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Optimal control of some degenerate pseudo-parabolic variational inequality involving state constraint

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Abstract

Necessary conditions for optimal controls of some degenerate pseudo - parabolic variational inequality are established by some well-known penalty arguments involving finite co-dimensionality of certain sets. Some analysis on the state equations are given by the theory of m-accretive operators in a Hilbert space.

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1 Introduction

We shall study the optimal control problems governed by nonlinear pseudo-parabolic equations of the form:

$$\begin{cases} \frac{dMy}{dt} + Ay + \beta(y) \ni Bu \; a.e \; in \; Q, \\ M^{1/2}y(0) = M^{1/2}y_0, \end{cases} \tag{1}$$

with the state constraint

$$F(y) \subset S. \tag{2}$$

The pay-off functional is given by

$$L(y,u) = \int_0^T [g(t,y) + h(u)]dt, \tag{3}$$

where $Q = \Omega \times (0,T), \Omega \subset \mathbb{R}^N$, is a bounded domain with smooth boundary.

For the data in (1)-(3), we have the following assumptions.

(H1) M is a nonnegative selfadjoint operator in $H = \text{Ł}^2(\Omega)$ with $D(M) \subset D(A+\beta)$.

(H2) $V \subset H$ is a real Hilbert space such that V is dense in H and $V \subset H \subset V'$ algebraically and topologically, where V' is the dual of V. Further, the injection of V into H is compact.

$A : V \to V'$ is a linear continuous and symmetric operator from V to V' satisfying the coercivity condition

$$(Ay, y) \geq w \parallel y \parallel_V^2 + \alpha \mid y \mid_H^2 \;\; for \; all \; y \in V, \tag{4}$$

where $w > 0$ and $\alpha \in \mathbb{R}$.

(H3) β is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with $0 \in \beta(0)$. Let $\phi(y) : H \to \mathbb{R} = (-\infty, +\infty]$ be the lower semicontinuous convex function defined by $\phi(y) = \int_\Omega j(y)dx$, where $j : \mathbb{R} \to \bar{\mathbb{R}}$ is such that $\partial j = \beta$. Moreover,

$$(Ay, \beta_\epsilon(y)) \geq 0 \;\; \forall \, y \in D(A), \;\, \epsilon > 0, \tag{5}$$

where $\beta_\epsilon(r) = \epsilon^{-1}(r - (1+\epsilon\beta)^{-1}r)$ for all $\epsilon > 0, r \in \mathbb{R}$. For every $\xi \in \beta$, there exists a constant $c > 0$ such that

$$\mid \xi(u) \mid \leq c(1+ \mid u \mid^{p+1}), 0 \leq p \leq 2/(N-2) \; if \; N > 2; 0 \leq p < +\infty \; if \; N = 1,2. \tag{6}$$

(H4) B is a linear continuous operator from a real Hilbert space U to H.

The norm and the scalar product of U will be denoted by $|\cdot|_U$ and $\langle\cdot,\cdot\rangle$, respectively. Let H be a real Hilbert space with the inner product $(\cdot,\cdot)$ and the norm $|\cdot|$.

(H5) Let Z be a Banach space with the dual Z^* strictly convex. $S \subset Z$ is a closed convex subset with finite codimensionality (cf.[22, 4, 5]). $F : L^2(0,T;V) \to Z$ is of class C^1.

(H6) The functional $h : U \to \overline{\mathbb{R}}$ is convex and lower semicontinuous (l. s. c), such that

$$h(u) \geq c_1 \mid u \mid_U^2 + c_2, \ \forall\, u \in U,$$

where $c_1 > 0, c_2 \in R$.

(H7) $g : [0,T] \times H \subseteq \mathbb{R}^+$ is measurable in t, and for every $\delta > 0$, there exists $L_\delta > 0$ independent of t such that $g(t,0) \in L^\infty(0,T)$ and

$$\mid g(t,y_1) - g(t,y_2) \mid \leq L_\delta \mid y_1 - y_2 \mid_H \ \ for\ all\ t \in [0,T], \ \mid y_1 \mid_H + \mid y_2 \mid_H \leq \delta.$$

Remark 1.1. *From a perturbation theorem for m-accretive operators ([8] Lemma 5) and (H2), (H3), we easily know that $A + \partial j$ is m-accretive in $H \times H$, we may write (1) as*

$$\begin{cases} \frac{dMy}{dt} + \partial\varphi(y(t)) \ni Bu \ a.e\ t \in (0,T), \\ M^{1/2}y(0) = M^{1/2}y_0, \end{cases} \tag{7}$$

where $\varphi(y) = \frac{1}{2}(Ay,y) + \phi(y), \quad y \in V$.

Remark 1.2. *M is not necessarily invertible and initial condition should not be given in the form*

$$y(0) = y_0.$$

Since the continuity with respect to t of the solution $y(\cdot,t)$ can not be expected.

As we know, by Barbu [19, 20] (see Chapter 4) and Theorem 1 of [17], we easily obtain.

Lemma 1.3. *Let (H1) − (H4) hold. Then, for any* $y_0 \in D(M) \cap V$, $u \in L^2(0,T;U)$, *(1) admits a unique solution* $y(x,t)$ *satisfying*

$$y \in L^\infty(0,T;D(M)\cap V),\ My,\ M^{1/2}y \in AC([0,T];H),$$
$$(d/dt)My,\ (d/dt)M^{1/2}y \in L^2(0,T;H).$$

Our optimal control problem is stated as follows.

Problem (P). Find a $(\bar{y},\bar{u}) \in A_{ad}$ such that

$$L(\bar{y},\bar{u}) = \inf_{(y,u)\in A_{ad}} L(y,u). \tag{8}$$

Here $A_{ad} = \{(y,u) \in (L^\infty(0,T;D(M)\cap V) \times L^2(0,T;U))|\ y$ is the solution of (7) with (2) $\}$. Any $u(\cdot) \in A_{ad}$ satisfying (2) is called an optimal control, and the corresponding $\bar{y}(\cdot) \equiv y(\cdot;\bar{u})$ is called an optimal state. The pair $(\bar{y},\bar{u})$ is called an optimal pair.

(1) cover cases of the following type: surface waves of long wavelength in liquids, acoustic-gravity waves in compressible fluids, hydromagnetic waves in cold plasma, acoustic waves in anharmonic crystals, etc. The wide applicability of these equations is the main reason why, during the last decades, they have attracted so much attention from mathematicians. Recently, some optimal control problems governed by pseudoparabolic equations have already been discussed. Linear optimal control problems for pseudoparabolic equations were considered by S. I. Lyashko [21, 15, 7, 18, 12, 13, 9]. However, these problems studied in [15, 7, 18, 12, 13, 9] do not involve state constraints and maximal monotone graph. On the other hand, optimal control governed by some parabolic variational inequalities (cf. [11, 14, 10, 3, 23, 19, 20, 1]) have already been discussed. Li and Yong [22] studied the maximal principle for optimal control governed by some nonlinear parabolic equations with two point boundary (time variable) state constraints. In Cases' work [2], the state constraint was considered, but the state equation did not involve monotone graph. He [25] studied the optimal control problems involving some special maximal monotone graph (Lipschitz continuous) with state constraint. Wang [4, 5] also discussed the optimal control problem governed by the state equation involving some maximal monotone graph. Xu [24] have established the maximal principle for optimal control to problem (P) under the following conditions on M and $A+\beta$:

$$D(M) \subset D(A+\beta) \ and \ M \ has \ the \ bounded \ inverse.$$

The present work in this paper is concerned with the optimal control problem governed by the pseudoparabolic equations where the operator M is not necessarily invertible. The method in [4, 5, 19] can not be used directly. The methods which be used primarily here are the theory of m-accretive operators in a Hilbert space. Especially the properties of subdifferentials will play an essential role in the approximating control process.

