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A Controlled Optimal Stochastic Production Planning Model

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Abstract

This paper considers the state of the system which is represented by a controlled stochastic process. We shall formulate a stochastic optimal control in which the stochastic differential equations of a type known as Ito equations are considered which are perturbed by Markov diffusion process. Our goal is to use the stochastic optimal control principle to completely solve production planning model for the demand rate.

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1 Introduction

In production planning, one of the most unstable variables is the inventory level. This is influenced by certain unavoidable environmental uncertainties such as: sudden random demand fluctuation, sales return, inventory spoilage, etc. They make ideal production policy for "wide" class of cost functional impossible [4].
To take care of these various sources of environmental randomness, we represent uncertainty by a filtered probability by n-dimensional Brownian motion w, defined on $(\Omega, \mathcal{F}, P)$ and satisfying the usual condition [3]. We move from deterministic problem to a stochastic one by considering the "noisy" environment in order to model their behavior fairly accurately by adding an additive noise term in the state dynamics [5].
The general form of production planning is then formulated by representing the inventory level by a stochastic process $\{X_t, t \geq 0\}$, defined on the probability space and generated by $\mathcal{F}_t$ with an overall noise rate that is distributed like white noise, σdW_t and whose dynamics is governed by the Ito stochastic differential equation

$$dX_t = (U_t - S)dt + \sigma dW_t \quad (1)$$

where σ is the intensity of the noise, S denotes the constant demand rate and U_t, the production function, is a non stochastic parameter controlled by the investor. The objective is to find an optimal policy which minimizes the associated expected cost functional

$$J(u) = E\left\{\int_0^T [c(U_t - \bar{u})^2 + h(X_t - \bar{x})^2]dt + BX_T\right\} \quad (2)$$

where $c(.)$ the cost is function and X_t is the solution of the stochastic differential equation (1). Let $V(x,t)$ denote the minimum expected value of the objective function from time t to the horizon T with $X_t = x$ and using the optimal policy from t to T. The function is given by

$$V(x,t) = \min_{U_t} E\left\{\int_t^T [c(U_t - \bar{u})^2 + h(X_t - \bar{x})^2]dt + BX_T\right\}. \quad (3)$$

In this paper, we assume the state variables to be observable and we use the Hamilton Jacobi Bellman (HJB) framework rather than stochastic maximize principles.

2 The Stochastic Production Planning Model

We consider a factory producing homogeneous goods and having an inventory warehouse. We define the following system variables and parameters that describe the state of the model as follows:

X_t= the inventory level at time t (state variable)

U_t = the production rate at time T (control variable)

$S(t)$ = the constant demand rate at time $t;\ s > 0$

T = the length of planning period

$\overline{x}$ = the factory – optimal inventory level

$\overline{u}$ = the factory- optimal production level

x_0 = the initial inventory level

h = the inventory holding cost coefficient

c = the production cost coefficient

B = the salvage value per unit of inventory at time T.

Z_t = the standard wiener process

σ = the constant diffusion coefficient

The dynamics of the stock flow is governed by the equation

$$\dot{x}(t) = u(t) - S(t),\ x(0) = x_0 \tag{4}$$

and the dynamics of the inventory level x_t is governed by the Ito stochastic differential equation

$$dX_t = (U_t - S)dt + \sigma dZ_t\ ,\ X(0) = x_0 \tag{5}$$

Subject to

$$V(x,t) = \min_{U_t} E\left[\int_t^T [c(U_t - \bar{u})^2 + h(X_t - \bar{x})^2]dt + BX_T\right] \tag{6}$$

Assume that $\bar{x} = \bar{u} = 0$ and $h = c = 1$, then, we restate (6) as

$$V(x,t) = \max E\left[\int_0^T -(U_t^2 + X_t^2)dt + BX_T\right] \quad (7)$$

The value function $V(x,t)$ satisfying the Hamilton-Jacobi-Bellman (HJB) equation

$$0 = \max\left[-(u^2 + x^2) + V_t + V_x(u - S) + \frac{1}{2}\sigma^2 V_{xx}\right] \quad (8)$$

with the boundary condition

$$V(x,t) = Bx \quad (9)$$

It is now possible to maximize the expression by taking its derivative with respect to u and setting it to zero which yields

