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Bounds for the first eigenvalue of spherically symmetric surfaces

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Abstract

The bounds for the first eigenvalue of geodesic balls in spherically symmetric surfaces have been considered in this paper. These lower and upper bounds are C^0-independent on the metric coefficients. Under special conditions shown in the paper, we obtain sharper lower or upper bounds for the first eigenvalues of geodesic balls of spherically symmetric surfaces than those of Barroso-Bessa's for 2-dimensional case.

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1 Introduction

Here we would like to give sharper lower or upper bounds for the first Dirichlet eigenvalues of the Laplacian on geodesic balls of spherically symmetric surfaces. In order to state our result, we need to introduce some notations

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first. As in [2], for a given manifold M and a fixed vector $\xi \in T_pM$, $|\xi| = 1$, we can define the path of linear transformations $\mathbb{A}(t, \xi)$ and the curvature operator $R(t)$ (see page 3 of [2] for the detail), and they satisfy the Jacobi equation $\mathbb{A}'' + R(t)\mathbb{A} = 0$ with initial conditions $\mathbb{A}(0, \xi) = 0$, $\mathbb{A}'(0, \xi) = I$, and on the set $\exp_p(\mathcal{D}_p)$ the Riemannian metric of M can be expressed by

$$ds^2(\exp_p(t\xi)) = dt^2 + |\mathbb{A}(t, \xi)d\xi|^2,$$

where $\exp_p$ denotes the exponential map from $\mathcal{D}_p$ to $\exp_p(\mathcal{D}_p)$, and $\mathcal{D}_p = \{t\xi \in T_pM | 0 \leq t < d_\xi, |\xi| = 1\}$ with $d_\xi = \sup\{t > 0 | \mathrm{dist}_M(p, \gamma_\xi(t)) = t, \gamma_\xi(t)$ is the unique minimal geodesic satisfying
$\gamma_\xi(0) = p$ and $\gamma'_\xi(0) = \xi\}$. Clearly, $\mathcal{D}_p$ is the maximal domain of T_pM s.t. $\exp_p : \mathcal{D}_p \to \exp_p(\mathcal{D}_p)$ is a diffeomorphism.

As in [2] and other literatures, one can define the notion of spherically symmetric manifolds as follows.

Definition 1.1. *A manifold M is said to be spherically symmetric if the matrix $\mathbb{A}(t, \xi)$ satisfies $\mathbb{A}(t, \xi) = f(t)I$, for a function $f \in C^2([0, R])$, $R \in (0, \infty]$ with $f(0) = 0$, $f'(0) = 1$, $f|(0, R) > 0$.*

Let M^2 be a 2-dimensional complete spherically symmetric surface, and let $B_{M^2}(r)$ be the geodesic ball of radius $r > 0$ and centered at the base point of M^2. Denote by $\lambda_1(B_{M^2}(r))$ the first Dirichlet eigenvalue of the Laplace operator on $B_{M^2}(r)$ (we write $B_M(r)$ for short). By applying Cheng's conclusions in [5, 6] and Barroso-Bessa's result in [2], we can get a sharper upper or lower bound for $\lambda_1(B_{M^2}(r))$. In fact, we can prove the following theorem.

Theorem 1.2. *Let $B_M(r) \subset M^2$ be a ball defined as above in a spherically symmetric surface with metric $dt^2 + f^2(t)d\theta^2$, where $f \in C^2([0, R])$ with $f(0) = 0$, $f'(0) = 1$, $f(t) > 0$ for all $t \in (0, R]$. For every non-negative function $u \in C^0([0, r])$, set*

$$h(t, u) = \frac{u(t)}{\int_t^r \int_0^\sigma \frac{f(s)}{f(\sigma)} \cdot u(s)dsd\sigma}. \tag{1.1}$$

Then we have
(I) if in addition $f''(t) \geq 0$, then

$$\sup_t h(t, u) \geq \lambda_1(B_M(r)) \geq \left(\frac{c(0)}{r}\right)^2,$$

where $c(0)$ is the first zero of the Bessel function J_0. The first equality holds if and only if $u(t)$ is a first positive eigenfunction of $B_M(r)$ and $\lambda_1(B_M(r)) = h(t,u)$, and the second equality holds if and only if $B_M(r)$ is isometric to $B_{\mathbb{R}^3}(r)$, the geodesic ball with radius r in Euclidean space $\mathbb{R}^3$, or equivalently, $f''(t) = 0$, i.e. $f(t) = t$.
(II) if in addition $f''(t) \leq 0$, then

$$\left(\frac{c(0)}{r}\right)^2 \geq \lambda_1(B_M(r)) \geq \inf_t h(t,u),$$

where $c(0)$ is the first zero of the Bessel function J_0. The first equality holds if and only if $B_M(r)$ is isometric to $B_{\mathbb{R}^3}(r)$, or equivalently, $f''(t) = 0$, i.e. $f(t) = t$. The second equality holds if and only if $u(t)$ is a first positive eigenfunction of $B_M(r)$ and $\lambda_1(B_M(r)) = h(t,u)$.

Remark 1.3. Although the additional conditions concerning the sign of $f''(t)$ seem a little strong and unnatural, however, it has interesting meaning and tells us how the first eigenvalue $\lambda_1(B_M(r))$ with sufficiently small $r > 0$ changes when the sign of $f''(t)$ changes from positive to negative. Especially, $f(t) = t$ is the critical state, in this state, $\lambda_1(B_M(r)) = (c(0)/r)^2$, and the corresponding geodesic ball $B_M(r)$ is isometric to $B_{\mathbb{R}^3}(r)$ in $\mathbb{R}^3$.