The plan of this paper is as follows. Section 2 gives an approximating control process. In section 3, we state and prove the necessary conditions on optimality for the problem (P).

2 The approximating control process

Let (y^*, u^*) be optimal for the problem (P). Then

$$\begin{cases} \frac{dMy^*}{dt} + Ay^* + \partial\phi(y^*(t)) \ni Bu^* \ a.e \ t \in (0,T), \\ M^{1/2}y^*(0) = M^{1/2}y_0, \end{cases} \tag{9}$$

with

$$F(y^*) \in S,$$

and

$$L(y^*, u^*) = \inf \ L(y,u) \ over \ (y,u) \in A_{ad}.$$

Now consider the following approximating equations for (7):

$$\begin{cases} (\epsilon + M)\frac{dy}{dt} + C_\epsilon J_\epsilon^M y = Bu \ a.e \ in \ Q, \\ y(0) = y_0, \end{cases} \tag{10}$$

where $C = \partial\varphi, C_\epsilon = \epsilon^{-1}(I - J_\epsilon^C)$ and $J_\epsilon^C = (I + \epsilon C)^{-1}$.

In virtue of (H1), for every fixed $\epsilon > 0$, it is seen that the operator $P_\epsilon(t) \equiv (\epsilon + M)^{-1}C_\epsilon J_\epsilon^M$ is locally Lipschitz continuous in H and the mapping $t \to P_\epsilon(t)y$ is strongly continuous on $[0,T]$ for every $u \in H$. Then we easily obtain that any $y_0 \in D(M) \cap V$, $u \in L^2(0,T;U)$, (10) has a unique solution $W^{1,2}([0,T];H) \cap C([0,T];H) \cap L^2(0,T;V)$ and $dMy/dt = M(dy/dt)$.

We also have the following result on (10).

Lemma 2.1. *For $\epsilon > 0$ given, $u_n \in L^2(0,T;U), u_n \to \tilde{u}$ weakly in $L^2(0,T;U)$. Then there exists some subsequence of $\{y_n\}$, still denoted by itself, such that $y_n \to \tilde{y}$ strongly in $C([0,T];H)$, where $\tilde{y}, y_n$ is the solutions of (10) corresponding to $\tilde{u}$ and u_n, respectively.*

Proof. Multiplying (10) by $J_\epsilon^M y_n(t)$ and using the selfadjointness of M, we see

$$\begin{aligned}\epsilon(y_n'(t), J_\epsilon^M y_n(t)) + \frac{d}{dt} \mid M_\epsilon^{1/2} y_n(t) \mid^2 + 2(C_\epsilon J_\epsilon^M y_n(t), J_\epsilon^M y_n(t)) \\ = 2(Bu_n, J_\epsilon^M y_n(t)).\end{aligned}$$

From the identity $J_\epsilon^M w = \epsilon C_\epsilon J_\epsilon^M w + J_\epsilon^C J_\epsilon^M w$ yields

$$\begin{aligned}&\epsilon\frac{d}{2dt} \mid (J_\epsilon^M)^{1/2} y_n(t) \mid^2 + \frac{d}{dt} \mid M_\epsilon^{1/2} y_n(t) \mid^2 + \epsilon \mid C_\epsilon J_\epsilon^M y_n(t) \mid^2 + c \mid J_\epsilon^C J_\epsilon^M y_n(t) \mid^2 \\ &\leq c \mid u_n \mid_U^2 + \frac{1}{4}\epsilon \mid C_\epsilon J_\epsilon^M y_n(t) \mid^2 + \frac{1}{4} c \mid J_\epsilon^C J_\epsilon^M y_n(t) \mid^2,\end{aligned}$$

where $M_\epsilon = \epsilon^{-1}(I - J_\epsilon^M)$. It is well known that M_ϵ is m-accretive on H and Lipschitz continuous with $1/\epsilon$ as a Lipschitz constant and $\mid M_\epsilon y \mid \leq \mid M^0 y \mid$ with $M_\epsilon y \to M^0 y$ as $\epsilon \to 0$ for every $y \in D(M)$. For an m-accretive operator M, $My (y \in D(M))$ is closed and convex; so My has a unique element of the least norm which is denoted by $M^0 y$. Integrating the above inequality from 0 to T , we see

$$\begin{aligned}&\epsilon \mid (J_\epsilon^M)^{1/2} y_n(t) \mid_{C([0,T];H)}^2 + \mid M_\epsilon^{1/2} y_n(t) \mid_{C([0,T];H)} \\ &+ \epsilon^{1/2} \mid C_\epsilon J_\epsilon^M y_n(t) \mid_{L^2(0,T;H)} + \mid J_\epsilon^C J_\epsilon^M y_n \mid_{L^2(0,T;H)} \leq c.\end{aligned} \tag{11}$$

Multiplying (10) by $J_\epsilon^M y_n'(t)$, we see

$$\epsilon(y_n'(t), J_\epsilon^M y_n'(t)) + \mid M_\epsilon^{1/2} y_n'(t) \mid^2 + \frac{d}{dt}\varphi_\epsilon(J_\epsilon^M y_n(t)) \leq (Bu_n, J_\epsilon^M y_n'(t)).$$

Here we have used the fact $(C_\epsilon J_\epsilon^M y_n(t), J_\epsilon^M y_n'(t)) = \frac{d}{dt}\varphi_\epsilon(J_\epsilon^M y_n(t))$, thus

$$\begin{aligned}&\epsilon(y_n'(t), J_\epsilon^M y_n'(t)) + \mid M_\epsilon^{1/2} y_n'(t) \mid^2 + \frac{d}{dt}\varphi_\epsilon(J_\epsilon^M y_n(t)) \\ &\leq c \mid (J_\epsilon^M)^{1/2} Bu_n \mid^2 + \frac{1}{4}\epsilon \mid (J_\epsilon^M)^{1/2} y_n'(t) \mid^2 \\ &\leq c \mid u_n \mid_U^2 + \frac{1}{4}\epsilon \mid (J_\epsilon^M)^{1/2} y_n'(t) \mid^2 .\end{aligned}$$

Integrating the above inequality from 0 to T, and φ_ϵ is Fréchet differentiable on H(see [4]), we see

$$\epsilon \mid (J_\epsilon^M)^{1/2} y_n'(t) \mid_{L^2(0,T;H)} + \mid M_\epsilon^{1/2} y_n'(t) \mid_{L^2(0,T;H)} + \varphi_\epsilon(J_\epsilon^M y_n(t)) \leq c. \tag{12}$$

