Theoretical Mathematics & Applications, vol.3, no.2, 2013, 71-78
ISSN: 1792-9687 (print), 1792-9709 (online)
Scienpress Ltd, 2013

Statistical Lacunary Invariant Summability

Nimet Pancaroglu[1] **and Fatih Nuray**[2]

Abstract

In this paper, we define statistical lacunary invariant summability and strongly lacunary q-invariant convergence $0 < q < \infty$ and investigate some relations between lacunary invariant statistical convergence, statistical lacunary invariant summability and strongly q- invariant convergence.

Mathematics Subject Classification: 40A05, 46A25
Keywords: statistical convergence, lacunary sequence, lacunary statistical convergence, invariant convergence, lacunary invariant statistical convergence

1 Introduction

Let σ be a mapping of the positive integers into itself. A continuous linear functional φ on l_∞, the space of real bounded sequences, is said to be an invariant mean or a σ mean, if and only if,

1. $\phi(x) \geq 0$, when the sequence $x = (x_n)$ is such that $x_n \geq 0$ for all n,

[1] Afyon Kocatepe University.
[2] Afyon Kocatepe University.

Article Info: *Received* : April 16, 2013. *Revised* : May 31, 2013
Published online : June 25, 2013

2. $\phi(e) = 1$,where $e = (1, 1, 1....)$,

3. $\phi(x_{\sigma(n)}) = \phi(x)$ for all $x \in l_\infty$.

The mappings ϕ are assumed to be one-to-one and such that $\sigma^m(n) \neq n$ for all positive integers n and m, where $\sigma^m(n)$ denotes the mth iterate of the mapping σ at n. Thus ϕ extends the limit functional on c, the space of convergent sequences, in the sense that $\phi(x) = \lim x$ for all $x \in c$. In case σ is translation mappings $\sigma(n) = n + 1$, the σ mean is often called a Banach limit and V_σ, the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences.

If $x = (x_n)$, set $Tx = (Tx_n) = (x_{\sigma(n)})$. It can be shown that

$$V_\sigma = \{x = (x_n) : \lim_m v_{mn}(x) = L \ uniformly \ in \ n, L = \sigma - \lim x\}$$

where,

$$v_{mn}(x) = \frac{x_n + Tx_n + ... + T^m x_n}{m+1}.$$

For example, the sequence $x = (x_n)$ defined as

$$x_n = \begin{cases} 1 & \text{n is odd} \\ 0 & \text{n is even} \end{cases}$$

is σ-convergent to $\frac{1}{2}$ for $\sigma(n) = n + 1$, but not convergent. The concept of statistical convergence for sequences of real numbers was introduced Fast[1], Salat[7] and others. Let $K \subseteq \mathbb{N}$ and $K_n = \{k \leq n : k \in K\}$. Then the natural density of K is defined by $\delta(K) = \lim_n n^{-1}|K_n|$ if the limit exists, where $|K_n|$ denotes the cardinality of K_n.

A sequence $x = (x_k)$ real numbers is said to be statistically convergent to L if for each $\varepsilon > 0$,

$$\lim_n \frac{1}{n}|\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0$$

By a lacunary sequence we mean an increasing integer sequence $\theta = (k_r)$ such that $k_0 = 0$ and $h_r := k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout this paper the intervals determined by θ will be denoted by $I_r := (k_{r-1}, k_r]$. By using lacunary sequences, Freedman, Sember and Raphael[2] defined the sequence space N_θ as follows: For any lacunary sequence $\theta = (k_r)$,

$$N_\theta = \left\{x = (x_k) : for \ some \ L, \lim_r \frac{1}{h_r} \sum_{k \in I_r} |x_k - L| = 0\right\}$$

In [3], lacunary statistically convergent sequence is defined as follows:

Let θ be a lacunary sequence; the number sequence (x_k) is lacunary statistically convergent to L provided that for every $\varepsilon > 0$,

$$\lim_r \frac{1}{h_r}|\{k \in I_r : |x_k - L| \geq \varepsilon\}| = 0.$$

Definition 1.1. [6] *A set E of positive integers said to have uniform invariant density of zero if and only if the number of elements of E which lie in the set $\{\sigma(m), \sigma^2(m), ..., \sigma^n(m)\}$ is $o(n)$ as $n \to \infty$, uniformly in m.*

