Theoretical Mathematics & Applications, vol.3, no.3, 2013, 87-106
ISSN: 1792- 9687 (print), 1792-9709 (online)
Scienpress Ltd, 2013

A Mathematical Study of Electro-Magneto-Thermo-Voigt Viscoelastic Surface Wave Propagation under Gravity Involving Time Rate of Change of Strain

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Abstract

This research is a mathematical investigation of the propagation of surface wave in a Voigt viscoelastic medium. A mathematical model for wave propagation in electro-magneto-thermo heterogeneous viscoelastic isotropic half space under gravity involving time rate of change of strain of n^{th} order is purposed. A solution to the partial differential equation of motion is assumed and is shown to satisfy the two necessary boundary conditions. The frequency equations for surface waves (Love, Rayleigh and Stoneley waves) are obtained with the help of Biot's theory of incremental deformations. Heterogeneities in the medium are assumed to vary exponentially with depth. The problem investigated by Bullen [Cambridge University Press, pp. 85-99, 1965] has been reduced as particular case.

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Article Info: *Received:* June 10, 2013. *Revised:* August 15, 2013.
Published online : September 1, 2013

Mathematics Subject Classification: 74H45; 74F10; 74J15; 74J20; 74K25
Keywords: viscoelastic solid medium; gravity; inhomogeneities; variable density; temperature; magnetic field; electric field

1 Introduction

In the earlier times, sufficient interest has been led on wave propagation in that material whose mechanical characteristics and density are functions of space i.e. heterogeneous engineering material. It is perhaps due to lack of information (experimental or empirical) available in the literature concerning the precise mechanical behaviour of heterogeneous media, that not much work has been reported in this area of applied mechanics. Wave propagation in inhomogeneous medium is a challenge for both theoretical research and engineering practice. With the rapid development of science and technology, wave motion study of the heterogeneous medium (atmosphere, ocean, earth-crust, functionally graded materials and cycle grid structure, etc.) seems much more important.

Although most wave equations assume propagation in an elastic medium, it is well known that many solids do not exactly obey the laws of the theory of elasticity. The purpose of this research, therefore, is to assume a non-elastic medium that represented by a Voigt viscoelastic element, and investigate the conditions necessary for the propagation of a surface wave. To the earthquake seismologist and to those concerned with predicting the effects of explosives in solids, the surface wave is one of the most important types of waves that have been observed. With accurate earthquake seismograms of surface waves, the thickness of the superficial layer of the earth (the crust) may be determined. On a smaller scale, in seismic exploration, knowledge of the thickness of the weathered surface layer is of primary importance

The investigation of viscoelastic wave propagation was initiated by Sezawa [1], who was concerned primarily with purely dilatational plane waves, and obtained his solution using Fourier integrals. Many years ago Bromwich [2] determined the influence of gravity on wave propagation in an elastic solid medium. Love [3] investigated the effect of gravity on Rayleigh wave velocity. Earlier, Thomson [4] discussed transmission of elastic waves through a stratified solid medium. Haskell [5], Ewing, Jardtezky and Press [6] studied wave propagation. De and Sen [7] presented note on elastic waves. Sen and Acharya [8] investigated the effect of gravity on waves in a thermoelastic layer. Das et al. [9] studied magneto-visco-elastic surface waves in stressed conducting media. Recently, Kakar et al. [10-13] presented many papers on surface waves in viscoelastic media.

In this paper, we have considered that the surface waves are propagating in isotropic, viscoelastic heterogeneous medium under the effect of temperature, electric field, magnetic field and gravity. The problem of n^{th} order viscoelastic electro-magneto-thermo surface waves (Love, Stoneley and Rayleigh waves) under gravity involving time rate of strain in heterogeneous medium is studied in detail.

2 Governing equations

The governing equations of motion for 3D viscoelastic solid medium in Cartesian co-ordinates with Eq. (1) are [3]

$$\frac{\partial \tau_{11}}{\partial x} + \frac{\partial \tau_{12}}{\partial y} + \frac{\partial \tau_{13}}{\partial z} + \rho g \frac{\partial w}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2}, \tag{1a}$$

$$\frac{\partial \tau_{12}}{\partial x} + \frac{\partial \tau_{22}}{\partial y} + \frac{\partial \tau_{23}}{\partial z} + \rho g \frac{\partial w}{\partial y} = \rho \frac{\partial^2 v}{\partial t^2}, \tag{1b}$$

$$\frac{\partial \tau_{13}}{\partial x}+\frac{\partial \tau_{23}}{\partial y}+\frac{\partial \tau_{33}}{\partial z}-\rho g\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=\rho\,\frac{\partial^2 w}{\partial t^2}. \tag{1c}$$

where $\tau_{ij}=\tau_{ji}$, $\forall$ i, j are the stress components and ρ is the density of the medium. We have assumed both the mediums are perfect electric conductor, therefore the governing equations of motion for such mediums are

$$\vec{\nabla}\cdot\vec{\mathrm{E}}=0,\ \vec{\nabla}\cdot\vec{\mathrm{B}}=0,\ \vec{\nabla}\times\vec{\mathrm{E}}=-\vec{B}_{,t},\ \vec{\nabla}\times\vec{\mathrm{B}}=\mu_e\varepsilon_e\vec{E}_{,t}. \tag{2}$$

where, $\vec{\mathrm{E}}$, $\vec{\mathrm{B}}$, μ_e and ε_e are electric field, magnetic field induction, permeability and permittivity of the medium.

The value of magnetic field intensity is

$$\vec{\mathrm{H}}(0,0,\mathrm{H})=\vec{\mathrm{H}}_0+\vec{\mathrm{H}}_i \tag{3}$$

where, magnetic field $\vec{\mathrm{H}}$ is acting along y-axis. $\vec{\mathrm{H}}_i$ is the perturbation in the magnetic field intensity.

The stress-strain relations for viscoelastic medium, according to Voigt are [15]

$$\tau_{ij}=2D_\mu\, e_{ij}+(D_\lambda\,\Delta)\,\delta_{ij} \tag{4a}$$

Therefore, the stress-strain relations for general isotropic, thermo, magneto, electro, viscoelastic medium can be written as

$$\tau_{ij}=2D_\mu\, e_{ij}+(D_\lambda\,\Delta-D_\beta\,T++E_0^2\,\Delta D_{\mathrm{E}_e}+H_0^2\,\Delta D_{\mathrm{m}_e})\,\delta_{ij} \tag{4b}$$

where , $\Delta=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}$, D_λ, D_μ, D_β, D_{m_e}, D_{E_e} are elastic constants.