On the other hand, from (4) and (H3), we get

$$c \mid J_\epsilon^C J_\epsilon^M y_n(t) \mid^2 \leq \varphi(J_\epsilon^C J_\epsilon^M y_n(t)) \leq \varphi_\epsilon(J_\epsilon^M y_n(t)) \ for\ any\ t \in [0,T]. \quad (13)$$

and

$$\mid J_\epsilon^M y_n(t) \mid \leq \epsilon \mid C_\epsilon J_\epsilon^M y_n(t)) \mid + \mid J_\epsilon^C J_\epsilon^M y_n(t) \mid . \quad (14)$$

This implies

$$\epsilon \mid y_n'(t) \mid_{L^2(0,T;H)} \leq \epsilon \int_0^T (y_n'(t), J_\epsilon^M y_n'(t))dt + \epsilon^2 \mid M_\epsilon^{1/2} y_n'(t) \mid_{L^2(0,T;H)} \leq c. \quad (15)$$

Multiplying (10) by $M_\epsilon J_\epsilon^M y_n(t)$, we see

$$\begin{aligned} 2\epsilon(y_n'(t), M_\epsilon J_\epsilon^M y_n(t)) + \frac{d}{dt} \mid M_\epsilon y_n(t) \mid^2 &+ 2(C_\epsilon J_\epsilon^M y_n(t), M_\epsilon J_\epsilon^M y_n(t)) \\ &= 2(Bu_n, M_\epsilon J_\epsilon^M y_n(t)), \end{aligned} \quad (16)$$

thus

$$\epsilon \frac{d}{dt} \mid M^{1/2} J_\epsilon^M y_n(t)) \mid^2 + \frac{d}{dt} \mid M_\epsilon y_n(t) \mid^2 \leq c + \mid M_\epsilon y_n(t) \mid^2,$$

Integrating this inequality over $(0,t], t \leq T$ and using Gronwall's inequality, yields

$$\mid M_\epsilon y_n(t) \mid_{L^2(0,T;H)} \leq c \ \ \forall\ t \in [0,T]. \quad (17)$$

Hence we get

$$\mid y_n(t) \mid \leq \epsilon \mid M_\epsilon y_n(t) \mid + \mid J_\epsilon^M y_n(t) \mid \leq c \ \ \forall\ t \in [0,T]. \quad (18)$$

From (H2) and (H3), it follows that

$$\mid C_\epsilon J_\epsilon^M y_n(t) \mid_{L^2(0,T;H)} \leq c,$$

then in view of (10) and the above inequality give

$$\mid M \frac{d}{dt} y_n(t) \mid_{L^2(0,T;H)} \leq c, \quad (19)$$

which implies

$$\mid M y_n(t) \mid \leq c \ \ for\ all\ t \in [0,T], \quad (20)$$

and

$$\mid \frac{d}{dt} y_n(t) \mid_{L^2(0,T;H)} \leq c. \quad (21)$$

For every $m, n > 0$

$$\epsilon \frac{d}{dt} \mid y_m - y_n \mid^2 + \frac{d}{dt} \mid M^{1/2}(y_m - y_n) \mid^2 + 2(C_\epsilon J_\epsilon^M (y_m - y_n), y_m - y_n) \\ \leq c \mid u_m - u_n \mid_U^2 + \epsilon \mid y_m - y_n \mid^2,$$

by some calculations, we see

$$\epsilon \mid (y_m - y_n)(t) \mid^2 + \mid M^{1/2}(y_m - y_n)(t) \mid^2 + \int_0^t \mid (y_m - y_n) \mid^2 d\tau \\ \leq c \int_0^t \mid u_m - u_n \mid_U^2 d\tau \quad for\ all\ t \in [0, T].$$

Hence $\{M^{1/2} y_n\}$ and $\{y_n\}$ are Cauchy sequences in $C([0,T]; H) \cap L^2(0,T; H)$. By (H2), then there exists a function $\tilde{y} \in C([0,T]; D(M^{1/2}))$ such that as $n \to \infty$

$$y_n \to \tilde{y} \quad strongly\ in\ C([0,T]; H), \\ M^{1/2} y_n \to M^{1/2} \tilde{y} \quad strongly\ in\ C([0,T]; H).$$

This completes the proof. □

Next, we recall the approximation g^ϵ of g and h^ϵ of h as follows. For the details, we refer to [4, 5, 6, 19]. Let

$$g^\epsilon(t, y) = \int_{R^N} g(t, P_N y - \epsilon \Lambda_N s) \rho(s) ds, \quad \epsilon > 0$$

where ρ is a mollifier in R^N, $N = [\epsilon^{-1}]$. $P_N : L^2 \to X_N$ is the projection of $L^2(\Omega)$ on X_N which is the finite dimensional space generated by $\{e_i\}_{i=1}^N$, where $\{e_i\}_{i=1}^\infty$ is an orthonormal basis in $L^2(\Omega)$. $\Lambda_N : R^N \to X_N$ is the operator defined by $\Lambda_N(s) = \sum_{i=1}^N s_i e_i$, $s = (s_1, \cdots, s_N)$.

Let $h_\epsilon : U \to R$ be defined by

$$h_\epsilon(u) = \inf \Big\{ \frac{\| u - v \|_{L^2(0,T;U)}^2}{2\epsilon} + h(v) : \ v \in U \Big\}, \epsilon > 0.$$

Now we define the penalty $L_\epsilon : L^2(0,T;U) \to \mathbb{R}$ by

$$L_\epsilon(u) = \int_0^T [g^\epsilon(t, y_\epsilon) + h_\epsilon(u)] dt + \frac{1}{2} \mid u - u^* \mid_{L^2(0,T;U)}^2 + \frac{1}{2\epsilon^{1/2}} [\epsilon^{1/2} + d_S(F(y_\epsilon))]^2, \tag{22}$$

where y_ϵ is the solution of (10). $d_S(F(y_\epsilon))$ denotes the distance of $F(y_\epsilon)$ to S.

The approximating optimal control problems are as follows:

(P^ϵ) Minimize $L_\epsilon(u)$ over $u \in L^2(0,T;U)$.

From Lemma 2.1, we easily obtain the following existence of optimal solutions for (P^ϵ) (see [4, 5, 19]).

Theorem 2.2. *Problem (P^ϵ) has at least one optimal solution.*

The following results are useful in discussing the approximating control problems.

Lemma 2.3. *Let $u_\epsilon \to u$ weakly in $L^2(0,T;U)$ as $\epsilon \to 0$. Then there exists a subsequence $\{y_\epsilon\}_{\epsilon>0}$, still denoted it self*

$$y_\epsilon \to y \quad \textit{strongly in } C([0,T];H),$$

as $\epsilon \to 0$. where y_ϵ is the solutions of (10) corresponding to u_ϵ and y is the solutions of (7) corresponding to u.