$$V_x - 2u = 0 \quad (10)$$

The optimal production rate that minimizes the cost can be expressed as a function of the current value function in the form

$$u(x,t) = \frac{V_x(x,t)}{2} \quad (11)$$

Substituting equation (11) into (8) yields the equation

$$0 = \frac{V_x^2}{4} - x^2 + V_t - SV_x + \frac{1}{2}\sigma^2 V_{xx} \quad (12)$$

which is known as the Hamilton Jacobi-Bellman equation. This is a nonlinear partial differential equation which must be satisfied by the current value function $V(x,t)$ with boundary condition $V(x,t) = Bx$. Hence the optimal production rate that minimizes the total cost can be expressed as a function of the current value function in the form

$$u(x,t) = \frac{V_x(x,t)}{2}.$$

It is important to remark that if production rate were restricted to be non-negative, then equation (11) would be changed to

$$u(x,t) = \max\left[0, \frac{V_x(x,t)}{2}\right]. \quad (13)$$

3 Solution for the Production Planning Problem

In this section, we shall obtain the solution of the stochastic production planning problem. To solve the partial differential equation (12), we assume a solution of the form

$$V(x,t) = Q(t)x^2 + R(t)x + M(t) \tag{14}$$

Then

$$V_x = 2Qx + R, \tag{15}$$

$$V_{xx} = 2Q, \tag{16}$$

$$V_t = \dot{Q}x^2 + \dot{R}x + \dot{M} \tag{17}$$

where the dot denotes the differential with respect to time. Substituting (17) into (12), we get

$$\frac{(2Qx+R)^2}{4} - x^2 + \dot{Q}x^2 + \dot{R}x + \dot{M} - S(2Qx+R) + \frac{1}{2}\sigma^2(2Q) = 0$$

$$Q^2x^2 + QRx + \frac{R^2}{4} - x^2 + \dot{Q}x^2 + \dot{R}x + \dot{M} - 2SQx - RS + \sigma^2 Q = 0$$

$$\dot{Q}x^2 + Q^2x^2 - x^2 + \dot{R}x + QRx - 2SQx + \dot{M} + \frac{R^2}{4} - RS + \sigma^2 Q = 0$$

Collecting like terms, we have

$$x^2\left[\dot{Q} + Q^2 - 1\right] + x\left[\dot{R} + QR - 2SQ\right] + \dot{M} + \frac{R^2}{4} - RS + \sigma^2 Q = 0 \tag{18}$$

Since (18) must hold for any value of x, we have the following system of nonlinear ordinary differential equations

$$\dot{Q} = 1 - Q^2 \tag{19}$$

$$\dot{R} = 2SQ - RQ \tag{20}$$

$$\dot{M} = RS - \frac{R^2}{4} - \sigma^2 Q \tag{21}$$

Next, we solve this nonlinear system for the demand rate with the following boundary conditions:

$$Q(T) = 0, \; R(T) = 0, \; M(T) = 0 \tag{22}$$

To solve (19), we expand $\frac{\dot{Q}}{1-Q^2}$ by partial fraction to obtain

$$\frac{\dot{Q}}{2}\left[\frac{1}{1-Q}+\frac{1}{1+Q}\right]=1$$

which can be integrated to obtain

$$Q = \frac{y-1}{y+1} \tag{23}$$

where

$$y = e^{2(t-T)} \tag{24}$$

and the time horizon will be $y \in [e^{-2T}, 1]$.

Since the demand rate S is assumed to be a constant, the optimal production rate is given by

$$u(x,t) = S + \left[\frac{(y-1)x+(B-2S)\sqrt{y}}{y+1}\right] \tag{25}$$

Next, we can reduce (20) to

$$\dot{R}^0 + R^0 Q = 0, \quad R^0(T) = B - 2S$$

By change of variable defined by $R^0(T) = B - 2S$, then the solution is given by

$$\log R^0(T) - \log R^0(t) = -\int_t^T Q(\tau)d\tau$$

which can be simplified further to obtain

$$R = 2S + 2\frac{(B-2S)\sqrt{y}}{y+1} \tag{26}$$

Having obtained solution for $R \; and \; Q$, we can now express (21) as

$$M = -\int_t^T \left[R(\tau)S - \frac{(R(\tau))^2}{4} - \sigma^2 Q(\tau)\right] d\tau$$