Let initial temperature of both the medium is kept at constant absolute temperature T_0. Fourier's law of heat conduction is used to calculate T and it is given by

$$\mathrm{p}\nabla^2\mathrm{T}=C_v\,\frac{\partial T}{\partial t}+T_0\,G_L\,\frac{\partial}{\partial t}\left(\nabla^2\phi\right), \tag{5}$$

where, K be the thermal conductivity and obeys the law as given by $K=K_0\,e^{mz}$,

$\mathrm{p}=\frac{K_0}{\rho_0}$ and C_V be the specific heat of the body at constant volume.

3 Formulation of the problem

Consider a heterogeneous, thermo-electro-magneto-Voigt viscoelastic, isotropic, semi-finite media M_1 and M_2 (as shown in Figure 1).

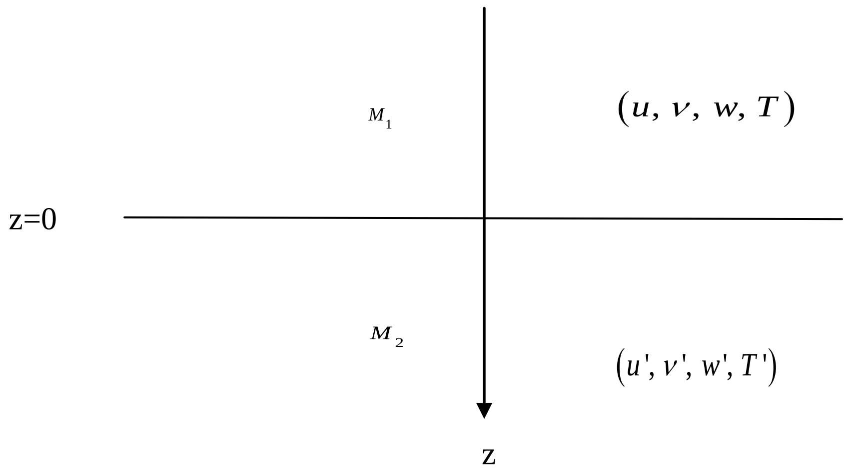

Figure 1: Geometry of the problem

The mechanical properties of M_1 are different from those of M_2. Let the components of displacements are u, v, w. The Cartesian co-ordinate system (x ,y , z) is located at the interface separating the two layers at z = 0. The z-axis is acting downward.

Introducing Eq. (4a) in Eq. (1a), Eq. (1b), Eq. (1c), we get

$$D_{\lambda}\frac{\partial\Delta}{\partial x}+\Delta\frac{\partial D_{\lambda}}{\partial x}+2D_{\mu}\frac{\partial^{2}u}{\partial x^{2}}+2\frac{\partial u}{\partial x}\frac{\partial D_{\mu}}{\partial x}-D_{\beta}\frac{\partial T}{\partial x}+D_{\mu}\frac{\partial}{\partial z}\left[\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right]+$$
$$\left[\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right]\frac{\partial D_{\mu}}{\partial z}H_{0}^{2}D_{m_e}\frac{\partial D}{\partial x}+\left[\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right]\frac{\partial D_{\mu}}{\partial z}E_{0}^{2}D_{E_e}\frac{\partial D}{\partial x}$$
$$+H_{0}^{2}D\frac{\partial D_{m_e}}{\partial x}+E_{0}^{2}D\frac{\partial D_{E_e}}{\partial x}+\rho g\frac{\partial w^{2}}{\partial x_{0}}=\rho\frac{\partial^{2}u}{\partial t^{2}}, \tag{6a}$$

$$D_{\mu}\nabla^{2}v+\frac{\partial v}{\partial x}\frac{\partial D_{\mu}}{\partial x}+\frac{\partial v}{\partial z}\frac{\partial D_{\mu}}{\partial z}=\rho\frac{\partial^{2}v}{\partial t^{2}}, \tag{6b}$$

$$D_\mu \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\frac{\partial D_\mu}{\partial x}+2D_\mu \frac{\partial^2 w}{\partial z^2}+2\frac{\partial w}{\partial z}\frac{\partial D_\mu}{\partial z}$$

$$+D_\lambda \frac{\partial \Delta}{\partial z}+\Delta\frac{\partial D_\lambda}{\partial z}-D_\beta \frac{\partial T}{\partial z}-T\frac{\partial D_\beta}{\partial z}+H_0^2 D_{\mathrm{m}_e}\frac{\partial D}{\partial z}+H_0^2 D\frac{\partial D_{\mathrm{m}_e}}{\partial z}$$

$$+E_0^2 D_{\mathrm{E}_e}\frac{\partial D}{\partial z}+E_0^2 D\frac{\partial D_{\mathrm{E}_e}}{\partial z}-\rho g\frac{\partial u}{\partial x}=\rho\frac{\partial^2 w}{\partial t^2}. \tag{6c}$$

We assume that the heterogeneities for the media M_1 and M_2 are given by

$$D\lambda=\sum_{K=0}^{n}\lambda_K e^{mz}\frac{\partial^K}{\partial t^K},\ D\mu=\sum_{K=0}^{n}\mu_K e^{mz}\frac{\partial^K}{\partial t^K},\ D\beta=\sum_{K=0}^{n}\beta_K e^{mz}\frac{\partial^K}{\partial t^K},\ \rho=\rho_0 e^{mz},$$

$$D_{\mathrm{m}_e}=\sum_{K=0}^{n}(\mu_e)_K e^{mz}\frac{\partial^K}{\partial t^K},D_\eta=\sum_{K=0}^{N}\eta_K e^{mz}\frac{\partial^K}{\partial t^K}\ D_{\mathrm{E}_e}=\sum_{K=0}^{n}(\varepsilon_e)_K e^{mz}\frac{\partial^K}{\partial t^K} \tag{7}$$