Proof. Rewrite (10) as follows:

$$\begin{cases} (\epsilon + M)\frac{dy_\epsilon(t)}{dt} + C_\epsilon J_\epsilon^M y_\epsilon(t) = Bu_\epsilon(t) \ a.e \ in \ Q, \\ y_\epsilon(0) = y_0. \end{cases} \tag{23}$$

Multiplying (23) by $J_\epsilon^M y_\epsilon(t)$, we see

$$\begin{aligned} 2\epsilon(y_\epsilon'(t), J_\epsilon^M y_\epsilon(t)) + \frac{d}{dt} \mid M_\epsilon^{1/2} y_\epsilon(t) \mid^2 + 2(C_\epsilon J_\epsilon^M y_\epsilon(t), J_\epsilon^M y_\epsilon(t)) \\ = 2(Bu_\epsilon, J_\epsilon^M y_\epsilon(t)), \end{aligned}$$

from the identity $J_\epsilon^M y_\epsilon(t) = \epsilon C_\epsilon J_\epsilon^M y_\epsilon(t) + J_\epsilon^C J_\epsilon^M y_\epsilon(t)$ and (H3) yield

$$(C_\epsilon J_\epsilon^M y_\epsilon(t), J_\epsilon^M y_\epsilon(t)) \geq \epsilon \mid C_\epsilon J_\epsilon^M y_\epsilon(t) \mid^2 + c \mid J_\epsilon^C J_\epsilon^M y_\epsilon(t) \mid^2,$$

thus,

$$\begin{aligned} &\frac{d}{dt} \mid M_\epsilon^{1/2} y_\epsilon(t) \mid^2 + \epsilon \frac{d}{dt}(y_\epsilon(t), J_\epsilon^M y_\epsilon(t)) + 2\epsilon \mid C_\epsilon J_\epsilon^M y_\epsilon(t) \mid^2 + 2c \mid J_\epsilon^C J_\epsilon^M y_\epsilon(t) \mid^2 \\ &\leq \frac{1}{4}\epsilon \mid (J_\epsilon^M)^{1/2} y_\epsilon(t) \mid^2 + c \mid (J_\epsilon^M)^{1/2} Bu_\epsilon \mid^2 . \end{aligned}$$

Integrating the above inequality from 0 to $t(t \in (0,T])$, we have

$$| M_\epsilon^{1/2} y_\epsilon(t) |_{C(0,T;H)} \leq c, \tag{24}$$

and

$$\epsilon^{1/2} | C_\epsilon J_\epsilon^M y_\epsilon(t) |_{L^2(0,T;H)} \leq c, \quad | J_\epsilon^C J_\epsilon^M y_\epsilon(t) |_{L^2(0,T;H)} \leq c, \tag{25}$$

which implies

$$| J_\epsilon^M y_\epsilon(t) |_{L^2(0,T;H)} \leq c. \tag{26}$$

Multiplying (23) by $J_\epsilon^M y_\epsilon'(t)$, we see

$$\begin{aligned} &\epsilon(y_\epsilon'(t), J_\epsilon^M y_\epsilon'(t)) + | M_\epsilon^{1/2} y_\epsilon'(t) |^2 + \frac{d}{dt}\varphi_\epsilon(J_\epsilon^M y_n(t)) \\ &\leq (Bu_\epsilon, J_\epsilon^M y_\epsilon'(t)) \leq c \mid (J_\epsilon^M)^{1/2} Bu_\epsilon |^2 + \frac{1}{4}\epsilon \mid (J_\epsilon^M)^{1/2} y_\epsilon' |^2 . \end{aligned}$$

Here we have used the fact $(C_\epsilon J_\epsilon^M y_\epsilon(t), J_\epsilon^M y_\epsilon'(t)) = \frac{d}{dt}\varphi_\epsilon(J_\epsilon^M y_\epsilon(t))$.

Integrating the above inequality over $(0,T]$, and φ_ϵ is Frechet differentiable on H(see [4]) we see

$$| M_\epsilon^{1/2} y_\epsilon'(t) |_{L^2(0,T;H)} \leq c, \ \varphi_\epsilon(J_\epsilon^M y_\epsilon(t)) \leq c, \quad \epsilon \int_0^T (y_\epsilon'(t), J_\epsilon^M y_\epsilon'(t)) dt \leq c. \tag{27}$$

This implies

$$\epsilon \mid y_\epsilon'(t) |_{L^2(0,T;H)} \leq \epsilon \int_0^T (y_\epsilon'(t), J_\epsilon^M y_\epsilon'(t)) dt + \epsilon^2 \mid M_\epsilon^{1/2} y_\epsilon'(t) |_{L^2(0,T;H)} \leq c. \tag{28}$$

Multiplying (23) by $M_\epsilon J_\epsilon^M y_\epsilon(t)$, we see

$$\begin{aligned} 2\epsilon(y_\epsilon'(t), M_\epsilon J_\epsilon^M y_\epsilon(t)) + \frac{d}{dt} \mid M_\epsilon y_\epsilon(t) |^2 + 2(C_\epsilon J_\epsilon^M y_\epsilon(t), M_\epsilon J_\epsilon^M y_\epsilon(t)) \\ = 2(Bu_\epsilon, M_\epsilon J_\epsilon^M y_\epsilon(t)), \end{aligned}$$

thus

$$\epsilon \frac{d}{dt} \mid M^{1/2} J_\epsilon^M y_\epsilon(t)) |^2 + \frac{d}{dt} \mid M_\epsilon y_\epsilon(t) |^2 \leq c \mid Bu_\epsilon |^2 + \mid M_\epsilon y_\epsilon(t) |^2,$$

Integrating this inequality over $(0,t], t \leq T$ and Gronwall's inequality yield

$$| M_\epsilon y_\epsilon(t) | \leq c. \tag{29}$$

Hence we get

$$| y_\epsilon(t) | \leq \epsilon \mid M_\epsilon y_\epsilon(t) \mid + \mid J_\epsilon^M y_\epsilon(t) | \leq c \ \ \forall \ t \in [0,T]. \tag{30}$$

In view of (23) and (28), we finally obtain

$$| My'_\epsilon(t) |_{L^2([0,T];H)} \leq c, \quad | My_\epsilon(t) | \leq c \ \ \forall \ t \in [0,T]. \tag{31}$$

For every $m, n > 0$, we have

$$\begin{aligned}(\epsilon_m y'_{\epsilon_m} - \epsilon_n y'_{\epsilon_n}, y_{\epsilon_m} - y_{\epsilon_n}) + \frac{d}{2dt} \mid M^{1/2}(y_{\epsilon_m} - y_{\epsilon_n}) \mid^2 + (C_{\epsilon_m} J^M_{\epsilon_m} y_{\epsilon_m} \\ - C_{\epsilon_n} J^M_{\epsilon_n} y_{\epsilon_n}, y_{\epsilon_m} - y_{\epsilon_n}) \\ \leq c \mid B(u_{\epsilon_m} - u_{\epsilon_n}) \mid^2 + \frac{1}{4}\epsilon \mid y_{\epsilon_m} - y_{\epsilon_n} \mid^2,\end{aligned}$$

