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On the absolute continuity of vector measures

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Abstract

In this short paper we use the notion of multiplier to obtain a characterization of the absolute continuity of a vector measure on a compact group.

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1 Introduction

Let A be a Banach algebra over the field $\mathbb{K}$ which can be $\mathbb{R}$ or $\mathbb{C}$. Let G a locally compact group and $\mathcal{K}(G,S)$ the space of $S-$valued continuous on G with compact support. An $A-$valued measure is a linear map $m : \mathcal{K}(G,\mathbb{K}) \rightarrow A$, continuous in the following sense : for every compact subset K of G, there exists a positive real constant α_K

$$\|m(f)\| \leq \alpha_K \sup\{|f(t)| : t \in K\} \text{ for all } f \in \mathcal{K}(G,\mathbb{K}). \quad (1)$$

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We may extend it to $\mathcal{K}(G, A)$ by the identity :

$$\tilde{m}(x\varphi) = xm(\varphi),\ x \in A,\ \varphi \in \mathcal{K}(G, \mathbb{K}) \tag{2}$$

and continue to write m for $\tilde{m}$. A measure so defined is called a *vector measure.*

2 Preliminary Notes

A vector measure m is said to be *dominated* if there exists a real positive measure μ such that :

$$\|m(f)\| \leq \int_G \|f(t)\|\, d\mu(t) \text{ for all } f \in \mathcal{K}(G, A). \tag{3}$$

The value $m(f)$ is rather written $m(f) := \int_G f(t)\, dm(t)$. If m is dominated, there exists a smallest such positive measure denoted by $|m|$ and called the *modulus* or the *variation* of m. A bounded vector measure is a vector measure dominated by a bounded real measure. A function $f : G \to A$ is $m-$integrable means that the function $t \mapsto \|f(t)\|$ is $|m|-$integrable. Measurability and negligibility of f are defined in the same way. Accordingly the space $L_p(G, m, A)$ is defined to be $L_p(G, |m|, A)$, $1 \leq p < \infty$ and is the completion of $\mathcal{K}(G, A)$ in the $p-$norm. In the other hand, the space $L_\infty(G, m, A)$ is the Banach space of the classes of essentially bounded functions with respect to $|m|$. A certain number of facts will be useful in the sequel. The following theorem can be found in [1, section 18.14].

Theorem 2.1. *A dominated vector measure m is absolutely continuous with respect to a positive measure μ if and only if its modulus $|m|$ is absolutely continuous with respect to μ.*

The next theorem due to Lebesgue can be found in [1, section 18.28].

Theorem 2.2. *Let μ be a positive measure. Every positive measure ν can be written uniquely in the form*

$$\nu = \omega + \pi \tag{4}$$

where ω is positive and absolutely continuous with respect to μ and π is positive and singular with respect to μ.

Now let $f * g$ of $f : G \to A$ and $g : G \to A$ be vector valued functions. If the function $y \mapsto f(y)g(y^{-1}x)$ is integrable with respect to the normalized Haar measure of G then the equality

$$f * g(x) = \int_G f(y)g(y^{-1}x)\, d\lambda(y) \tag{5}$$

defines the convolution f by g. The convolution of a vector measure m with a vector valued function f is defined by

$$m * f(x) = \int_G f(y^{-1}x) dm(y). \tag{6}$$

Let $C(G, A)$ the space of $A-$valued continuous functions on G. Then we have the following theorem [1, section 24.44].

Theorem 2.3. *Assume $f \in L_p(G, A)$ and $g \in L_q(G, A)$, $1 < p, q, < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ or $p = 1$ and $q = \infty$. Then the function $f * g \in \mathcal{C}(G, A)$ and $\|f * g\|_\infty \leq \|f\|_p \|g\|_q$.*

Here we assume that the group G is compact and we denote by Σ its unitary dual. For $\sigma \in \Sigma$, we chose once and for all $(\xi_1^\sigma, \ldots, \xi_{d_\sigma}^\sigma)$ as a canonical basis of the representation Hilbert space H_σ of σ where $d_\sigma = \dim H_\sigma$. We denote by $\mathcal{T}(G)$ the linear space spanned by the coefficients $u_{ij}^\sigma : t \mapsto \langle U_t^\sigma \xi_j^\sigma, \xi_i^\sigma \rangle$ where U^σ is an element from σ.
For the proof of the following theorem, see [2, section 35.11].

Theorem 2.4. *Suppose G is compact, $\mu \in M(G)$, the space of bounded complex measure on G, $1 < p \leq \infty$ and $\sup\{\|\mu * h\|_p : h \in \mathcal{T}(G), \|h\|_1 \leq 1\} < \infty$. Then there exists $\nu \in L_p(G)$ such that $\mu = \nu\lambda$, where λ is the normalized Haar measure of G.*

3 Main Results

In this paper p and q are called *conjugate* if $\frac{1}{p}+\frac{1}{q}=1,\ 1<p,q<\infty$ or $p=1$ and $q=\infty$. We prove the following theorem.

Theorem 3.1. *Let G be a compact group, m be a bounded vector measure on G, p,q conjugate with $p\geq 1$.*
The two assertions are equivalent.

1. *For each $h\in L^p(G,A), m*h\in\mathcal{C}(G,A)$*

2. *There exists $u\in L^q(G,A)$ such that $m=u\lambda$.*

Proof The implication $2\Longrightarrow 1$ is obvious because of the Theorem 2.2.
Case $p>1$.
Consider a bounded vector measure m such that $m*h\in\mathcal{C}(G,A)$ whenever $h\in L_p(G,A)$. The mapping $T: L_p(G,A)\to\mathcal{C}(G,A),\ h\mapsto Th=|m|*h$ is a multiplier (a continuous linear map which commutes with convolution and multiplication by elements of A). The mapping T may be seen as the restriction to $L_p(G,A)$ of the multiplier $L_1(G,A)\to L_1(G,A),\ h\mapsto m*h$ such that $m*h\in\mathcal{C}(G,A)$. The group G being compact, $\mathcal{C}(G,A)\subset L_r(G,A),\ 1\leq r\leq\infty$. Thus $\sup\{\||m|*h\|_q:\ h\in\mathcal{T}(G),\ \|h\|_1<1\}<\infty$, using inclusions of various spaces in another and comparing their norms. From the Theorem 2.4 we conclude that there exists $v\in L_q(G,\mathbb{C})$ such that $|m|=v\lambda$. Hence due to the Theorem 2.1 there exists $u\in L_q(G,A)$ such that $m=u\lambda$.
Case $p=1$.
An $(L_\infty(G,A),\mathcal{C}(G,A))-$multiplier T is an $(L_\infty(G,A),L_\infty(G,A))-$multiplier. So there exists a bounded vector measure m such that for every $h\in L_\infty(G,A)$, $Th=m*h$. Following the theorem 2.2 the modulus $|m|$ of m may be written $|m|=\mu+\nu$ where μ is positive and absolutely continuous and ν is singular and positive with respect to λ. We can show that $\nu=0$ as done in the proof of [2, 35.13]. Therefore m is absolutely continuous with respect to λ, say $m=u\lambda$, $u\in L_1(G,A)$. □

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