nu > 0$  and  $M_1, M_2 \geq 0$  are constants.

(i) *If  $M_1 = M_2 = 0$ ,  $F(0) > 0$  and  $F'(0) > 0$  then there is a*

$$t_1 \leq t_2 = \frac{F(0)}{\nu F'(0)} \text{ such that } F(t) \rightarrow \infty \text{ as } t \rightarrow t_1.$$

(ii) If  $M_1 + M_2 > 0$ ,  $F(0) > 0$  and  $F'(0) > -\gamma_2 \nu^{-1} F(0)$ , then there is a  $t_1 \leq t_2$  such that  $F(t) \rightarrow \infty$  as  $t \rightarrow t_1$ , where  $\gamma_{1,2} = -M_1 \pm \sqrt{M_1^2 + \nu M_2^2}$  and

$$t_2 = \frac{1}{2\sqrt{M_1^2 + \nu M_2^2}} \ln \frac{\gamma_1 F(0) + \nu F'(0)}{\gamma_2 F(0) + \nu F'(0)}.$$

**Theorem 3.** Assume that  $k \geq 0$ ,  $f(u) \in C(R)$ ,  $u_0 \in H^2$ ,  $u_1 \in L^1$ , and  $\Lambda^{-1}u_0, \Lambda^{-1}u_1 \in L^2$ ,  $F(u) = \int_0^u f(s) ds$ ,  $F(u_0) \in L^1$ , and there exists a constant  $\alpha > 0$  such that

$$uf(u) \leq (2 + \alpha + k)F(u) + \frac{\alpha}{2}u^2, \quad \forall u \in R. \quad (9)$$

Then the solution  $u(x, t)$  of problem (1)-(2) blows up in finite time if one of the following conditions is valid:

- (i)  $E(0) < 0$ ,
- (ii)  $E(0) = 0$  and  $(\Lambda^{-1}u_0, \Lambda^{-1}u_1) + (u_0, u_1) > 0$ ,
- (iii)  $E(0) > 0$  and

$$(\Lambda^{-1}u_0, \Lambda^{-1}u_1) + (u_0, u_1) > \sqrt{2 \frac{4\alpha + k + 2}{\alpha + 2} E(0) (\|\Lambda^{-1}u_0\|^2 + \|u_0\|^2)}.$$

*Proof.* Suppose that the maximal time of existence of the solution for problem (1)-(2) is infinite. A contradiction will be obtained by Lemma 2. Multiplying both sides of (1) by  $\Lambda^{-2}u_t$ , integrating over  $R$ , we arrive at

$$\frac{d}{dt} E(t) = 0, \quad t > 0,$$

where

$$E(t) = \|\Lambda^{-1}u_t\|^2 + \|u\|^2 + \|u_t\|^2 + \|u_{xx}\|^2 + 2k \int_0^t \|u_\tau\|^2 d\tau + 2 \int_R F(u) dx = E(0).$$

Let

$$F(t) = \|\Lambda^{-1}u\|^2 + \|u\|^2 + \beta(t + \tau)^2 \quad (10)$$

where  $\beta$  and  $\tau$  are nonnegative constants to be specified later. Obviously we have

$$F'(t) = 2[(\Lambda^{-1}u, \Lambda^{-1}u_t) + (u, u_t) + \beta(t + \tau)]. \quad (11)$$

Using the Schwartz inequality and the inequality

$$(a_1b_1 + a_2b_2 + \dots + a_nb_n)^2 \leq (a_1^2 + a_2^2 + \dots + a_n^2)(b_1^2 + b_2^2 + \dots + b_n^2),$$

where  $a_i, b_i \geq 0$ ,  $i = 1, 2, \dots, n$ , we have

$$\begin{aligned} (F'(t))^2 &\leq 4 \left[ \|\Lambda^{-1}u\|^2 + \|u\|^2 + \beta(t + \tau)^2 \right] \left[ \|\Lambda^{-1}u_t\|^2 + \|u_t\|^2 + \beta \right] \\ &= 4F(t) \left[ \|\Lambda^{-1}u_t\|^2 + \|u_t\|^2 + \beta \right]. \end{aligned} \quad (12)$$