Using the identities $w = J^M_{\epsilon_m} w + \epsilon_m M_{\epsilon_m} w \ for \ every \ w \in H, etc.$, we see

$$\begin{aligned}&(C_{\epsilon_m} J^M_{\epsilon_m} y_{\epsilon_m} - C_{\epsilon_n} J^M_{\epsilon_n} y_{\epsilon_n}, y_{\epsilon_m} - y_{\epsilon_n}) \\ &= (C_{\epsilon_m} J^M_{\epsilon_m} y_{\epsilon_m} - C_{\epsilon_n} J^M_{\epsilon_n} y_{\epsilon_n}, J^C_{\epsilon_m} J^M_{\epsilon_m} y_{\epsilon_m} - J^C_{\epsilon_m} J^M_{\epsilon_n} y_{\epsilon_n}) \\ &+ (C_{\epsilon_m} J^M_{\epsilon_m} y_{\epsilon_m} - C_{\epsilon_n} J^M_{\epsilon_n} y_{\epsilon_n}, \epsilon_m C_{\epsilon_m} J^M_{\epsilon_m} y_{\epsilon_m} - \epsilon_n C_{\epsilon_m} J^M_{\epsilon_n} y_{\epsilon_n}) \\ &+ (C_{\epsilon_m} J^M_{\epsilon_m} y_{\epsilon_m} - C_{\epsilon_n} J^M_{\epsilon_n} y_{\epsilon_n}, \epsilon_m M_{\epsilon_m} y_{\epsilon_m} - \epsilon_n M_{\epsilon_n} y_{\epsilon_n}) \\ &\geq c \mid J^C_{\epsilon_m} J^M_{\epsilon_m} y_{\epsilon_m} - J^C_{\epsilon_m} J^M_{\epsilon_n} y_{\epsilon_n} \mid^2 - c(\epsilon_m + \epsilon_n) \\ &\geq c \mid y_{\epsilon_m} - y_{\epsilon_n} \mid^2 - c(\epsilon_m + \epsilon_n).\end{aligned}$$

Here using the estimate

$$\mid y_{\epsilon_m} - y_{\epsilon_n} \mid \leq \mid J^M_{\epsilon_m} y_{\epsilon_m} - J^M_{\epsilon_n} y_{\epsilon_n} \mid + \epsilon_m \mid M_{\epsilon_m} y_{\epsilon_m} \mid + \epsilon_n \mid M_{\epsilon_n} y_{\epsilon_n} \mid .$$

From (28) and (30) we see

$$\mid \int_0^T (\epsilon_m y'_{\epsilon_m} - \epsilon_n y'_{\epsilon_n}, y_{\epsilon_m} - y_{\epsilon_n}) dt \mid \leq c(\epsilon_m^{1/2} + \epsilon_n^{1/2}).$$

Hence combining them, we get

$$\begin{aligned}\mid M^{1/2}(y_{\epsilon_m} - y_{\epsilon_n}) \mid^2 + \int_0^t \mid y_{\epsilon_m} - y_{\epsilon_n} \mid^2 ds \leq c \int_0^t \mid (u_{\epsilon_m} - u_{\epsilon_n}) \mid_U^2 ds \\ + c(\epsilon_m^{1/2} + \epsilon_n^{1/2}).\end{aligned}$$

Combining this with (H2), we know that $\{M^{1/2} y_{\epsilon_n}\}$ and $\{y_{\epsilon_n}\}$ are Cauchy sequences in $C([0,T]; H) \cap L^2(0,T;V)$. Then there exists a function

$$y \in C([0,T]; D(M^{1/2}))$$

such that as $n \to \infty, \epsilon_n \to 0$,

$$\begin{aligned} y_{\epsilon_n} \to y \ \ &strongly \ in \ C([0,T];H) \cap L^2(0,T;V), \\ M^{1/2} y_{\epsilon_n} \to M^{1/2} y \ \ &strongly \ in \ C([0,T];H). \end{aligned} \tag{32}$$

$$J^M_{\epsilon_n} y_{\epsilon_n} \to y \ \ strongly \ in \ C([0,T];H), \quad \epsilon_n y'_{\epsilon_n} \to 0 \ weakly \ in \ L^2(0,T;H). \tag{33}$$

Note that

$$M^{1/2} J^M_{\epsilon_n} y_{\epsilon_n} \to M^{1/2} y \ \ strongly \ in \ L^2(0,T;H), \tag{34}$$

Indeed, we see

$$\begin{aligned} \mid M^{1/2} J^M_{\epsilon_n} y_{\epsilon_n}(t) - M^{1/2} y \mid^2 (t) &\leq 2 \mid M^{1/2} (J^M_{\epsilon_n} y_{\epsilon_n}(t) - y_{\epsilon_n})(t) \mid^2 \\ &+ \mid M^{1/2} (y_{\epsilon_n}(t) - y(t)) \mid^2 \\ = -2\epsilon_n (M_{\epsilon_n} y_{\epsilon_n}(t), M_{\epsilon_n} y_{\epsilon_n}(t) - M y_{\epsilon_n}(t)) &+ \mid M^{1/2} (y_{\epsilon_n}(t) - y) \mid^2 \\ &\leq c\epsilon_n + \mid M^{1/2} (y_{\epsilon_n}(t) - y) \mid^2 \to 0, \end{aligned}$$

for all $t \in [0,T]$. In virtue of (28), (30) and (31), weakly closedness of d/dt and β, it is shown that

$$dy_{\epsilon_n}/dt \to dy/dt \ \ weakly \ in \ L^2(0,T;H), \tag{35}$$

$$M dy_{\epsilon_n}/dt \to M dy/dt \ \ weakly \ in \ L^2(0,T;H). \tag{36}$$

Therefore, $y \in AC(0,T;D(M))$ and $dy/dt \in L^2(0,T;D(M))$. We easily see that $y(t) \in D(C)$ *a.e.* $t \in (0,T)$ and there exists a function $\xi \in L^\infty(0,T;H)$ such that as $\epsilon_n \to 0$,

$$C_{\epsilon_n} J^M_{\epsilon_n} y_{\epsilon_n} \to \xi \ weakly \ star \ in \ L^\infty(0,T;H),$$

and $\xi(t) \in Cy (= Ay + \beta(y))$ *a.e.* $t \in (0,T)$. From (32), we have $M^{1/2} y(0) = M^{1/2} y_0$. Thus letting $\epsilon \to 0$ in (23), it follows that

$$\begin{cases} \frac{dMy(t)}{dt} + Ay + \xi(t) = Bu(t) \ a.e \ in \ Q, \\ M^{1/2} y(0) = M^{1/2} y_0. \end{cases} \tag{37}$$

□

Lemma 2.4. *Let $y_0 \in D(M) \cap V, u \in L^2(0,T;U)$, then $y_\epsilon \to y$ strongly in $C([0,T];H) \cap L^2(0,T;V)$ as $\epsilon \to 0$. where y_ϵ is the solutions of (10) corresponding to u and y be the solutions of (1) corresponding to u with the initial condition $M^{1/2}y(0) = M^{1/2}y_0$. Furthermore,*

$$\mid M^{1/2}(y_\epsilon - y) \mid_{C([0,T];H)} + \mid y_\epsilon - y \mid_{L^2(0,T;H)} \leq c\epsilon^{1/2}. \tag{38}$$

Proof. By the same argument in the proof of Lemma 2.3 we have $y_\epsilon \to y$ *strongly in* $C([0,T];H)$. We have for all ϵ and λ,

$$\begin{cases} \frac{d}{dt}(\epsilon y_\epsilon - \lambda y_\lambda) + \frac{dM(y_\epsilon(t) - y_\lambda(t))}{dt} + C_\epsilon J_\epsilon^M y_\epsilon(t) - C_\lambda J_\lambda^M y_\lambda(t) = 0 \ a.e \ in \ Q, \\ y_\epsilon(0) - y_\lambda(0) = 0. \end{cases} \tag{39}$$

