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A Generalization of Van der Pol Equation of degree five

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Abstract

In this paper we make an analysis of a generalization of van der Pol equation of degree five without periodic orbits in a domain on the plane. We use a Gasull's result and Dulac's theorem.

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1 Introduction

It is important to make in differential equations the study of periodic orbits

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in the plane. Certain systems do not have limit cycles. It should be considered: Bendixson's theorem and critical points. (See [4, 5]). In this paper we are interested in studying a generalization of a van der Pol equation of degree five that has a periodic orbit but in a circular domain of radius one and center in the origin there is no this limit cycle (See [1]). We use the theorem of Bendixson–Dulac (See [3]) and paper of Gasull (See [1]).

2 Preliminary Notes

Theorem 2.1. *(Bendixson–Dulac theorem)([3]) Let* $f_1(x_1, x_2)$, $f_2(x_1, x_2)$ *and* $h(x_1, x_2)$ *be functions* C^1 *in a simply connected domain* $D \subset \mathbb{R}^2$ *such that* $\frac{\partial(f_1 h)}{\partial x_1} + \frac{\partial(f_2 h)}{\partial x_2}$ *does not change sign in* D *and vanishes at most on a set of measure zero. Then the system*

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2), \\ \dot{x}_2 = f_2(x_1, x_2), \quad (x_1, x_2) \in D, \end{cases} \tag{1}$$

does not have periodic orbits in D.

According to this theorem, to rule out the existence of periodic orbits of the system (1) in a simply connected region D, we need to find a function $h(x_1, x_2)$ that satisfies conditions of Bendixson–Dulac theorem, such function h is called a Dulac function.

Our goal is the study of a dynamical system on the plane that does not have periodic orbits in a circular domain of radius one.

3 Method to Obtain Dulac functions

A Dulac function for the system (1) satisfies the equation

$$f_1 \frac{\partial h}{\partial x_1} + f_2 \frac{\partial h}{\partial x_2} = h \left(c(x_1, x_2) - \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right) \right) \tag{2}$$

Theorem 3.1. (See [3]) *For the system of differential equations* (1)*, if* (2) *(for some function c which does not change of sign and it vanishes only on a subset of measure zero) has a solution h on D such that h does not change sign and vanishes only on a subset of measure zero, then h is a Dulac function for* (1) *on D.*

Theorem 3.2. (See [1]) *Assume that there exist a real number s and an analytic function h in R^2 such that*

$$f_1\frac{\partial h}{\partial x_1}+f_2\frac{\partial h}{\partial x_2}=h\left(c(x_1,x_2)-s\left(\frac{\partial f_1}{\partial x_1}+\frac{\partial f_2}{\partial x_2}\right)\right) \tag{3}$$

does not change sign in an open region $W\subset R^2$ with regular boundary and vanishes only in a null measure Lebesgue set. Then the limit cycles of system (1) are either totally contained in $\mathfrak{h}_0:=\{h=0\}$, or do not intersect $\mathfrak{h}_0$. Moreover the number N of limit cycles that do not intersect $\mathfrak{h}_0$ satisfies $N=0$ if $s=0$.

4 Main Results

These are the main results of the paper.

Theorem 4.1. *Let $f(x_1,x_2),g(x_1)$ be functions C^1 in a simply connected domain $D=\{h\le 0\}\subset\mathbb{R}^2$ where $h(x_1,x_2)=\psi(x_1)+ax_2^2+bx_2+c$ and $\psi(x_1)$ is a function C^1 in $\mathbb{R}$, $a,b,c\in\mathbb{R}$ with the following conditions $x_2\psi'(x_1)-(g(x_1)+f(x_1,x_2)x_2)(2ax_2+b)$ which does not change sign and it vanishes only in a null measure Lebesgue subset and $b^2-4a(\psi(x_1)+c)\ge 0$. Then the system*

$$\begin{cases}\dot{x}_1=x_2,\\ \dot{x}_2=-g(x_1)-f(x_1,x_2)x_2,\end{cases} \tag{4}$$

does not have periodic orbits on D.

Proof. Applying Theorem 3.2 to (4) (this system has critical point on $x_2=g(x_1)=0$). From (3) we see function $f(x_1,x_2)$ and values of s satisfy the equation

$$x_2 h_{x_1} - (g(x_1) + f x_2) h_{x_2} = h(c(x_1, x_2) + s(\frac{\partial f}{\partial x_2}(x_1, x_2) x_2 + f)) \qquad (5)$$

for some $h, c(x_1, x_2)$ with hc does not change of sign (except in a set of measure 0). Obviously h is a Dulac function in certain cases. We propose (instead of try to solve equation (5)) the function $h = \psi(x_1) + a x_2^2 + b x_2 + c$ for adequate ψ such that h has a closed curve of level 0. When $h = 0$, we have $x_2 = \frac{-b \pm \sqrt{b^2 - 4a(\psi(x_1)+c)}}{2a}$. Then $b^2 - 4a(\psi(x_1) + c) \geq 0$. We try to find the domain for which the system does not have periodic orbits. We have $h_{x_1} = \psi'(x_1), \quad h_{x_2} = 2a x_2 + b$. So we have

$$x_2 \psi'(x_1) - (g(x_1) + f x_2)(2a x_2 + b) - sh(\frac{\partial f}{\partial x_2}(x_1, x_2) x_2 + f(x_1, x_2))$$

which does not change sign and it vanishes only in a null measure Lebesgue subset. Making $s = 0$ (this system would not have periodic orbits inside the domain with boundary $h = 0$) we get that $x_2 \psi'(x_1) - (g(x_1) + f(x_1, x_2) x_2)(2a x_2 + b)$ which does not change sign and it vanishes only in a null measure Lebesgue subset. □

Example 4.2. Consider the system

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -x_1 + (d^2 - x_1^2)(1 + x_2^2) x_2, \quad d \geq 1. \end{cases} \qquad (6)$$

Taking $h(x_1, x_2) = x_1^2 + x_2^2 - 1$ we obtain that the associated equation given in (3) with $s = 0$ is $hc(x_1, x_2) = 2x_2^2(d^2 - x_1^2)(1 + x_2^2)$. So, this function does not change sign and it is zero only at $x_2 = 0, x_1 = \pm d$. The system does not contain periodic orbits on $D = \{x_1^2 + x_2^2 \leq 1\}$. By (6), we have $\ddot{x}_1 + (x_1^2 - d^2)(\dot{x}_1^2 + 1)\dot{x}_1 + x_1 = 0$. This equation is generalized by $\ddot{x}_1 + f(x_1, \dot{x}_1)\dot{x}_1 + g(x_1) = 0$. The last equation was studied in [2]. □

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