We get from equation (1)

$$\begin{aligned} F''(t) &= 2 \|\Lambda^{-1}u_t\|^2 + 2 \|u_t\|^2 + 2(\Lambda^{-1}u, \Lambda^{-1}u_{tt}) + 2(u, u_{tt}) + 2\beta \\ &= 2 \|\Lambda^{-1}u_t\|^2 + 2 \|u_t\|^2 + 2\beta + 2(u, u_{tt} + \Lambda^{-2}u_{tt}) \\ &= 2 \|\Lambda^{-1}u_t\|^2 + 2 \|u_t\|^2 + 2\beta - 2(u, u + u_{xxxx} + ku_t + f(u)) \\ &= 2 \|\Lambda^{-1}u_t\|^2 + 2 \|u_t\|^2 + 2\beta - 2 \|u\|^2 - 2 \|u_{xx}\|^2 - 2k(u, u_t) - 2 \int_R uf(u) dx. \end{aligned} \quad (13)$$

By the aid of the Cauchy inequality and energy equality  $E(t)$ , we have

$$\begin{aligned} 2k(u, u_t) &\leq k(\|u\|^2 + \|u_t\|^2) \\ &= k \left[ E(0) - \|\Lambda^{-1}u_t\|^2 - \|u_{xx}\|^2 - 2k \int_0^t \|u_\tau\|^2 d\tau - 2 \int_R F(u) dx \right]. \end{aligned} \quad (14)$$

From (10)-(14) we obtain that

$$\begin{aligned}
F(t)F''(t) - \left(1 + \frac{\alpha}{4}\right)(F'(t))^2 &\geq F(t)F''(t) - (4 + \alpha)F(t)\left[\|\Lambda^{-1}u_t\|^2 + \|u_t\|^2 + \beta\right] \\
&\geq F(t)\left[(k - \alpha - 2)\|\Lambda^{-1}u_t\|^2 + (-\alpha - 2)\|u_t\|^2\right. \\
&\quad \left.+ (k - 2)\|u_{xx}\|^2 + (-\alpha - 2)\beta + 2k^2 \int_0^t \|u_\tau\|^2 d\tau\right. \\
&\quad \left.+ \int_R (2kF(u) - 2uf(u) - 2u^2) dx - kE(0)\right]. \quad (15)
\end{aligned}$$

From energy equality  $E(t)$ , we have

$$\begin{aligned}
&(k - \alpha - 2)\|\Lambda^{-1}u_t\|^2 + (-\alpha - 2)\|u_t\|^2 + (k - 2)\|u_{xx}\|^2 \\
&\geq (-\alpha - 2)\left(\|\Lambda^{-1}u_t\|^2 + \|u_t\|^2 + \|u_{xx}\|^2\right) \\
&\geq (\alpha + 2)\left(\|u\|^2 + 2k \int_0^t \|u_\tau\|^2 d\tau + 2 \int_R F(u) dx - E(0)\right).
\end{aligned}$$

Thus, from the above inequality and inequalities (15) and (9), we get

$$\begin{aligned}
F(t)F''(t) - \left(1 + \frac{\alpha}{4}\right)(F'(t))^2 &\geq F(t)\left[-(\alpha + 2)\beta - (2 + \alpha + k)E(0)\right. \\
&\quad \left.+ (2k(2 + \alpha) + 2k^2) \int_0^t \|u_\tau\|^2 d\tau\right. \\
&\quad \left.+ \int_R (2(2 + \alpha + k)F(u) + \alpha u^2 - 2uf(u)) dx\right] \\
&\geq -[(2 + \alpha)\beta + (2 + \alpha + k)E(0)]F(t). \quad (16)
\end{aligned}$$

(1) If  $E(0) < 0$ , taking  $\beta = \frac{(2 + \alpha + k)}{2 + \alpha}E(0) > 0$ , then

$$F(t)F''(t) - \left(1 + \frac{\alpha}{4}\right)(F'(t))^2 \geq 0.$$

We may now choose  $\tau$  so large that  $F'(0) > 0$ . From Lemma 2 we know that