Which we can further simplify to

$$M = \int \frac{1}{2y}\left[SR - \frac{R^2}{4} - \sigma^2 Q\right] d\tau \tag{27}$$

The optimal production rate in equation (25) equals the demand rate plus a correction term which depends on the level of inventory and the distance from the horizon time T. Since $(y-1) < 0 \;\; \forall t < T$, then for a lower value of x, it is clear that the optimal production rate is likely to be positive. However, if x is very high, the correction term will become smaller than $-S$, and the optimal control will be negative. Hence, if inventory is too high, the factory can save money by disposing a part of the inventory resulting in lower holding $cost$.

4 A Stochastic Advertising Problem

We consider a modification of the Vidale-Wolfe advertising model whose market share X_t is governed by the dynamics

$$dX_t = \left(rU_t(x)\sqrt{1-X_t} - \partial X_t\right)dt + \sigma(X_t)dz_t,\ \ X_0 = x_0 \qquad (28)$$

Subject to

$$\max E\left[\int_0^\infty e^{-\rho t}(\pi X_t - U_t^2)\right]dt \qquad (29)$$

where U_t is the rate of advertising at time t, dz_t represents a standard white noise and $\sigma(X_t)$ are functions to be suitably specified next.

An important consideration in choosing the function $\sigma(x)$ should be that the solution X_t to the Ito equation (28) remains inside the interval $[0,1]$. Merely requiring that the initial condition $x_0 \epsilon [0,1]$ is no longer sufficient in the stochastic case.

The interval under consideration is therefore $(0,1)$ with boundaries $x = 0$ and $x = 1$. We require that $u(x)$ and $\sigma(x)$ be continuous functions that satisfy Lipschitz condition on every closed subinterval of $(0,1)$. We require that

$$\sigma(x) > 0, x\epsilon(0,1) \text{ and } \sigma(0) = \sigma(1) = 0$$

(30)

and

$$u(x) \geq 0,\ x\epsilon(0,1] \text{ and } u(0) > 0. \qquad (31)$$

Using these conditions, we have

$$ru(0)\sqrt{1-0} - \partial 0 = ru(0) > 0, \text{ and } ru(1)\sqrt{1-1} - \partial < 0,$$

So the Ito equation (28) will have a solution X_t such that $0 < X_t < 1$ a.s. Since our solution for the optimal advertising $U^*(x)$ would turn out to satisfy (31), we will have the optimal market share X_t^* lie in the interval $(0,1)$.

Let $V(x)$ denote the expected value of the discounted profit from time t to infinity. Since $T = \infty$, the future looks the same from any time t and therefore the value function does not depend on it. We can write the HJB equation as

$$\rho V(x) = \max_u \left[\pi x - u^2 + V_x\left(ru\sqrt{1-x} - \delta x\right) + V_{xx}\frac{1}{2}\sigma^2\right] \qquad (32)$$

Taking the derivative of the right hand side of (4.5) with respect to u and setting it to zero, we get

$$U(x) = \frac{1}{2} r V_x \sqrt{1-x} \tag{33}$$

Substituting equation (33) into (32), we have

$$\begin{aligned}\rho V(x) &= \pi x - \left(\tfrac{1}{2} r V_x \sqrt{1-x}\right)^2 + V_x\left(\tfrac{1}{2} r^2 V_x (1-x) - \delta x\right) + \tfrac{1}{2} V_{xx}\sigma^2(x) \\ &= \pi x - \tfrac{1}{4} r^2 V_x^2 (1-x) + \tfrac{1}{2} r^2 V_x^2 (1-x) - V_x \delta x + \tfrac{1}{2} V_{xx}\sigma^2(x) \\ &= \pi x - \frac{r^2 V_x^2 (1-x) + 2 r^2 V_x^2 (1-x)}{4} - V_x \delta x + \tfrac{1}{2} V_{xx}\sigma^2(x)\end{aligned}$$

Collecting like terms yield the HJB equation

$$\rho V(x) = \pi x + \frac{1}{4} r^2 V_x^2 (1-x) - V_x \delta x + \frac{1}{2} V_{xx}\sigma^2(x) \tag{34}$$