Definition 1.2. [6] *A complex number sequence $x = (x_k)$ is said to be σ statistically convergent to L if for every $\varepsilon > 0$,*

$$\lim_n \frac{1}{n}|\{k \leq n : |x_{\sigma^k(m)} - L| \geq \varepsilon\}| = 0$$

uniformly in $m = 1, 2 \ldots$

Definition 1.3. [9] *Let be $\theta = (k_r)$ be a lacunary sequence; the number sequence $x = (x_k)$ is $S_{\sigma\theta}$ convergent to L provided that for every $\varepsilon > 0$,*

$$\lim_r \frac{1}{h_r}|\{k \in I_r : |x_{\sigma^k(m)} - L| \geq \varepsilon\}| = 0$$

uniformly in $m = 1, 2 \ldots$

2 Main Results

Now we define some new concepts by using the notions of uniform invariant density and lacunary invariant statistical convergence.

Definition 2.1. *A sequence $x = (x_k)$ is said to be lacunary invariant summable to L if*

$$\lim_r t_{rm}(x) = L$$

uniformly in $m = 1, 2 \ldots$ where $t_{rm}(x) = \frac{1}{h_r} \sum_{j \in I_r} x_{\sigma^j(m)}$.

This will be denoted $\sigma\theta - \lim_r x_r = L$.

Definition 2.2. *A sequence $x = (x_k)$ is said to be statistically lacunary invariant summable(or statistically lacunary σ summable) to L if for every $\varepsilon > 0$,*

$$\lim_n \frac{1}{n}|\{r \leq n : |t_{rm}(x) - L| \geq \varepsilon\}| = 0$$

uniformly in m.

In other words, a sequence $x = (x_k)$ is statistically lacunary invariant summable to L if and only if the sequence $(t_{rm}(x))$ is statistically convergent to L. In this case we write $S_{\theta\sigma} - \lim x = L$. We denote the set of all statistically lacunary invariant summable sequences by $S_{\theta\sigma}$.

Definition 2.3. [8] *A sequence $x = (x_k)$ is said to be strongly lacunary q-invariant convergent $(0 < q < \infty)$ to the limit L if*

$$\lim_r \frac{1}{h_r} \sum_{j \in I_r} |x_{\sigma^j(m)} - L|^q = 0$$

uniformly in $m = 1, 2, ...$ and we write it as $x_k \to L([V_{\theta\sigma}]_q)$. In this case L is called the $[V_{\theta\sigma}]_q$ limit of x. We denote the set all strongly lacunary q-invariant convergent sequences by $[V_{\theta\sigma}]_q$.

Now we prove some relations between lacunary invariant statistical convergence, statistical lacunary invariant summability, strong lacunary q-invariant convergence and statistical convergence.

In the first theorem we investigate a relation between lacunary invariant statistical convergence and statistical lacunary invariant summability.

Theorem 2.4. *If sequence $x = (x_k)$ is bounded and lacunary invariant statistically convergent to L then it is statistically lacunary invariant summable to L.*

Proof. Let $x = (x_k)$ be bounded and lacunary invariant statistically convergent to L; them we write, for each $m \geq 1$,

$$K_{\theta\sigma}(\varepsilon) = \{k_{r-1} \leq j \leq k_r : |x_{\sigma^j(m)} - L| \geq \varepsilon\}.$$

Hence,

$$
\begin{aligned}
|t_{rm}(x) - L| =& \left| \frac{1}{h_r} \sum_{j \in I_r} x_{\sigma^j(m)} - L \right| \\
=& \left| \frac{1}{h_r} \sum_{j \in I_r} (x_{\sigma^j(m)} - L) \right| \\
\leq& \left| \frac{1}{h_r} \sum_{j \in K_{\theta\sigma}(\varepsilon)} (x_{\sigma^j(m)} - L) \right| \\
\leq& \frac{1}{h_r} (\sup_{j,m} |x_{\sigma^j(m)} - L|) |K_{\theta\sigma}(\varepsilon)| \to 0
\end{aligned}
$$

as $r \to \infty$, which implies that $t_{rm}(x) \to L$ uniformly in m. That is, x is lacunary invariant convergent to L and hence statistically lacunary invariant summable to L. This completes proof of theorem. □

Now we shall give relation between lacunary invariant statistical convergence and strong lacunary q-invariant convergence in the following theorem.

Theorem 2.5. *i)If $0 < q < \infty$ and a sequence $x = (x_k)$ is strongly lacunary q-invariant convergent to the limit L, then it is lacunary invariant statistically convergent to L.*

ii)If (x_k) is bounded and lacunary invariant statistically convergent to L then $x_k \to L([V_{\theta\sigma}]_q)$.