and

$$D'\lambda=\sum_{K=0}^{n}\lambda'_K e^{lz}\frac{\partial^K}{\partial t^K},\ D\mu=\sum_{K=0}^{n}\mu'_K e^{lz}\frac{\partial^K}{\partial t^K},\ \rho'=\rho'_0 e^{lz},\ D\beta=\sum_{K=0}^{n}\beta_K e^{lz}\frac{\partial^K}{\partial t^K},$$

$$D'_{\mu_e}=\sum_{K=0}^{n}(\mu'_e)_K e^{mz}\frac{\partial^K}{\partial t^K},D'_\eta=\sum_{K=0}^{N}\eta'_K e^{lz}\frac{\partial^K}{\partial t^K},D'_{\varepsilon_e}=\sum_{K=0}^{n}(\varepsilon'_e)_K e^{mz}\frac{\partial^K}{\partial t^K} \tag{8}$$

where λ_0, M_0, λ'_0, $\mu'_0\varepsilon'_0$ are elastic constants, whereas β_0, β'_0 are thermal parameters are ρ_0, ρ'_0, m, n are constants. λ_K, μ_K, ε_K (K = 0,1,2, ... n) are the parameters associated with K^{th} order viscoelasticity and β_K, $(\varepsilon_e)_K$ and $(\mu_e)_K$ (K = 1, 2, ..., n) are the thermal, electric and magnetic parameters associated with K^{th} order. T is the absolute temperature over the initial temperature T_0. In a thermo viscoelastic solid, the thermal parameters β_K (K = 0, 1, ... n) are given by $\beta_K = (3\lambda_K + 2\mu_K)\,\alpha_t$, where, α_t be the coefficient of linear expansion of solid.

$$\begin{aligned}&\left(G_\lambda+G_\mu+H_0^2G_{\mathrm{m}_e}+E_0^2G_{\mathrm{E}_e}\right)\frac{\partial\Delta}{\partial x}+G_\mu\nabla^2 u\\&+mG_\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)-G_\beta\frac{\partial T}{\partial x}+\rho_0 g\frac{\partial w}{\partial x}=\rho_0\frac{\partial^2 u}{\partial t^2},\end{aligned} \tag{9a}$$

$$G_\mu \nabla^2 v + mG_\mu \frac{\partial v}{\partial z} = \rho_0 \frac{\partial^2 v}{\partial t^2}, \tag{9b}$$

$$(G_\lambda + G_\mu + H_0^2 G_{m_e} + E_0^2 G_{E_e}) \frac{\partial \Delta}{\partial z} + G_\mu \nabla^2 w + \Delta G_\lambda m + 2G_\mu m \frac{\partial w}{\partial z}$$

$$- G_\beta \frac{\partial T}{\partial z} - mG_B T + mH_0^2 DG_{m_e} + mE_0^2 DG_{E_e} - \rho_0 g \frac{\partial u}{\partial x} = \rho_0 \frac{\partial^2 w}{\partial t^2}. \tag{9c}$$

where,

$$G_\lambda = \sum_{K=0}^{n} \lambda_K \frac{\partial^K}{\partial t^K}, \qquad G_\mu = \sum_{K=0}^{n} \mu_K \frac{\partial^K}{\partial t^K},$$

$$G_\beta = \sum_{K=0}^{n} \beta_K \frac{\partial^K}{\partial t^K}, \qquad G_{\varepsilon_e} = \sum_{K=0}^{n} (\varepsilon_e)_K \frac{\partial^K}{\partial t^K},\ G_{\mu_e} = \sum_{K=0}^{n} (\mu_e)_K \frac{\partial^K}{\partial t^K}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}, \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}. \tag{10}$$

To investigate the surface wave propagation along the direction of Ox, we introduce displacement potential ϕ (x, z, t) and ψ (x, z, t) which are related to the displacement components as follows:

$$u = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial z}, \ w = \frac{\partial \phi}{\partial z} + \frac{\partial \psi}{\partial x}. \tag{11}$$

The displacement potential ϕ (x, z, t) and ψ (x, z, t) in Eq. (11) satisfy the following Laplace equation (known as dilation and rotation and are associated with P and SV waves)

$$\nabla^2 \phi = u_{,x} + w_{,z} = \Delta, \qquad \nabla^2 \psi = w_{,x} - u_{,z} = 2\Omega. \tag{11a}$$

Substituting Eq. (11) in Eqs (9a), (9b) and (9c), we get

$$G_R \nabla^2 \phi + mG_S \left(2\phi_{,z} + \psi_{,x}\right) - G_L T + g\psi_{,x} = \phi_{,tt}, \tag{12a}$$

$$G_S \nabla^2 v + mG v_{,z} = v_{,tt}, \tag{12b}$$

$$G_S \nabla^2 \psi + mG_P \phi_{,x} + 2m G_S \psi_{,z} - G_q \nabla^4 \psi - g\phi_{,x} = \psi_{,tt}, \tag{12c}$$

Where,

$$U_{KR}^2 = \frac{\lambda_K + 2\mu_K + H_0^2(\mathrm{m}_e)_K + E_0^2(E_e)_K}{\rho_0}, \qquad U_{KS}^2 = \frac{\mu_K}{\rho_0},\ U_{KL}^2 = \frac{\beta_K}{\rho_0},$$

$$U_{KP}^2 = \frac{\lambda_K + H_0^2(\mathrm{m}_e)_K + E_0^2(\mathrm{E}_e)_K}{\rho_0},$$

and

$$\mathrm{G_R} = \sum_{K=0}^{n} U_{KR}^2 \frac{\partial^K}{\partial t^K},\ \mathrm{G_S} = \sum_{K=0}^{n} U_{KS}^2 \frac{\partial^K}{\partial t^K},\ \mathrm{G_P} = \sum_{K=0}^{n} U_{KP}^2 \frac{\partial^K}{\partial t^K},$$

$$\mathrm{G_L} = \sum_{K=0}^{n} U_{KL}^2 \frac{\partial^K}{\partial t^K},\ G_q = \sum_{K=0}^{n} U_{Kq}^2 \frac{\partial^K}{\partial t^K}. \tag{13}$$

By using Eq. (5), temperature T can be calculated.