$F(t)$  becomes infinite at a time  $T_1$  at most equal to

$$T_2 = \frac{4F(0)}{\alpha F'(0)} < \infty.$$

(2) If  $E(0) = 0$ , taking  $\beta = 0$ , then we get from (16) that

$$F(t)F''(t) - \left(1 + \frac{\alpha}{4}\right)(F'(t))^2 \geq 0.$$

Also  $F'(0) > 0$  by assumption (ii). Thus, we obtain from Lemma 2 that  $F(t)$  becomes infinite at a time  $T_1$  at most equal to

$$T_2 = \frac{4F(0)}{\alpha F'(0)} < \infty.$$

(3) If  $E(0) > 0$ , then taking  $\beta = 0$ , inequality (16) becomes

$$F(t)F''(t) - \left(1 + \frac{\alpha}{4}\right)(F'(t))^2 \geq -(2 + \alpha + k)E(0)F(t). \quad (17)$$

Define  $J(t) = (F(t))^{-\nu}$ , where  $\nu = \frac{\alpha}{4}$ . Then

$$\begin{aligned} J'(t) &= -\nu(F(t))^{-\nu-1}F'(t), \\ J''(t) &= -\nu(F(t))^{-\nu-2}\left[F(t)F''(t) - (1 + \nu)(F'(t))^2\right] \\ &\leq \nu(2 + \alpha + 4\nu)E(0)(F(t))^{-\nu-1} \end{aligned} \quad (18)$$

where inequality (17) is used. Assumption (iii) implies  $J'(0) < 0$ . Let

$$t^* = \sup\{t : J'(\tau) < 0, \tau \in (0, t)\}. \quad (19)$$

By the continuity of  $J'(t)$ ,  $t^*$  is positive. Multiplying (18) by  $2J'(t)$  yields

$$\begin{aligned} \left[(J'(t))^2\right]' &\geq -2\nu^2(2 + \alpha + 4\nu)E(0)(F(t))^{-2\nu-2}F'(t) \\ &= 2\nu^2 \frac{(2 + \alpha + 4\nu)}{2\nu + 1} E(0) \left[(F(t))^{-2\nu-1}\right]', \quad \forall t \in [0, t^*]. \end{aligned} \quad (20)$$



Integrating (20) with respect to  $t$  over  $[0, t)$  gives

$$\begin{aligned} (J'(t))^2 &\geq (J'(0))^2 + 2\nu^2 \frac{(2+\alpha+4\nu)}{2\nu+1} E(0)(F(t))^{-2\nu-1} \\ &\quad - 2\nu^2 \frac{(2+\alpha+4\nu)}{2\nu+1} E(0)(F(0))^{-2\nu-1} \\ &\geq (J'(0))^2 - 2\nu^2 \frac{(2+\alpha+4\nu)}{2\nu+1} E(0)(F(0))^{-2\nu-1}. \end{aligned}$$

By assumption (iii)

$$(J'(0))^2 - 2\nu^2 \frac{(2+\alpha+4\nu)}{2\nu+1} E(0)(F(0))^{-2\nu-1} > 0.$$

Hence by continuity of  $J'(t)$ , we obtain

$$J'(t) \leq - \left[ (J'(0))^2 - 2\nu^2 \frac{(2+\alpha+4\nu)}{2\nu+1} E(0)(F(0))^{-2\nu-1} \right]^{\frac{1}{2}} \quad (21)$$

for  $0 \leq t < t^*$ . By the definition of  $t^*$ , it follows that inequality (21) holds for all  $t \geq 0$ . Therefore,

$$J(t) \leq J(0) - \left[ (J'(0))^2 - 2\nu^2 \frac{(2+\alpha+4\nu)}{2\nu+1} E(0)(F(0))^{-2\nu-1} \right]^{\frac{1}{2}} t, \quad \forall t > 0.$$

So,  $J(T_1) = 0$  for some  $T_1$  and

$$0 < T_1 \leq T_2 = \frac{J(0)}{\left[ (J'(0))^2 - 2\alpha^2 \frac{(2+\alpha+k)}{4k+8} E(0)(F(t))^{-\frac{\alpha+2}{2}} \right]^{\frac{1}{2}}}.$$

Thus,  $F(t)$  becomes infinite at a time  $T_1$ .