Proof. (i)If $0 < q < \infty$ and $x_k \to L([V_{\theta\sigma}]_q)$, then as $r \to \infty$, for each $m \geq 1$,

$$
\begin{aligned}
0 \leftarrow \frac{1}{h_r} \sum_{j \in I_r} |x_{\sigma^j(m)} - L|^q \geq& \frac{1}{h_r} \sum_{\substack{j \in I_r \\ |x_{\sigma^j(m)} - L| \geq \varepsilon}} |x_{\sigma^j(m)} - L|^q \\
\geq& \frac{\varepsilon^q}{h_r} |K_{\sigma\theta}(\varepsilon)|.
\end{aligned}
$$

That is,

$$
\lim_{r \to \infty} \frac{1}{h_r} |\{k_{r-1} \leq j \leq k_r : |x_{\sigma^j(m)} - L| \geq \varepsilon\}| = 0
$$

uniformly in m. Hence $x = (x_k)$ is invariant statistically convergent to L.

(ii)Suppose that $x = (x_k)$ is bounded and lacunary invariant statistically convergent to L. Then for $\varepsilon > 0$, we have

$$\lim_{r\to\infty} \frac{1}{h_r} |\{j \in I_r : |x_{\sigma^j(m)} - L| \geq \varepsilon\}| = 0$$

uniformly in m. Since $x \in \ell_\infty$, there exists $M > 0$ such that $|x_{\sigma^j(m)} - L| \leq M$, $(j = 1, 2\ldots)$ and $(m = 1, 2\ldots)$ and we have

$$\begin{aligned}\frac{1}{h_r}\sum_{j\in I_r} |x_{\sigma^j(m)} - L|^q &= \frac{1}{h_r}\sum_{\substack{j\in I_r\\ j\notin K_{\sigma\theta}(\varepsilon)}} |x_{\sigma^j(m)} - L|^q + \frac{1}{h_r}\sum_{\substack{j\in I_r\\ j\in K_{\sigma\theta}(\varepsilon)}} |x_{\sigma^j(m)} - L|^q \\ &= S_1(r) + S_2(r)\end{aligned}$$

where

$$S_1(r) = \frac{1}{h_r}\sum_{\substack{j\in I_r\\ j\notin K_{\sigma\theta}(\varepsilon)}} |x_{\sigma^j(m)} - L|^q$$

$$S_2(r) = \frac{1}{h_r}\sum_{\substack{j\in I_r\\ j\in K_{\sigma\theta}(\varepsilon)}}^{n} |x_{\sigma^j(m)} - L|^q$$

Now if $j \notin K_{\sigma\theta}(\varepsilon)$ then $S_1(r) < \varepsilon^q$. For $j \in K_{\sigma\theta}(\varepsilon)$, we have

$$S_2(r) \leq (\sup |x_{\sigma^j(m)} - L|)\frac{|K_{\sigma\theta}(\varepsilon)|}{h_r} \leq M\frac{|K_{\sigma\theta}(\varepsilon)|}{h_r} \to 0,$$

Since,

$$\frac{1}{h_r}\sum_{j\in I_r} |x_{\sigma^j(m)} - L|^q = 0$$

uniformly in m. We have $x_k \to L([V_{\theta\sigma}]_q)$.

This completes the proof of the theorem. □

In the next result we classify statistically lacunary invariant summable sequences through the lacunary invariant summability of subsequences.

Theorem 2.6. *A sequence $x = (x_k)$ is statistically lacunary invariant summable to L if and only if there exists a set $K = \{(r_i) : r_i < r_{i+1}\} \subseteq \mathbb{N}$ such that $\delta(K) = 1$ and $\theta\sigma - \lim x_{r_n} = L$.*

Proof. Suppose that there exists a set $K = \{(r_i) : r_i < r_{i+1}\} \subseteq \mathbb{N}$ such that $\delta(K) = 1$ and $\theta\sigma - \lim x_{r_n} = L$. Then there is a positive integer N such that for $n > N$ and for each $m \geq 1$,

$$|t_{r_n m}(x) - L| < \varepsilon \tag{1}$$

Put $K_\varepsilon(\theta\sigma) = \{n \in \mathbb{N} : |t_{r_n m}(x) - L| \geq \varepsilon\}$ and $K' = \{r_{N+1}, r_{N+2}, ...\}$. Then $\delta(K') = 1$ and $K_\varepsilon(\theta\sigma) \subseteq \mathbb{N} - K'$ which implies that

$$\lim_{n\to\infty} \frac{1}{n}|\{r \leq n : |t_{r_n m}(x) - L| \geq \varepsilon\}| = 0$$

uniformly in m. Hence $x = (x_k)$ is statistically lacunary invariant summable to L.