Further, similar relations in medium M_2 can be found out by replacing λ_K, μ_K, β_K, ε'_K, ρ_0 by λ'_K, μ'_K, β'_K, ε'_K, ρ'_0 and so on.

4 Solution of the problem

Now our main objective to solve Eq. (12a), Eq. (12b), Eq. (12c) and Eq. (5), for this, we seek the solutions in the following forms.

$$(\phi, \psi, T, v) = [f(z), V(z), T_1(z), h(z)]\, e^{i\alpha(x-ct)} \tag{14}$$

Using Eq. (14) in Eq. (12a), Eq. (12b), Eq. (12c) and Eq. (5), we get a set of differential equations for the medium M_1 as follows:

$$\frac{d^2 f}{dz^2} + 2mf_1^2 \frac{df}{dz} + h_1^2 f + (i\alpha\, mf_1^2 + i\alpha g J_1^2)\, j - g_1^2 T_1 = 0,$$

$$\frac{d^2 h}{dz^2} + m\frac{dh}{dz} + K_1^2 h = 0,$$

$$\frac{d^2 g}{dz^2} + 2m\frac{dg}{dz} + K_1^2 j + (i\alpha\, ml_1^2 - i\alpha g N_1^2)\, f = 0,$$

$$\frac{d^2T_1}{dz^2}+AT_1+B\left(\frac{d^2f}{dz^2}-\alpha^2 f\right)=0\,, \tag{15}$$

where,

$$f_1{}^2=\frac{\sum_{K=0}^{n}U_{KS}^2\left(-i\alpha c\right)^K}{\sum_{K=0}^{n}U_{KR}^2\left(-i\alpha c\right)^K}, \qquad h_1{}^2=\frac{\alpha^2c^2}{\sum_{K=0}^{n}U_{KR}^2\left(-i\alpha c\right)^K}-\alpha^2,$$

$$K_1{}^2=\frac{\alpha^2c^2}{\sum_{K=0}^{n}U_{KS}^2\left(-i\alpha c\right)^K}-\alpha^2, \quad J_1^2=\frac{1}{\sum_{K=0}^{n}U_{KT}^2\left(i\alpha c\right)^K}$$

$$l_1{}^2=\frac{\sum_{K=0}^{n}U_{KP}^2\left(-i\alpha c\right)^K}{\sum_{K=0}^{n}U_{KS}^2\left(-i\alpha c\right)^K}, \qquad g_1{}^2=\frac{\sum_{K=0}^{n}U_{KL}^2\left(-i\alpha c\right)^K}{\sum_{K=0}^{n}U_{KR}^2\left(-i\alpha c\right)^K}, \tag{16}$$

$$N_1^2=\frac{1}{\sum_{K=0}^{n}U_{KS}^2\left(i\alpha c\right)^K}, \quad A=\frac{C_\nu\, i\alpha c}{p}-\alpha^2, \quad B=\frac{i\alpha c T_0}{p}G_L$$

and those for the medium M_2 are given by

$$\frac{d^2f}{dz^2}+2lf_1^{'2}\frac{df}{dz}+h_1^{'2}f+(i\alpha l\, f_1^{'2}+i\alpha g J_1^{'2})\,j-g_1^{'2}T_1=0,$$

$$\frac{d^2h}{dz^2}+l\frac{dh}{dz}+K_1^{'2}h=0,$$

$$\frac{d^2g}{dz^2}+2l\frac{dg}{dz}+K_1^{'2}j+(i\alpha l.l_1^{'2}-i\alpha g N_1^{'2})\,f=0,$$

$$\frac{d^2T_1}{dz^2}+A'T_1+B'\left(\frac{d^2f}{dz^2}-\alpha^2f\right)=0, \tag{17}$$

where,

$$f_1'^2 = \frac{\sum_{K=0}^{n} U_{KS}'^2 \left(-i\alpha c\right)^K}{\sum_{K=0}^{n} U_{KR}'^2 \left(-i\alpha c\right)^K}, \qquad h_1'^2 = \frac{\alpha^2 c^2}{\sum_{K=0}^{n} U_{KR}'^2 \left(-i\alpha c\right)^K} - \alpha^2,$$

$$K_1'^2 = \frac{\alpha^2 c^2}{\sum_{K=0}^{n} U_{KS}'^2 \left(-i\alpha c\right)^K} - \alpha^2, \quad J_1'^2 = \frac{1}{\sum_{K=0}^{n} U_{KT}'^2 \left(i\alpha c\right)^K},$$

$$N_1'^2 = \frac{1}{\sum_{K=0}^{n} U_{KS}'^2 \left(i\alpha c\right)^K}, \qquad l_1'^2 = \frac{\sum_{K=0}^{n} U_{KP}'^2 \left(-i\alpha c\right)^K}{\sum_{K=0}^{n} U_{KS}'^2 \left(-i\alpha c\right)^K},$$

$$g_1'^2 = \frac{\sum_{K=0}^{n} U_{KL}'^2 \left(-i\alpha c\right)^K}{\sum_{K=0}^{n} U_{KR}'^2 \left(-i\alpha c\right)^K}, \qquad B' = \frac{i\alpha c T_0}{p'} G'_L \tag{18}$$

Eq. (15) and Eq. (17) must have exponential solutions in order that f, j, T_1, h will describe surface waves, and they must become varnishing small as $z \to \infty$. Hence for the medium M_1

$$\phi\,(x, z, t) = \left\{A_1 e^{-\lambda_1 z} + B_1 e^{-\lambda_2 z} + C_1 e^{-\lambda_3 z}\right\} e^{i\alpha(x-ct)}$$

$$\psi\,(x, z, t) = \left\{A_2 e^{-\lambda_1 z} + B_2 e^{-\lambda_2 z} + C_2 e^{-\lambda_3 z}\right\} e^{i\alpha(x-ct)}$$

$$T\,(x, z, t) = \left\{A_3 e^{-\lambda_1 z} + B_3 e^{-\lambda_2 z} + C_3 e^{-\lambda_3 z}\right\} e^{i\alpha(x-ct)}$$

$$v\,(x, z, t) = C e^{-\lambda_4 z + i\alpha(x-ct)} \tag{19a}$$

For finite disturbances as $z \to \infty$, for medium M_1 must hold Re $(\lambda_i)>0$ for i=1,2,3,4,5. Similarly for the medium M_2 are given by