A solution of equation (34) is obtained as

$$V(x) = \bar{\lambda} x + \frac{\bar{\lambda}^2 r^2}{4\rho} \tag{35}$$

where

$$\bar{\lambda} = \frac{-(\rho+\delta) + \sqrt{(\rho+\delta)^2 + \sigma^2 \pi}}{\frac{r^2}{2}} \; . \tag{36}$$

We obtain the explicit formula for the optimal feedback control as

$$U^*(x) = \frac{1}{2} r \bar{\lambda} \sqrt{1-x} \, . \tag{37}$$

Also the advertising rate is given by

$$U^*(x) = \frac{1}{2} r \bar{\lambda} \sqrt{1-x}.$$

The minimum value of the advertising rate U^* is zero at $x = 1$ and the maximum value of U^* is $\frac{1}{2} r \bar{\lambda} > 0$ at $x = 0$.

We can easily characterize U^* as

$$U_t^* = U^*(X_t) = \begin{cases} > \bar{u} \; if \; X_t < \bar{x} \\ = \bar{u} \; if \; X_t = \bar{x} \\ < \bar{u} \; if \; X_t > \bar{x} \end{cases} \tag{38}$$

where

$$\bar{x} = \frac{r^2\frac{\bar{\lambda}}{2}}{r^2\frac{\bar{\lambda}}{2}+\delta} \tag{39}$$

and

$$\bar{u} = \frac{1}{2}r\bar{\lambda}\sqrt{1-x} \tag{40}$$

Eventually the market share process hovers around the equilibrium level x.

5 An Optimal Consumption investment Problem

Consider investing a part of Rich's wealth in a risky security or stock that earns an expected rate of return that equal $\alpha > \delta$. The problem of Rich, known now as Rich investor is to optimally allocate his wealth between the risky free savings account and the risky stock over time and also consume overtime so as to maximize his total utility of consumption. To formulate the stochastic optimal control model of Rich's investor, this investment shall be modeled.

Assume that B_0 is the initial price of a unit of investment in the savings account earning an interest at the positive rate r, then we can write the rate of change of the accumulated amount B_t at time t as

$$\frac{dB_t}{dt} = rB_t,\ B_0 = B(0) \tag{41}$$

Equation (5.1) may be expressed in differential form as

$$dB_t = rB_t dt\ ,\ \ B_0 = B(0) \tag{42}$$

Using the separation of variable technique for ordinary differential equation, we solve the equation (42) to obtain the accumulated amount as a function of time as

$$B_t = B_0 e^{rt} \tag{43}$$

The stock price, S_t, is a stochastic differential equation as follows

$$\frac{dS_t}{S_t} = \alpha dt + \sigma dz_t,\ \ S_0 = S(0) \tag{44}$$

Equation (44) may be expressed in differential form as

$$dS_t = \alpha S_t dt + \sigma S_t dz_t,\ S_0 = S(0) \tag{45}$$

where α is the average rate of return on stock, σ is the standard derivation associated with the return and z_t is a standard Wiener process.

In order to complete the formulation of Rich's stochastic optimal control problem, we need the following additional notations

$W_t =$ the wealth at time t,

$C_t =$ the consumption rate at time t,

$Q_t =$ the fraction of the wealth kept in the saving account at time t,

$U(c) =$ the utility of consumption when consumption is at the rate c; the function $U(c)$ is assumed to be increasing and concave,

$\rho =$ the rate of discount applied to consumption utility,

$B =$ the bankruptcy parameter to explained later.

We now develop the dynamics of the wealth process. Since the investment decision Q_t is unconstrained, this means that Rich can deposit in, as well as borrow money from, the saving account at the rate r. Hence it is possible to obtain rigorously the equation for the wealth process involving an intermediate variable, namely, the number N_t of shares of stock owned at time t, we shall not do so. Therefore, we shall write the wealth equation in the form as

$$dW_t = Q_t W_t \alpha dt + Q_t W_t \sigma dz_t + (1 - Q_t) r W_t dt - C_t dt \quad (46)$$

$$= (\alpha - r) Q_t W_t dt + (rW_t - C_t) dt + Q_t W_t \sigma dz_t,\ W_0 = W(0) \quad (47)$$

where the term $\alpha Q_t W_t$ represents the expected return from the risky investment $Q_t W_t$ during the period from t to $t + dt$, the term $\sigma Q_t W_t dt$ represents the risk involved in investing $Q_t W_t$ in stock, the term $r(1 - Q_t) W_t dt$ represents the amount of interest earned on the balance of $(1 - Q_t) W_t$ in saving account and finally $C_t dt$ represents the amount of consumption during the interval from t to $t + dt$.