Multiplying (39) by $y_\epsilon(t) - y_\lambda(t)$, we have

$$\begin{aligned} 2(\epsilon y_\epsilon' - \lambda y_\lambda', y_\epsilon - y_\lambda)(t) + \frac{d \mid M^{1/2}(y_\epsilon(t) - y_\lambda(t)) \mid^2}{dt} \\ + 2(C_\epsilon J_\epsilon^M y_\epsilon(t) - C_\lambda J_\lambda^M y_\lambda(t), y_\epsilon(t) - y_\lambda(t)) = 0. \end{aligned} \tag{40}$$

From (H2), (H3) and (30), the third term of the left hand side of (40), using the identities $w = J_\epsilon^M w + \epsilon M_\epsilon w \ for \ every \ w \in H, etc.$, we see

$$\begin{aligned} &(C_\epsilon J_\epsilon^M y_\epsilon - C_\lambda J_\lambda^M y_\lambda, y_\epsilon - y_\lambda) \\ &= (C_\epsilon J_\epsilon^M y_\epsilon - C_\lambda J_\lambda^M y_\lambda, J_\epsilon^C J_\epsilon^M y_\epsilon - J_\epsilon^C J_\lambda^M y_\lambda) \\ &+ (C_\epsilon J_\epsilon^M y_\epsilon - C_\lambda J_\lambda^M y_\lambda, \epsilon C_\epsilon J_\epsilon^M y_\epsilon - \lambda C_\epsilon J_\lambda^M y_\lambda) \\ &+ (C_\epsilon J_\epsilon^M y_\epsilon - C_\lambda J_\lambda^M y_\lambda, \epsilon M_\epsilon y_\epsilon - \lambda M_\lambda y_\lambda) \\ &\geq c \mid J_\epsilon^C J_\epsilon^M y_\epsilon - J_\epsilon^C J_\lambda^M y_\lambda \mid^2 - c(\epsilon + \lambda) \\ &\geq c \mid y_\epsilon - y_\lambda \mid^2 - c(\epsilon + \lambda). \end{aligned}$$

From (28) and (30) we see

$$\mid \int_0^T (\epsilon y_\epsilon' - \lambda y_\lambda', y_\epsilon - y_\lambda) dt \mid \leq c(\epsilon^{1/2} + \lambda^{1/2}).$$

Hence combining them, we get

$$\mid M^{1/2}(y_\epsilon - y_\lambda) \mid^2 + \int_0^T \mid y_\epsilon - y_\lambda \mid^2 ds \leq c(\epsilon^{1/2} + \lambda^{1/2}).$$

Letting $\lambda \to 0$ in the above inequality, we get (38). □

Lemma 2.5. *Let u_ϵ be optimal for the problem (P^ϵ) and y_ϵ be the solution of (10) corresponding to u_ϵ. For $\epsilon \to 0$, then*

$$y_\epsilon \to y_* \ \ \textit{strongly in } C([0,T];H) \cap L^2(0,T;V).$$

$$u_\epsilon \to u_* \ \ \textit{strongly in } L^2(0,T;U).$$

y_ϵ be the solutions of (10) corresponding to u_ϵ, y_* be the solutions of (1) corresponding to u_* with the initial condition $M^{1/2}y(0) = M^{1/2}y_0$.

Proof. For any $\epsilon > 0$, we have

$$L_\epsilon(u_\epsilon) \le L_\epsilon(u_*) = \int_0^T [g^\epsilon(t, y_\epsilon) + h_\epsilon(u_*)]dt + \frac{1}{2\epsilon^{1/2}}[\epsilon^{1/2} + d_S(F(y_\epsilon))]^2.$$

By Lemma 2.3, $y_\epsilon \to y_*$ strongly in $C([0,T];H)$, We have

$$g^\epsilon(t, y_\epsilon) \to g(t, y_*) \ for\ all\ t \in [0,T],$$

and

$$h_\epsilon(u_\epsilon) \to h(u_*).$$

So

$$\lim_{\epsilon\to 0} \int_0^T g^\epsilon(t, y_\epsilon)dt = \int_0^T g(t, y_*)dt, \ \ \lim_{\epsilon\to 0} \int_0^T h_\epsilon(u_*)dt = \int_0^T h(u_*)dt.$$

Similarly, By (38) and (H5), we see

$$\frac{1}{2\epsilon^{1/2}}[\epsilon^{1/2} + d_S(F(y_\epsilon))]^2 \le \frac{1}{2\epsilon}[\epsilon^{1/2} + \parallel F(y_\epsilon) - F(y_*) \parallel_Z]^2 \le c\epsilon^{1/2} \to 0 \ \ as\ \epsilon \to 0,$$

thus,

$$\limsup_{\epsilon\to 0} L_\epsilon(u_\epsilon) \le L(u_*). \tag{41}$$

On the other hand, since $\{u_\epsilon\}_{\epsilon>0}$ is bounded in $L^2(0,T;U)$, there exists $u_1 \in L^2(0,T;U)$ such that, on some subsequence $\{\epsilon\}_{\epsilon>0}$, still denoted by itself, as $\epsilon \to 0$,

$$u_\epsilon \to u_1 \ weakly\ in \ \ L^2(0,T;U),$$

and so, by Lemma 11,

$$y_\epsilon \to y_1 = y(u_1) \ strongly\ in \ \ C([0,T];H).$$

By (41), one can check easily that

$$\frac{1}{2\epsilon^{1/2}}[\epsilon^{1/2} + d_S(F(y_\epsilon))]^2 \leq c.$$

Thus, $d_S(F(y_\epsilon)) \to 0$ as $\epsilon \to 0$. Since S is closed and convex, $F(y_1) = \lim_{\epsilon \to 0} F(y_\epsilon) \in S$. Since the function $u \to \int_0^T h(u)dt$ is weakly lower semicontinuous on $L^2(0,T;U)$, we see

$$\liminf_{\epsilon \to 0} L_\epsilon(u_\epsilon) \geq L(u_1) \geq L(u_*).$$

together with (41), yields

$$\lim_{\epsilon \to 0} L_\epsilon(u_\epsilon) = L(u_*).$$

Arguing as in the proof of Lemma 11, we have $M^{1/2}y_\epsilon(0) \to M^{1/2}y_*(0)$ *in* H. Hence $y_1 = y_*,\ u_1 = u_*$. This completes the proof. $\square$

3 Necessary conditions on optimality

Let ∂g be the generalized gradient of $y \to g(t,y)$ and ∂h be the subdifferential of h (see [4, 5, 19]). $Y^* = (H^s(\Omega))' + V'$ is the dual of $Y = H^s(\Omega) \cap V$ with $s > N/2$.

Firstly, we consider the Cauchy problem

$$\begin{cases} (\epsilon + M)\frac{dp_\epsilon}{dt} - Ap_\epsilon - \dot{\beta}^\epsilon(y_\epsilon)p_\epsilon - [F'(y_\epsilon)]^*\xi_\epsilon = \lambda_\epsilon \nabla g^\epsilon(t, y_\epsilon) \ \ in \ \ (0,T), \\ p_\epsilon(T) = 0, \end{cases} \tag{42}$$

where $\dot{\beta}^\epsilon = (\beta^\epsilon)'$. $\beta_\epsilon = \epsilon^{-1}(I - (I + \epsilon\beta)^{-1})$, $\beta^\epsilon = \int_{-\infty}^{\infty}[\beta_\epsilon(r - \epsilon^2\theta) - \beta_\epsilon(-\epsilon^2\theta)]\rho(\theta)$ $+\beta_\epsilon(0)$ and ρ is a C_0^∞ -mollifier on $\mathbb{R}$.