$$\phi\,(x, z, t) = \left\{A'_1 e^{-\lambda'_1 z} + B'_1 e^{-\lambda'_2 z} + C'_1 e^{-\lambda'_3 z}\right\} e^{i\alpha(x-ct)}$$

$$\psi\ (x, z, t)= \left\{A'_2\, e^{-\lambda'_1 z} + B'_2\, e^{-\lambda'_2 z} + C'_2\, e^{-\lambda'_3 z}\right\} e^{i\alpha(x-ct)}$$

$$T\ (x, z, t)= \left\{A'_3\, e^{-\lambda'_1 z} + B'_3\, e^{-\lambda'_2 z} + C'_3\, e^{-\lambda'_3 z}\right\} e^{i\alpha(x-ct)}$$

$$v\ (x, z, t)= C' e^{-\lambda'_4 z + i\alpha(x-ct)} \tag{19b}$$

For finite disturbances as $z \to -\infty$, for medium M_2 must hold $Re(\lambda'_i)<0$ for $i=1,2,3,4,5$. Where, λ_j and λ'_j ($j = 1, 2, 3$) are the real roots of the Eqns.

$$\lambda^6 + \xi_1 \lambda^5 + \xi_2 \lambda^4 + \xi_3 \lambda^3 + \xi_4\lambda^2 + \xi_5\lambda + \xi_6 = 0, \tag{20}$$

where,

$\xi_1 = 2m \{1 + f_1^2\}$,

$\xi_2 = K_1^2 + A + 4m^2 + h_1^2 + Bg_1^2$,

$\xi_3 = 2mA + 2f_1^2\, m\, (K_1^2 + A) + 2mh_1^2 + 2mBg_1^2$,

$\xi_4 = AK_1^2 + 4m^2A\, f_1^2 + (K_1^2 + A)\, h_1^2 + \alpha^2\, m^2\, l_1^2\, f_1^2 + BK_1^2\, g_1^2 - \alpha^2\, Bg_1^2$,

$\xi_5 = 2mAK_1^2\, f_1^2 + 2mAh_1^2 - 2m\, \alpha^2\, Bg_1^2$,

$\xi_6 = AK_1^2\, h_1^2 + A\alpha^2\, m^2\, l_1^2\, f_1^2 - \alpha^2 B\, K_1^2\, g_1^2$. (21)

$$\lambda'^6+\xi'_1 \lambda'^5+\xi'_2 \lambda'^4+\xi'_3 \lambda'^3+\xi'_4\lambda'^2+\xi'_5\lambda'+\xi'_6= 0 \tag{22}$$

where,

$\xi'_1 = 2l\, \{1 + f_1'^2\}$,

$\xi'_2 = K_1'^2 + A' + 4l^2 + h_1'^2 + B'g_1'^2$,

$\xi'_3 = 2lA' + 2lf_1'^2\, (K_1'^2 + A) + 2lh_1'^2 + 2l\, B'g_1'^2$,

$\xi'_4 = A'K_1'^2 + 4l^2A'\, f_1'^2 + (K_1'^2 + A')\, h_1'^2 + \alpha^2\, l^2\, l_1'^2\, f_1'^2 + B'\, K_1'^2\, g_1'^2 - \alpha^2\, B'\, g_1'^2$,

$\xi'_5 = 2lA'K_1'^2\, f_1'^2 + 2lA'h_1'^2 - 2l\, \alpha^2\, B'g_1'^2$,

$\xi'_6 = A'K_1'^2\, h_1'^2 + A'\alpha^2\, l^2\, l_1'^2\, f_1'^2 - \alpha^2 B'\, K_1'^2\, g_1'^2$. (23)

$\lambda_4 = \{m + (m^2 - 4\, K_1^2)^{1/2}\}/ 2$,

$\lambda'_4 = \{l + (l^2 - 4\, K_1'^2)^{1/2}\}/ 2$

where the symbol used in eqns. (21) and (23) are given by eqns. (16) and (18). The constants A_j, B_j, C_j (j = 1, 2, 3) are related with A'_j, B'_j, C'_j (j = 1, 2, 3) in Eq. (19a) and Eq. (19b) by means of first equations in Eq. (15) and Eq. (17). Equating the coefficients of $e^{-\lambda_1 z}$, $e^{-\lambda_2 z}, e^{-\lambda_3 z}, e^{-\lambda'_1 z}$, $e^{-\lambda'_2 z}, e^{-\lambda'_3 z}$ to zero, after substituting Eq. (19a) and Eq. (19b) in the first and 3rd equations of Eq. (15) and Eq. (17) respectively, we get

$$A_2 = \gamma_1 A_1, \quad B_2 = \gamma_2 B_1, \quad C_2 = \gamma_3 C_1,$$

and

$$A_3 = \delta_1 A_1, \quad B_3 = \delta_2 B_1, \quad C_3 = \delta_3 C_1, \tag{24}$$

where,

$$\gamma_j = \frac{-i\alpha m l_1^2}{\lambda_j^2 - 2m\lambda_j + K_1^2} \text{ (j = 1, 2, 3)},$$

$$\delta_j = \frac{1}{g_1^2}\left[\lambda_j^2 - 2m f_1^2 \lambda_j + h_1^2 + i\,\alpha\, m f_1^2 \gamma_j\right], \quad j = 1, 2, 3.$$

Similar result holds for medium M_2 and usual symbols replacing by dashes respectively.