Therefore,  $F(t)$  becomes infinite at a time  $T_1$  under either assumptions (i), (ii), or (iii). We have a contradiction with the fact that the maximal time of existence is infinite. Hence the maximal time of existence is finite. This completes the proof.  $\square$

## References

- [1] J. Scott Russell, Report on water waves, *British Assoc. Report*, 1844.
- [2] V.V.Varlamov, Eigenfunction expansion method and the long-time asymptotics for the damped Boussinesq equation, *Discrete Contin. Dyn. Syst.*, **7**(4), (2001), 675-702.
- [3] N. Polat and A. Ertaş, Existence and Blow up of Solution of Cauchy problem for the generalized damped multidimensional Boussinesq equation, *J. Math. Anal. Appl.*, **349**, (2009), 10-20.
- [4] N. Polat, D. Kaya and H.İ. Tutarlar, Blow-up of solutions for the damped Boussinesq equation, *Z. Naturforsch.*, **60a**, (2005), 473-476.
- [5] X. Chen and G. Chen, Asymptotic behavior and blow-up of solutions to a nonlinear evolution equation of fourth order, *Nonlinear Anal.*, **68**, (2008), 892-904.
- [6] N. Polat and D. Kaya, Existence, Asymptotic Behaviour, and Blow up of Solution for a class of Nonlinear Wave Equations with Dissipative and Dispersive Term, *Z. Naturforsch.*, **64a**, (2009), 1-12.
- [7] S. Wang and G. Chen, Cauchy problem of the generalized double dispersion equation, *Nonlinear Anal.*, **64**, (2006), 159-173.
- [8] N. Polat and E. Pişkin, Asymptotic behavior of a solution of the Cauchy problem for the generalized damped multidimensional Boussinesq equation, *Appl. Math. Lett.*, **25**, (2012), 1871-1874.
- [9] G. Schneider and C.W. Eugene, Kawahara dynamics in dispersive media, *Physica D: Nonlinear Phenomena*, **152-153**, (2001), 384-394.
- [10] P. Daripa, Higher-order Boussinesq equations for two-way propagation of shallow water waves, *Eur. J. Mech. B-Fluid*, **25**, (2006), 1008-1021.
- [11] Y. Wang and C. Mu, Global existence and blow-up of the solutions for the multidimensional generalized Boussinesq equation, *Math. Meth. Appl. Sci.*, **30**, (2007), 1403-1417.

- [12] Y. Wang, B. Guo, Blow-up solution for a generalized Boussinesq equation, *Appl. Math. Mech-Engl.*, **28** (11), (2007), 1437-1443.
- [13] N. Duruk, A. Erkip and H.A. Erbay, A higher-order Boussinesq equation in locally non-linear theory of one-dimensional non-local elasticity, *IMA Journal of App. Math.*, **74**, (2009), 97-106.
- [14] A. Esfahani, L.G. Farah and H. Wang, Global existence and blow-up for the generalized sixth-order Boussinesq equation, *Nonlinear Anal.*, **75**, (2012), 4325-4338.
- [15] A. Esfahani, L. G. Farah, Local well-posedness for the sixth-order Boussinesq equation, *J. Math. Anal. Appl.*, **385**, (2012), 230-242.
- [16] N. Polat and E. Pişkin, Existence and asymptotic behavior of solution of Cauchy problem for the damped sixth-order Boussinesq equation, *Acta. Math. Appl. Sin-E*, (doi:10.1007/s10255-012-0174-2).
- [17] V.K. Kalantarov and O.A. Ladyzhenskaya, The occurrence of collapse for quasilinear equations of parabolic and hyperbolic type, *J. Soviet Math.*, **10**, (1978), 53-70.
- [18] H.A. Levine, Instability and nonexistence of global solutions of nonlinear wave equations of the form  $Pu_{tt} = Au + F(u)$ , *Trans. Amer. Math. Soc.*, **192**, (1974), 1-21.