Lemma 3.1. *Problem (42) has a unique absolutely continuous function* $p_\epsilon \in L^2(0,T;V) \cap C([0,T];H)$ *with* $p_\epsilon' \in L^2(0,T;V')$*, such that*

$$\epsilon \mid p_\epsilon(t) \mid_2^2 + \mid M^{1/2}p_\epsilon(t) \mid_2^2 + \int_0^T \parallel p_\epsilon(t) \parallel_V^2 dt \leq c \ \ \forall\ \epsilon > 0,\ t \in [0,T], \tag{43}$$

$$\int_Q \mid p_\epsilon \dot{\beta}^\epsilon(y_\epsilon) \mid dxdt \leq c \ \ \forall\ \epsilon > 0. \tag{44}$$

Proof. From (H1) − (H3), $\dot{\beta}^\epsilon(y_\epsilon) \geq 0$, it is see that $C = (\epsilon + M)^{-1}(A + \dot{\beta}^\epsilon(y_\epsilon)) : V \to V'$ is demicontinuous monotone operator that satisfies

$$(C\omega, \omega) \geq w \parallel \omega \parallel^p + c \quad \forall \omega \in V,$$
$$\parallel C\omega \parallel_* \leq c(1 + \parallel \omega \parallel^{p-1}),$$

where $w > 0$ and $p \geq 2$. It follows by Theorem 1.9′ of [19] that (42) has a unique solution $p_\epsilon \in L^2(0,T;V) \cap C([0,T];H)$ with $p'_\epsilon \in L^2(0,T;V')$. Multiplying (42) by $p_\epsilon(t)$ and integrating over $[t,T]$, we see

$$\epsilon \mid p_\epsilon(t) \mid^2 + \mid M^{1/2} p_\epsilon(t) \mid^2 + \int_t^T \parallel p_\epsilon(s) \parallel_V^2 ds \leq c.$$

Thus we obtain (43). Multiplying (42) by $\zeta(p_\epsilon)$ and integrating on Q, where ζ is a smooth monotonically increasing approximation of the sign function such that $\zeta(0) = 0$. For instance

$$\zeta = \zeta_\lambda(r) = \int_{-\infty}^{\infty} (\zeta_\lambda(r - \lambda\theta) - \zeta_\lambda(-\lambda\theta))\rho(\theta) d\theta,$$

where $\zeta_\lambda(r) = r \mid r \mid^{-1}$ for $\mid r \mid \geq \lambda, \zeta_\lambda(r) = \lambda^{-1} r$ for $\mid r \mid < \lambda$, and ρ is a $C_0^\infty-$mollifier. Then $(Ap_\epsilon(t), \zeta(p_\epsilon(t))) \geq 0$, therefore,

$$\int_Q \dot{\beta}^\epsilon(y_\epsilon)\zeta(p_\epsilon)p_\epsilon dxdt \leq \int_Q \mid \nabla_y g^\epsilon(t, y_\epsilon)\zeta(p_\epsilon) \mid dxdt, \ \forall\ \epsilon > 0.$$

Then, letting ζ tend to the sign function, we get (44). □

We state the main results of the necessary conditions on optimality as follows.

Theorem 3.2. *Suppose that (H1) − (H7) hold. Let (y_*, u_*) be an optimal pair of problem (P). Then there exist the function $p \in L^\infty(0,T;H) \cap L^2(0,T;V) \cap BV([0,T];Y^*)$, the measure $\mu \in (L^\infty(Q))'$ and $\lambda_0 \in \mathbb{R}$, $\xi_0 \in Z^*$ satisfying*

$$\frac{d}{dt} Mp - Ap - \mu - [F'(y_*)]^* \xi_0 \in L^\infty(0,T;H), \tag{45}$$

$$\begin{cases} \frac{d}{dt} Mp(t) - Ap(t) - \mu - [F'(y_*)]^*\xi_0 \in \lambda_0 \partial g(t, y_*) \ a.e. \ in \ (0,T), \\ M^{1/2} p(T) = 0, \end{cases} \tag{46}$$

$$\langle \xi_0, w - F(y_*) \rangle \leq 0 \ for \ all \ w \in S, \tag{47}$$

$$B^* p \in \lambda_0 \partial h(u_*)(t), a.e. \ t \in (0,T), \tag{48}$$

$$(\lambda_0, \xi_0) \neq 0. \tag{49}$$

Proof. Since (y_ϵ, u_ϵ) is optimal for problem (P^ϵ), we see

$$L_\epsilon(u_\epsilon^\rho) \geq L_\epsilon(u_\epsilon) \ for\ any\ \rho > 0, v \in L^2(0,T;V),$$

Here $u_\epsilon^\rho = u_\epsilon + \rho v$. Thus

$$\frac{L_\epsilon(u_\epsilon^\rho) - L_\epsilon(u_\epsilon)}{\rho} \geq 0. \tag{50}$$

By some calculation, we have

$$\lim_{\rho\to 0}\int_0^T \frac{g^\epsilon(t, y_\epsilon^\rho) - g^\epsilon(t, y_\epsilon)}{\rho} = \int_0^T (\nabla g^\epsilon(t, y_\epsilon), z_\epsilon)dt,$$

where $z_\epsilon \in C([0,T];H) \cap L^2(0,T;V) \cap W^{1,2}([0,T];H)$ is the solution to the linear equation

$$\begin{cases} (\epsilon + M)\frac{dz}{dt} + Az + \dot{\beta}^\epsilon(y_\epsilon)z = Bv \ \ in \ \ (0,T), \\ z(0) = 0, \end{cases} \tag{51}$$

Thus, we also have

$$\lambda_\epsilon\Big[\int_0^T \langle\nabla g^\epsilon(t, y_\epsilon), z_\epsilon\rangle dt + \int_0^T \langle\nabla h_\epsilon(u_\epsilon), v\rangle dt\Big] + \langle\xi_\epsilon, F'(y_\epsilon)z_\epsilon\rangle \\ \geq \int_0^T \langle u_* - u_\epsilon, v\rangle dt. \tag{52}$$

where

$$\lambda_\epsilon = \frac{\epsilon^{1/2}}{d_S(F(y_\epsilon)) + \epsilon^{1/2}},\ \xi_\epsilon = \begin{cases} \nabla d_S(F(y^\epsilon)), & if\ F(y^\epsilon) \notin \ S, \\ 0 & otherwise, \end{cases}$$

and $\xi_\epsilon \in \partial d_S(F(y^\epsilon))$. Since S is convex and closed, we see

$$\parallel \xi_\epsilon \parallel_{Z^*} = \begin{cases} 1, & if\ F(y^\epsilon) \notin \ S, \\ 0 & otherwise. \end{cases}$$

and

$$1 \leq \varphi_\epsilon^2 + \parallel \xi_\epsilon \parallel_{Z^*}^2 \leq 2.$$

Thus, we see

$$\lambda_\epsilon \to \lambda_0, \quad \xi_\epsilon \to \xi_0 \ weakly\ in\ Z^*.$$

It follows from lemma 2.5 that $y_\epsilon \to y_*$ strongly in $C([0,T];H) \cap L^2(0,T;V)$.