5 Boundary conditions

There are two boundary conditions

(i) The displacement components, temperature and temperature flux at the boundary surface between the media M_1 and M_2 must be continuous at all times and positions.

i.e. $$\left[u,\ v,\ w,\ T,\ p\frac{\partial T}{\partial z}\right]_{M_1} = \left[u,\ v,\ w,\ T, p'\frac{\partial T}{\partial z}\right]_{M_2}$$

(ii) The stress components τ_{31}, τ_{32}, τ_{33} must be continuous at the boundary z = 0.

i.e. $[\tau_{31}, \tau_{32}, \tau_{33}]_{M_1} = [\tau_{31}, \tau_{32}, \tau_{33}]_{M_2}$ at z = 0, respectively

where,

$$\tau_{31} = D_\mu \left(2\frac{\partial^2\phi}{\partial x\partial z} + \frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial z^2}\right), \qquad \tau_{32} = D_\mu \frac{\partial \nu}{\partial z},$$

$$\tau_{33} = D_\lambda \nabla^2\phi + 2D_\mu\left(\frac{\partial^2\phi}{\partial z^2} + \frac{\partial^2\phi}{\partial x\partial z}\right) - D_B T + D_{\mathrm{m}_e} H_0{}^2\nabla^2\phi + D_{\mathrm{E}_e} E_0{}^2\nabla^2\phi. \qquad (25)$$

Applying the boundary conditions, we get

$$A_1(1 - i\gamma_1\zeta_1) + B_1(1 - i\gamma_2\zeta_2) + C_1(1 - i\gamma_3\zeta_3) - A'_1(1 - i\gamma'_1\zeta'_1)$$
$$- B'_1(1 - i\gamma'_2\zeta'_2) - C'_1(1 - i\gamma'_3\zeta'_3) = 0 \qquad (26a)$$

$$C = C' \qquad (26b)$$

$$A_1(\gamma_1 + i\zeta_1) + B_1(\gamma_2 + i\zeta_2) + C_1(\gamma_3 + i\zeta_3) - A'_1(\gamma'_1 + i\zeta'_1)$$
$$- B'_1(\gamma'_2 + i\zeta'_2) - C'_1(\gamma'_3 + i\zeta'_3) = 0 \qquad (26c)$$

$$\delta_1 A_1 + \delta_2 B_1 + \delta_3 C_1 = \delta'_1 A'_1 + \delta'_2 B'_1 + \delta'_3 C'_1 \qquad (26d)$$

$$p\lambda_1\delta_1 A_1 + p\lambda_2\delta_2 B_1 + p\lambda_3\delta_3 C_1 - p'\lambda'_1\delta'_1 A'_1 + p'\lambda'_2\delta'_2 B'_1 - p'\lambda'_3\delta'_3 C'_1 = 0 \qquad (26e)$$

$$\mu^*_K[(2i\zeta_1 + \gamma_1 + \zeta_1^2\gamma_1)A_1 + (2i\zeta_2 + \gamma_2 + \zeta_2^2\gamma_2)B_1 + (2i\zeta_3 + \gamma_3 + \zeta_3^2\gamma_3)C_1]$$
$$= \mu'^*_K[(2i\zeta'_1 + \gamma'_1 + \zeta_1'^2\gamma'_1)A'_1 + (2i\zeta'_2 + \gamma'_2 + \zeta_2'^2\gamma'_2)B'_1$$
$$+ (2i\zeta'_3 + \gamma'_3 + \zeta_3'^2\gamma'_3)C'_1] \qquad (26f)$$

$$\mu^*_K[-\lambda_4 C] = \mu'^*_K[-\lambda'_4 C'] \qquad (26g)$$

$$A_1[(\lambda^*_K + (\mu_e)^*_K H_0^2 (\varepsilon'_e)^*_K E_0^2)(\zeta_1^2 - 1) + 2\mu^*_K(\zeta_1^2 - i\zeta_1) - \beta^*_K\delta_1]$$
$$+ B_1[(\lambda^*_K + (\mu_e)^*_K H_0^2 (\varepsilon'_e)^*_K E_0^2)(\zeta_2^2 - 1) + 2\mu^*_K(\zeta_2^2 - i\zeta_2) - \beta^*_K\delta_2]$$
$$+ C_1[(\lambda^*_K + (\mu_e)^*_K H_0^2 (\varepsilon'_e)^*_K E_0^2)(\zeta_3^2 - 1) + 2\mu^*_K(\zeta_3^2 - i\zeta_3) - \beta^*_K\delta_3]$$
$$= A'_1[(\lambda'^*_K + (\mu'_e)^*_K H_0^2 (\varepsilon'_e)^*_K E_0^2)(\zeta_1'^2 - 1) + 2\mu'^*_K(\zeta_1'^2 - i\zeta'_1) - \beta'^*_K\delta'_1]$$
$$+ B'_1[(\lambda'^*_K + (\mu'_e)^*_K H_0^2 (\varepsilon'_e)^*_K E_0^2)(\zeta_2'^2 - 1) + 2\mu'^*_K(\zeta_2'^2 - i\zeta'_2) - \beta'^*_K\delta'_2]$$
$$+ C'_1[(\lambda'^*_K + (\mu'_e)^*_K H_0^2 (\varepsilon'_e)^*_K E_0^2)(\zeta_3'^2 - 1) + 2\mu'^*_K(\zeta_3'^2 - i\zeta'_3) - \beta'^*_K\delta'_3] \qquad (26h)$$

where,

$$\zeta_j = \frac{\lambda_j}{\alpha}, \quad \zeta'_j = \frac{\lambda'_j}{\alpha}, \quad j = 1, 2, 3$$

$$\lambda^*_K = \sum_{K=0}^{n} \lambda_K \left(-i\alpha c\right)^K, \quad \mu^*_K = \sum_{K=0}^{n} \mu_K \left(-i\alpha c\right)^K, \quad \beta^*_K = \sum_{K=0}^{n} \beta_K \left(-i\alpha c\right)^K,$$

$$(\mu_e)^*_K = \sum_{K=0}^{n} (\mu_e)_K \left(-i\alpha c\right)^K, \quad (\varepsilon_e)^*_K = \sum_{K=0}^{n} (\varepsilon_e)_K \left(-i\alpha c\right)^K$$

and

$$\lambda'^*_K = \sum_{K=0}^{n} \lambda'_K \left(-i\alpha c\right)^K, \quad \mu'^*_K = \sum_{K=0}^{n} \mu'_K \left(-i\alpha c\right)^K, \quad \beta'^*_K = \sum_{K=0}^{n} \beta'_K \left(-i\alpha c\right)^K,$$

$$(\mu'_e)^*_K = \sum_{K=0}^{n} (\mu'_e)_K \left(-i\alpha c\right)^K, \quad (\varepsilon'_e)^*_K = \sum_{K=0}^{n} (\varepsilon'_e)_K \left(-i\alpha c\right)^K$$