We should remark that when $n = 2$, taking $u \equiv 1$ in (1.2) we get (1.1). In the following table, when $M^2 = \mathbb{S}^2$ (the 2-dimensional sphere), we compare our estimates (denote by YGC) for $\lambda_1(r) = \lambda_1(B_{\mathbb{S}^2}(r))$ for $r = \pi/8$, $\pi/4$, $3\pi/8$, $\pi/2$, $5\pi/8$ by taking $u(t) = \cos(\frac{t\pi}{2r})$ with the estimates obtained by Betz-Camera-Gzyl (BCG) in [1] and Barroso-Bessa (BB) in [4] respectively for the 2-dimensional case.

	$r =$	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$	$\frac{5\pi}{8}$
BCG	$\lambda_1(r)$	≥ 25.77	≥ 6.31	≥ 2.70	≥ 1.44	≥ 0.85
BB	$\lambda_1(r)$	≥ 35.85	≥ 8.78	≥ 3.76	$= 2$	≥ 1.01
		≤ 39.06	≤ 9.77	≤ 4.34	≤ 2.44	≤ 1.56
YGC	$\lambda_1(r)$	≥ 35.85	≥ 8.78	≥ 3.76	$= 2$	≥ 1.01
		≤ 37.50	≤ 9.375	≤ 4.167	≤ 2.34	≤ 1.50

2 Proof of Theorem 1.2

First, we recall the following two theorems due to S.Y. Cheng.

Theorem 2.1. (Cheng, [5]) *Let M be a complete, n-dimensional Riemannian manifold, all of whose sectional curvatures are less than or equal to a given constant k. Let $\lambda_1(V_n(k,\delta))$ denote the lowest Dirichlet eigenvalue of the the disk $V_n(k,\delta)$ with radius $\delta > 0$ in $\mathbb{M}_k$, a space form of constant sectional curvature k. Then for any $p \in M$, and $\delta > 0$ for which the geodesic ball $B_p(\delta) \subset M$ satisfies*

$$B_p(\delta) \subseteq \exp_p(\mathcal{D}_p),$$

we have

$$\lambda_1(B_p(\delta)) \geq \lambda_1(V_n(k,\delta)),$$

and equality holds if and only if $B(p,\delta)$ is isometric to $V_n(k,\delta)$.

Theorem 2.2. (Cheng, [6]) *Suppose that M is a complete n-dimensional Riemannian manifold and Ricci curvature of $M \geq (n-1)k$. Then, for $x_0 \in M$ we have*

$$\lambda_1(B_{x_0}(r_0)) \leq \lambda_1(V_n(k,r_0))$$

and equality holds if and only if $B_{x_0}(r_0)$ is isometric to $V_n(k,r_0)$.

Now, we can prove Theorem 1.2 as follows.

Proof of Theorem 1.2. By theorem 1.2 in [2], we can obtain the first eigenvalue $\lambda_1(B_M(r))$ of the geodesic ball $B_M(r)$ of the spherically symmetric surface M^2 satisfies

$$\sup_t h(t,u) \geq \lambda_1(B_M(r)) \geq \inf_t h(t,u), \tag{2.1}$$

where $h(t,u)$ is defined by (1.1). Moreover, equality in (2.1) holds if and only if u is the first positive eigenfunction of $B_M(r)$ and $\lambda_1(B_M(r)) = h(t,u)$.

On the other hand, since the Riemannian metric is given by $dt^2 + f^2(t)d\theta^2$, we can easily obtain an orthonormal cotangent frame field $\{w^1, w^2\}$ with $w^1 = dt$ and $w^2 = f(t)d\theta$ for the cotangent space T_p^*M. Then we can get the

Gauss curvature K of the spherically symmetric surface M in our case is $K = -f''(t)/f(t)$.

Hence, we can divide into two cases to discuss the bounds for the first Dirichlet eigenvalue $\lambda_1(B_M(r))$ of geodesic ball $B_M(r)$ on M^2.

Case (I). If $f''(t) \geq 0$, which implies the sectional curvature of M is less than or equal to 0, then by Theorem 2.1 together with the bounds in (2.1), we have

$$\sup_t h(t,u) \geq \lambda_1(B_M(r)) \geq \left(\frac{c(0)}{r}\right)^2, \tag{2.2}$$

where $c(0)$ is the first zero of the Bessel function J_0. The conditions under which the equalities hold in (2.2) are shown in Theorem 1.2 (I) and can be easily obtained by Theorem 2.1 and theorem 1.2 in [2].

Case (II). If $f''(t) \geq 0$, which implies the sectional curvature of M is greater than or equal to 0, and furthermore, the Ricci curvature is greater than or equal to 0, then by Theorem 2.2 together with the bounds in (2.1), we have

$$\left(\frac{c(0)}{r}\right)^2 \geq \lambda_1(B_M(r)) \geq \inf_t h(t,u), \tag{2.3}$$

where $c(0)$ is the first zero of the Bessel function J_0. The conditions under which the equalities hold in (2.3) are shown in the Theorem 1.2 (II) and can be easily obtained Theorem 2.2 and theorem 1.2 in [2]. $\square$

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