Conversely, let $x = (x_k)$ be statistically lacunary invariant summable to L. For $r = 1, 2 \ldots$ and $m = 1, 2 \ldots$ put $K_p(\theta\sigma) = \{j \in \mathbb{N} : |t_{r_j m}(x) - L| \geq \frac{1}{p}\}$ and $M_p(\theta\sigma) = \{j \in \mathbb{N} : |t_{r_j m}(x) - L| < \frac{1}{p}\}$. Then

$$\lim_{n\to\infty} \frac{1}{n}|\{j \leq n : |t_{r_j m}(x) - L| \geq \frac{1}{p}\}| = 0$$

and

$$M_1(\theta\sigma) \supset M_2(\theta\sigma) \supset M_3(\theta\sigma) \supset ..... \supset M_i(\theta\sigma) \supset M_{i+1}(\theta\sigma) \supset ... \tag{2}$$

and

$$\lim_{n\to\infty} \frac{1}{n}|\{j \leq n : |t_{r_j m}(x) - L| < \frac{1}{p}\}| = 1 \tag{3}$$

uniformly in m.

Now we have to show that for $j \in M_p(\theta\sigma)$, (x_{k_j}) is lacunary invariant summable to L. Suppose that (x_{k_j}) is not lacunary invariant summable to L. Therefore there is $\varepsilon > 0$ such that $|t_{r_j m}(x) - L| \geq \varepsilon$ for infinitely many terms. Let $M_\varepsilon(\theta\sigma) = \{j \in \mathbb{N} : |t_{r_j m}(x) - L| < \varepsilon\}$ and $\varepsilon > \frac{1}{p}$ $(p = 1, 2 \ldots)$. Then ,

$$\lim_{n\to\infty} \frac{1}{n}|\{j \leq n : |t_{r_j m}(x) - L| < \varepsilon\}| = 0 \tag{4}$$

uniformly in m and by (2), we have $M_p(\theta\sigma) \subset M_\varepsilon(\theta\sigma)$. Hence

$$\lim_{n\to\infty} \frac{1}{n}|\{j \leq n : |t_{r_j m}(x) - L| < \frac{1}{p}\}| = 0 \tag{5}$$

uniformly in m which contradicts (3) and therefore (x_{k_j}) is lacunary invariant summable to L. This completes proof of the theorem. □

Similarly we can prove the following dual statement.

Theorem 2.7. *A sequence $x = (x_k)$ is lacunary invariant statistically convergent to L if and only if there exists a set $K = \{(k_i) : k_i < k_{i+1}\} \subseteq \mathbb{N}$ such that $\delta_{\theta\sigma}(K) = 1$ and $\sigma - \lim x_{k_n} = L$.*

References

[1] H. Fast, Sur la convergence statistique, *Colloq.Math.*, **2**, (1951), 241-244.

[2] A.R. Freedman, J.J. Sember and M. Raphael, Some Cesaro type summability, *Proc. London Math. Soc.*, **37**, (1978), 508-520.

[3] J.A. Fridy and C. Orhan,Lacunary statistical convergence, *Pacific Journal of Math.*, **160**, (1993), 43-51.

[4] M. Mursaleen, H. Osama and H. Edely, On the invariant mean and statistical convergence, *Appl. Math. Lett.*, **22**, (2009), 1700-1704.

[5] M. Mursaleen and A. Alotaibi, Statistical lacunary summability and a Korovkin type approximation theorem, *Ann. Univ. Ferrara*, **57**, (2011), 373-381.

[6] F. Nuray and E. Savaş, Invariant statistical convergence and A-invariant statistical convergence, *Indian J. Pure Appl. Math.*, **25**, (1994), 267-274.

[7] T. Salat, On statistically convergent sequences of real numbers, *Math. Slovaca*, **30**, (1980), 139-150.

[8] E. Savaş, Strongly σ-Convergence Sequences, *Bull. Calcutta Math. Soc.*, **81**, (1989), 295-300.

[9] E. Savaş and F. Nuray, On σ- statistically convergence and lacunary σ-statistically convergence, *Math. Slovaca*, **43**(3), (1993), 309-315.

[10] E. Savaş, Strongly σ- Summable and Strongly σ-Convergence Sequences, *Bull. Calcutta Math. Soc.*, **81**, (1989), 295-300.