From Eq. (26b) and Eq. (26h), we have C = C' = 0. Thus there is no propagation of displacement v. Hence SH-waves do not occur in this case. Finally, eliminating the constants A_1, B_1, C_1, A'_1, B'_1, C'_1 from the remaining equations, we get

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{vmatrix} = 0 \qquad (27)$$

where,

$a_{11} = 1 - i\gamma_1 \zeta_1$, $a_{12} = 1 - i\gamma_2\zeta_2$, $a_{13} = 1 - i\gamma_3\zeta_3$, $a_{14} = (i\,\gamma'_1\,\zeta'_1 - 1)$,

$a_{15} = (i\,\gamma'_2\,\zeta'_2 - 1)$, $a_{16} = (i\,\gamma'_3\,\zeta'_3 - 1)$,

$a_{21} = \gamma_1 + i\zeta_1$, $a_{22} = \gamma_2 + i\zeta_2$, $a_{23} = \gamma_3 + i\zeta_3$, $a_{24} = (\gamma'_1 + i\,\zeta'_1)$,

$a_{25} = (\gamma'_2 + i\zeta'_2)$, $a_{26} = (\gamma'_3 + i\zeta'_3)$,

$a_{31} = \delta_1$, $a_{32} = \delta_2$, $a_{33} = \delta_3$, $a_{34} = -\delta'_1$, $a_{35} = -\delta'_2$, $a_{36} = -\delta'_3$,

$a_{41} = p\lambda_1 \delta_1$, $a_{42} = p\lambda_2 \delta_2$, $a_{43} = p\lambda_3 \delta_3$, $a_{44} = -p' \lambda'_1 \delta'_1$,

$a_{45} = -p' \lambda'_2 \delta'_2$, $a_{46} = -p' \lambda'_3 \delta'_3$,

$a_{51} = \mu^*_K (2i\,\zeta_1 + \gamma_1 + \gamma_1 \zeta_1^2)$, $a_{52} = \mu^*_K (2i\,\zeta_2 + \gamma_2 + \gamma_2 \zeta_2^2)$,

$a_{53} = \mu^*_K (2i\,\zeta_3 + \gamma_3 + \gamma_3 \zeta_3^2)$, $a_{54} = \mu'^*_K (2i\,\zeta'_1 + \gamma'_1 + \gamma'_1 \zeta_1'^2)$,

$a_{55} = \mu'^*_K (2i\,\zeta'_2 + \gamma'_2 + \gamma'_2 \zeta_2'^2)$,

$a_{56} = \mu'^*_K (2i\,\zeta'_3 + \gamma'_3 + \gamma'_3 \zeta_3'^2)$,

$a_{61} = (\lambda^*_K + (\mu_e)^*_K H_0^2 + (\varepsilon_e)^*_K E_0^2)(\zeta_1^2 - 1) + 2\mu^*_K (\zeta_1^2 - i\zeta_1) - \beta^*_K \delta_1$,

$a_{62} = (\lambda^*_K + (\mu_e)^*_K H_0^2 + (\varepsilon_e)^*_K E_0^2)(\zeta_2^2 - 1) + 2\mu^*_K (\zeta_2^2 - i\zeta_2) - \beta^*_K \delta_2$,

$a_{63} = (\lambda^*_K + (\mu_e)^*_K + (\varepsilon_e)^*_K E_0^2 H_0^2)(\zeta_3^2 - 1) + 2\mu^*_K (\zeta_3^2 - i\zeta_3) - \beta^*_K \delta_3$,

$a_{64} = (\lambda'^*_K + (\mu'_e)^*_K H_0^2 + (\varepsilon_e)^*_K E_0^2)(\zeta_1'^2 - 1) + 2\mu'^*_K (\zeta_1'^2 - i\zeta'_1) - \beta'^*_K \delta'_1$,

$a_{65} = (\lambda'^*_K + (\mu'_e)^*_K H_0^2 + (\varepsilon_e)^*_K E_0^2)(\zeta_2'^2 - 1) + 2\mu'^*_K (\zeta_2'^2 - i\zeta'_2) - \beta'^*_K \delta'_2$,

$a_{66} = (\lambda'^*_K + (\mu'_e)^*_K H_0^2 + (\varepsilon_e)^*_K E_0^2)(\zeta_3'^2 - 1) + 2\mu'^*_K (\zeta_3'^2 - i\zeta'_3) - \beta'^*_K \delta'_3$. (28)

From Eq. (27), we obtain velocity of surface waves in common boundary between two viscoelastic, heterogeneous solid media under the influence of thermal, electric and magnetic field, where the viscosity is of general nth order involving time rate of change of strain.

6 Particular cases

Stoneley Waves:

Eq. (27) determine the wave velocity equation for Stoneley waves in the case of general magneto-thermo viscoelastic, heterogeneous solid media of nth order involving time rate of strain. Clearly from Eq. (27), it is follows that the wave velocity equation for Stoneley waves depends upon the heterogeneity of the material medium, temperature, electric, magnetic and viscous field. This equation, of course, is in well agreement with the corresponding classical result, when the effects of thermal, electric, magnetic and viscous field and heterogeneity are absent.

Rayleigh Waves:

To investigate the possibility of Rayleigh waves in a electro, magneto, thermo viscoelastic, heterogeneous elastic media, we replace media M_2 by vacuum, in the proceeding problem; we also note the SH-waves do not occur in this case.