Now, since $\{Ap_\epsilon\}$ is bounded in $L^2(0,T;V')$ and $\dot{\beta}^\epsilon(y_\epsilon)p_\epsilon$ is bounded in $L^1(0,T;L^1(\Omega))$, $[F'(y_\epsilon)]^*\xi_\epsilon$ is bounded in $L^2(0,T;V')$, we may infer that $\{p_\epsilon\}$ is bounded in $L^1(0,T;L^1(\Omega)+V')$ and so, $\{(\epsilon+M)p'_\epsilon\}$ is bounded in $L^1(0,T;Y^*)$. Since the injection of H into Y^* is compact and the set $\{p_\epsilon\}$ is bounded in H for any $t\in[0,T]$. By the same arguments as those in [19, 4, 5], there exists $p\in L^\infty([0,T];H)\cap L^2(0,T;V)\cap BV([0,T];Y^*)$ and $\mu\in(L^\infty(Q))^*$ such that, on some subsequence ϵ, still denoted itself

$$p_\epsilon(t)\to p \textit{ strongly in } Y^*,\ \forall\, t\in[0,T].$$

Here $BV([0,T];Y^*)$ is the space of all Y^*-valued functions $p:[0,T]\to Y^*$ with bounded variation on $[0,T]$.

On the other hand, by (43) we see

$$p_\epsilon\to p \textit{ weakly star in } L^\infty(0,T;H),\ \textit{weakly in } L^2(0,T;V). \tag{53}$$

Note that $V\hookrightarrow H$ is compact, for every $\lambda>0$ there is $\delta(\lambda)>0$ such that

$$\mid p_\epsilon(t)-p(t)\mid_2\le\parallel p_\epsilon(t)-p(t)\parallel_V+\delta(\lambda)\parallel p_\epsilon(t)-p(t)\parallel_{Y^*}\ \ \forall\, t\in[0,T].$$

This yields

$$p_\epsilon\to p \textit{ strongly in } L^2(0,T;H), \tag{54}$$

and

$$p_\epsilon(t)\to p(t) \textit{ weakly in } H\ \forall\, t\in[0,T]. \tag{55}$$

Moreover, by (44) we infer that there is $\mu\in(L^\infty(Q))^*$ such that, on some generalized subsequence ϵ,

$$\dot{\beta}^\epsilon(y_\epsilon)p_\epsilon\to\mu \textit{ weakly star in } (L^\infty(Q))^*, \tag{56}$$

$$\nabla g^\epsilon(t,y_\epsilon)\to\eta \textit{ weakly star in } L^\infty(0,T;H))^*,\ \eta(t)\in\partial g(t,y_*)\ a.e.\ t\in(0,T). \tag{57}$$

Since F is continuously differentiable from $L^2(0,T;V)$ to Z,

$$[F'(y_\epsilon)]^*\xi_\epsilon\to[F'(y_*)]^*\xi_0 \textit{ weakly } L^2(0,T;V').$$

From (43), we infer that $\mid(\epsilon+M)^{1/2}p_\epsilon(T)\mid^2=\langle(\epsilon+M)p_\epsilon(T),p_\epsilon(T)\rangle=\epsilon\mid p_\epsilon(T)\mid_2^2+\mid M^{1/2}p_\epsilon(T)\mid_2^2\le c$. Together with (55), we obtain

$$(\epsilon+M)^{1/2}p_\epsilon(T)\to M^{1/2}p(T)\ \ \textit{in } H.$$

Now letting $\epsilon \to 0$ in (42), it follows that

$$\frac{d}{dt}Mp - Ap - \mu - [F'(y_*)]^*\xi_0 \in L^\infty(0,T;H), \tag{58}$$

$$\begin{cases} \frac{d}{dt}Mp(t) - Ap(t) - \mu - [F'(y_*)]^*\xi_0 \in \lambda_0 \partial g(t,y_*) \ a.e. \ in \ (0,T), \\ M^{1/2}p(T) = 0, \end{cases} \tag{59}$$

It follows from (51), (52) and (42) that

$$-\int_0^T \langle B^* p_\epsilon, v\rangle dt + \lambda_\epsilon \int_0^T \langle \nabla h_\epsilon(u_\epsilon), v\rangle dt \geq \int_0^T \langle u_* - u_\epsilon, v\rangle dt,$$

for all $v \in L^2(0,T;V)$. By lemma 2.5, $u_\epsilon \to u_*$ strongly in $L^2(0,T;U)$, it follows

$$\int_0^T \langle \nabla h_\epsilon(u_\epsilon), v\rangle dt \to \int_0^T \langle \nabla\zeta(t), v\rangle dt, \quad \zeta(t) \in \partial h(u_*) \ a.e. \ in \ (0,T), \tag{60}$$

for all $v \in L^2(0,T;V)$. Thus,

$$-\int_0^T \langle B^* p, v\rangle dt + \lambda_0 \int_0^T \langle \zeta(t), v\rangle dt \geq 0 \ for \ all \ v \in L^2(0,T;V).$$

Since $\xi_\epsilon \in \partial d_S(F(y_\epsilon))$, we get $\langle \xi_\epsilon, w - F(y_\epsilon)\rangle \leq 0$ for all $w \in S$.

Now we claim that $(\lambda_0, \xi_0) \neq 0$. Indeed, if $\lambda_0 = 0$, we have that $\{\xi_\epsilon\}_{\epsilon>0}$ is bounded in Z^*. By (H3), S has finite codimentionality, so dose $S - F(y^*)$. Thus it follows that $\xi_\epsilon \to \xi_0$ weakly in Z^* and

$$\langle \xi_0, w - F(y^*)\rangle \leq 0 \ for \ all \ w \in S. \tag{61}$$

Finally, if $(\lambda_0, p) = 0$, it follows from (59) that $\mu + [F'(y^*)]^*\xi_0 = 0$. So in the case that $\mu \notin R([F'(y^*)]^*)$, we must have $(\lambda_0, p) \neq 0$. This together with (58), (59) and (61) completes the proof. □

Example 3.3. Consider the initial-boundary vary value controlled system

$$\begin{cases} -\triangle \frac{dy(x,t)}{dt} + (I - \triangle)y(x,t) + \beta(y(x,t)) \ni Bu(x,t) \ in \ \Omega \times (0,T), \\ \frac{\partial}{\partial\nu} y(x,t) = 0 \quad on \ \partial\Omega \times (0,T), \\ (-\triangle)^{1/2} y(x,0) = (-\triangle)^{1/2} y_0 \ in \ \Omega, \end{cases} \tag{62}$$

where $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary.

$$y_0 \in W = \{y \in H^2(\Omega) : \frac{\partial}{\partial\nu} y(x,t) = 0 \ a.e. \ on \ \partial\Omega\}, \ \beta(\cdot)$$

satisfies (H3). Thus, the results of Theorem 3.2 remain true. □

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