A Generalization of Van der Pol Equation of degree five

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Abstract

In this paper we make an analysis of a generalization of van der Pol equation of degree five without periodic orbits in a domain on the plane. We use a Gasull’s result and Dulac’s theorem.

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1 Introduction

It is important to make in differential equations the study of periodic orbits

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in the plane. Certain systems do not have limit cycles. It should be considered: Bendixson’s theorem and critical points. (See [4, 5]). In this paper we are interested in studying a generalization of a van der Pol equation of degree five that has a periodic orbit but in a circular domain of radius one and center in the origin there is no this limit cycle (See [1]). We use the theorem of Bendixson–Dulac (See [3]) and paper of Gasull (See [1]).

2 Preliminary Notes

**Theorem 2.1.** (Bendixson–Dulac theorem)([3]) Let \( f_1(x_1, x_2), f_2(x_1, x_2) \) and \( h(x_1, x_2) \) be functions \( C^1 \) in a simply connected domain \( D \subset \mathbb{R}^2 \) such that \( \frac{\partial f_1 h}{\partial x_1} + \frac{\partial f_2 h}{\partial x_2} \) does not change sign in \( D \) and vanishes at most on a set of measure zero. Then the system

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1, x_2), \\
\dot{x}_2 &= f_2(x_1, x_2), \quad (x_1, x_2) \in D,
\end{align*}
\]

does not have periodic orbits in \( D \).

According to this theorem, to rule out the existence of periodic orbits of the system (1) in a simply connected region \( D \), we need to find a function \( h(x_1, x_2) \) that satisfies conditions of Bendixson–Dulac theorem, such function \( h \) is called a Dulac function.

Our goal is the study of a dynamical system on the plane that does not have periodic orbits in a circular domain of radius one.

3 Method to Obtain Dulac functions

A Dulac function for the system (1) satisfies the equation

\[
f_1 \frac{\partial h}{\partial x_1} + f_2 \frac{\partial h}{\partial x_2} = h \left( c(x_1, x_2) - \left( \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right) \right)
\]
Theorem 3.1. (See [3]) For the system of differential equations (1), if (2) (for some function $c$ which does not change of sign and it vanishes only on a subset of measure zero) has a solution $h$ on $D$ such that $h$ does not change sign and vanishes only on a subset of measure zero, then $h$ is a Dulac function for (1) on $D$.

Theorem 3.2. (See [1]) Assume that there exist a real number $s$ and an analytic function $h$ in $\mathbb{R}^2$ such that

$$f_1 \frac{\partial h}{\partial x_1} + f_2 \frac{\partial h}{\partial x_2} = h \left( c(x_1, x_2) - s \left( \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right) \right)$$

(3)

does not change sign in an open region $W \subset \mathbb{R}^2$ with regular boundary and vanishes only in a null measure Lebesgue set. Then the limit cycles of system (1) are either totally contained in $h_0 := \{ h = 0 \}$, or do not intersect $h_0$. Moreover the number $N$ of limit cycles that do not intersect $h_0$ satisfies $N = 0$ if $s = 0$.

4 Main Results

These are the main results of the paper.

Theorem 4.1. Let $f(x_1, x_2), g(x_1)$ be functions $C^1$ in a simply connected domain $D = \{ h \leq 0 \} \subset \mathbb{R}^2$ where $h(x_1, x_2) = \psi(x_1) + ax_2^2 + bx_2 + c$ and $\psi(x_1)$ is a function $C^1$ in $\mathbb{R}$, $a, b, c \in \mathbb{R}$ with the following conditions $x_2 \psi'(x_1) - (g(x_1) + f(x_1, x_2)x_2)(2ax_2 + b)$ which does not change sign and it vanishes only in a null measure Lebesgue subset and $b^2 - 4a(\psi(x_1) + c) \geq 0$. Then the system

$$\begin{cases} 
\dot{x}_1 = x_2, \\
\dot{x}_2 = -g(x_1) - f(x_1, x_2)x_2,
\end{cases}$$

(4)

does not have periodic orbits on $D$.

Proof. Applying Theorem 3.2 to (4) (this system has critical point on $x_2 = g(x_1) = 0$). From (3) we see function $f(x_1, x_2)$ and values of $s$ satisfy the
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\[ x_2 h_{x_1} - (g(x_1) + f x_2) h_{x_2} = h(c(x_1, x_2) + s \frac{\partial f}{\partial x_2}(x_1, x_2) x_2 + f)) \]  

for some \( h, c(x_1, x_2) \) with \( hc \) does not change of sign (except in a set of measure 0). Obviously \( h \) is a Dulac function in certain cases. We propose (instead of try to solve equation (5)) the function \( h = \psi(x_1) + ax_2^2 + bx_2 + c \) for adequate \( \psi \) such that \( h \) has a closed curve of level 0. When \( h = 0 \), we have \( x_2 = \frac{-b \pm \sqrt{b^2 - 4a(\psi(x_1) + c)}}{2a} \). Then \( b^2 - 4a(\psi(x_1) + c) \geq 0 \). We try to find the domain for which the system does not have periodic orbits. We have \( h_{x_1} = \psi'(x_1), \ h_{x_2} = 2ax_2 + b \). So we have

\[ x_2 \psi'(x_1) - (g(x_1) + f x_2)(2ax_2 + b) - sh\left(\frac{\partial f}{\partial x_2}(x_1, x_2) x_2 + f(x_1, x_2)\right) \]

which does not change sign and it vanishes only in a null measure Lebesgue subset. Making \( s = 0 \) (this system would not have periodic orbits inside the domain with boundary \( h = 0 \)) we get that \( x_2 \psi'(x_1) - (g(x_1) + f(x_1, x_2)x_2)(2ax_2 + b) \) which does not change sign and it vanishes only in a null measure Lebesgue subset.

\[ \text{Example 4.2.} \] Consider the system

\[ \begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -x_1 + (d^2 - x_1^2)(1 + x_2^2)x_2, \end{cases} \]

Taking \( h(x_1, x_2) = x_1^2 + x_2^2 - 1 \) we obtain that the associated equation given in (3) with \( s = 0 \) is \( hc(x_1, x_2) = 2x_2^2(d^2 - x_1^2)(1 + x_2^2) \). So, this function does not change sign and it is zero only at \( x_2 = 0, x_1 = \pm d \). The system does not contain periodic orbits on \( D = \{x_1^2 + x_2^2 \leq 1\} \). By (6), we have \( \ddot{x}_1 + (x_1^2 - d^2)(\dot{x}_1^2 + 1)\dot{x}_1 + x_1 = 0 \). This equation is generalized by \( \ddot{x}_1 + f(x_1, \dot{x}_1)\dot{x}_1 + g(x_1) = 0 \). The last equation was studied in [2].

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References