Since the temperature difference across the boundary is always small so thermal condition given by

$$\frac{\partial T}{\partial z} + hT = 0 \text{ at } z = 0, \text{ respectively} \tag{28}$$

Thus Eq. (26f) and Eq. (26h) reduces to,

$$(2i\,\zeta_1 + \gamma_1 + \gamma_1\,\zeta_1^2)\,A_1 + (2i\,\zeta_2 + \gamma_2 + \gamma_2\,\zeta_2^2)\,B_1 + (2i\,\zeta_3 + \gamma_3 + \gamma_3\,\zeta_3^2)\,C_1 = 0 \tag{29a}$$

$$\begin{aligned}&[(\lambda_K^* + (\mu_e)^*_K\,H_0^2 + (\varepsilon_e)^*_K\,E_0^2)(\zeta_1^2 - 1) + 2\,\mu_K^*\,(\zeta_1^2 - i\zeta_1) - \beta_K^*\,\delta_1]\,A_1\\ &+ [(\lambda_K^* + (\mu_e)^*_K\,H_0^2 + (\varepsilon_e)^*_K\,E_0^2)(\zeta_2^2 - 1) + 2\,\mu_K^*\,(\zeta_2^2 - i\zeta_2) - \beta_K^*\,\delta_2]\,B_1\\ &+ [(\lambda_K^* + (\mu_e)^*_K\,H_0^2 + (\varepsilon_e)^*_K\,E_0^2)(\zeta_3^2 - 1) + 2\,\mu_K^*\,(\zeta_3^2 - i\zeta_3) - \beta_K^*\,\delta_3]\,C_1 = 0\end{aligned} \tag{29b}$$

From Eq. (28), we have

$$(\lambda_1 - h)\,\delta_1\,A_1 + (\lambda_2 - h)\,\delta_2\,B_1 + (\lambda_3 - h)\,\delta_3\,C_1 = 0 \tag{29c}$$

Eliminating A_1, B_1 and C_1 from Eqns. (29a), (29b) and (29c), we get

$$\det(b_{ij}) = 0,\ i, j = 1, 2, 3. \tag{30}$$

where,

$b_{11} = (2i\,\zeta_1 + \gamma_1 + \gamma_1\,\zeta_1^2)$, $b_{12} = (2i\,\zeta_2 + \gamma_2 + \gamma_2\,\zeta_2^2)$, $b_{13} = (2i\,\zeta_3 + \gamma_3 + \gamma_3\,\zeta_3^2)$,

$b_{21} = [(\lambda_K^* + (\mu_e)^*_K\,H_0^2 + (\varepsilon_e)^*_K\,E_0^2)(\zeta_1^2 - 1) + 2\,\mu_K^*\,(\zeta_1^2 - i\zeta_1) - \beta_K^*\,\delta_1]$,

$b_{22} = [(\lambda_K^* + (\mu_e)^*_K\,H_0^2 + (\varepsilon_e)^*_K\,E_0^2)(\zeta_2^2 - 1) + 2\,\mu_K^*\,(\zeta_2^2 - i\zeta_2) - \beta_K^*\,\delta_2]$,

$b_{23} = [(\lambda_K^* + (\mu_e)^*_K\,H_0^2 + (\varepsilon_e)^*_K\,E_0^2)(\zeta_3^2 - 1) + 2\,\mu_K^*\,(\zeta_3^2 - i\zeta_3) - \beta_K^*\,\delta_3]$,

$b_{31} = (\lambda_1 - h)\,\delta_1,$

$b_{32} = (\lambda_2 - h)\,\delta_2,$

$b_{33} = (\lambda_3 - h)\,\delta_3.$ (31)

Thus, Eq. (30) gives the wave velocity equation for Rayleigh waves in a heterogeneous, electro, magneto-thermo viscoelastic solid media of nth order involving time rate of strain.

From Eq. (30), it is follows that dispersion equation of Rayleigh waves depends upon the heterogeneity, the viscous, magnetic and thermal fields.

When the effects of thermal, electro, magnetic viscous field and heterogeneity are absent, this equation, of course, is in complete agreement with the corresponding classical result by Bullen [14].

Love Waves:

To investigate the possibility of love waves in a heterogeneous, viscoelastic solid media, we replace medium M_2 is obtained by two horizontal plane surfaces at a distance H-apart, while M_1 remains infinite. For medium M_1, the displacement component ν remains same as in general case given by equation (19). For the medium M_2, we preserve the full solution, since the displacement component along y-axis i.e. v no longer diminishes with increasing distance from the boundary surface of two media.

Thus

$$v' = C_1 e^{\lambda'_4 z + i\alpha(x-ct)} + C_2 e^{-\lambda'_4 z + i\alpha(x-ct)} \quad (32)$$

In this case, the boundary conditions are

(i) v and τ_{32} are continuous at z = 0

(ii) $\tau'_{32} = 0$ at z = –H.

Applying boundary conditions (i) and (ii) and using eqns (19) and (26), we get

$$C = C_1 + C_2 \quad (33)$$

$$-\mu_K^* \lambda_4 C = (\mu'_K)^* [\lambda'_4 C_1 - \lambda'_4 C_2] \tag{34}$$

$$C_1 e^{-\lambda'_4 H} - C_2 e^{\lambda'_4 H} = 0 \tag{35}$$

On eliminating the constants C, C_1 and C_2 from Eqns. (33), (34) and (35), we get

$$\tanh(\lambda'_4 H) = -\frac{\lambda_4 \mu_K^*}{\lambda'_4 (\mu'_K)^*}. \tag{36}$$

Thus Eq. (36) gives the wave velocity equation for Love waves in a heterogeneous, electro, magneto, thermo viscoelastic solid medium of nth order involving time rate of strain. Clearly it depends upon the heterogeneity, magnetic and viscous fields and independent of thermal field.

7 Conclusions

- The time rate of strain parameters influence the wave velocity of surface waves to an extent depending on the corresponding constants characterizing the electro-magneto thermo and viscoelasticity of the material. So the results of this analysis become useful in circumstances where these effects cannot be neglected. These velocities depend upon the wave number ‘ α ’ confirming that these waves are affected by heterogeneity of the material medium.
- It has been observed that temperature has no effect on Love waves. However, viscosity, gravity, magnetic fields, electric fields and heterogeneity of the medium effects the propagation of Love waves.
- The dispersion of Rayleigh waves is observed due to the presence of heterogeneity, temperature, gravity, magnetic field, electric field and viscosity of the medium.
- The wave velocity equation of Stoneley waves is very similar to the corresponding problem in the classical theory of elasticity. The dispersion of waves is due to the presence of heterogeneity, gravity, magnetic field, electric

field, temperature and viscoelasticity of the solid. Also, wave velocity equation of this generalized type of surface wave is in complete agreement with the corresponding classical result in the absence of all fields and heterogeneity.

- The solution of wave velocity equation for Stoneley waves cannot be determined by easy analytical methods however we can apply numerical techniques to solve this determinantal equation by choosing suitable values of physical constants for both media M_1 and M_2.

Acknowledgements. The authors are thankful to referees for their valuable comments.

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