We shall say that Rich goes bankrupt at time T, when his wealth falls to zero at that time T is a random variable called a stopping time, since it is observed exactly at the instant of time when wealth fall to zero. Rich's objective function is

$$\max_{C_t>0}\left\{E\left[\int_0^T e^{-\rho t}U(C_t)dt + Be^{-\rho T}\right]\right\} \tag{48}$$

Subject to

$$dW_t = (\alpha - r)Q_tW_tdt + (rW_t - C_t)dt + \sigma Q_tW_tdz_t,\ W_0 = W(0),\ C_t > 0 \tag{49}$$

Next we assumed that the value function $V(x)$ associated with the optimal policy starting with the wealth $W_t = x$ at time t and using the optimality principle, the Hamilton-Jacobi-Bellman (HJB) equation satisfying the value function $V(x)$ which has the form

$$\rho V(x) = \max_{Q.C>0}\left\{(\alpha - r)qxV_x + (rx - C)V_x + \frac{1}{2}q^2\sigma^2x^2V_{xx} + U(c)\right\},$$
$$V(0) = B \tag{50}$$

We further simplify the problem by assuming the following condition

$$U(c) = \sqrt{C} \tag{51}$$

$$\frac{dU}{dC}\ \Big|_{C=0} = \frac{1}{2\sqrt{C}}\ \Big|_{C=0} = \infty \tag{52}$$

We also assume $B = \infty$, together with condition (49) implied a strictly positive consumption level at all time and no bankruptcy.

Now differentiating equation (50) with respect to $Q\ and\ C$ and equating the resulting expression to zero, we get the following system of equations

$$(\alpha - r)xV_x + + \frac{1}{2}Q\sigma^2x^2V_{xx} = 0 \tag{53}$$

$$1 - CV_x = 0 \tag{54}$$

Solving the above system of equation with respect to the formulation $Q(x)$ and $C(x)$, we get

$$Q(x) = \frac{(\alpha - r)V_x}{x\sigma^2V_{xx}} \tag{55}$$

and

$$C(x) = \frac{1}{V_x} \tag{56}$$

Substituting (53) and (54) into (50) allows us to remove the max operator from (50) and provides us with the equation

$$\rho V(x) = -\frac{\gamma(V_x)^2}{V_{xx}} + \left(rx - \frac{1}{V_x}\right)V_x - \ln V_x \tag{57}$$

where

$$\gamma = \frac{(\alpha - r)^2}{2\sigma^2} \tag{58}$$

To solve the nonlinear differential equation (57), we assume that the solution has the form

$$V(x) = aln(kx) + H \tag{59}$$

where $a\ and\ H$ are determinable constant. We have

$$V_x = \frac{1}{ax},\ V_{xx} = -\frac{1}{ax^2} \tag{60}$$

Substituting equation (59) and (60) into (57), we obtain the constant $a, k\ and\ H$ as

$$a = \frac{1}{\rho},\ \ k = \rho,\ H = \frac{r - \rho + \gamma}{\rho^2} \tag{61}$$

Hence, substituting (61) into (59), we get

$$V(x) = \frac{1}{\rho} ln\rho x + \frac{r - \rho + \gamma}{\rho^2} \tag{62}$$

Using (62) into (57), the fraction of the wealth invested in the stock is given by

$$Q = \frac{(\alpha - r)}{\sigma^2}$$

and

$$C = \rho x.$$

6 Conclusion

We have analyzed the optimal control of a single dimension stochastic production planning model and optimal production obtained as a function of the stochastic inventory level and of time demand rate. Also, the existence of a complete solution to the associated HJB equation is established and the optimal policy is characterized. The optimal advertising rate is obtained as function of the market share and the optimal consumption rate and the fraction of the wealth invested in stock at any time is obtain using Rich’s stochastic optimal control problem.

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